BULL. AUSTRAL. MATH. SOC. 10F25
VOL. 32 (1985), 379-387.

ON A CONJECTURE OF LITTLEWOOD IN DIOPHANTINE APPROXIMATIONS

S. KRASS

A conjecture of Littlewood states that for arbitrary $\underline{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$, $n \geq 2$, and any $\varepsilon > 0$ there exist $m_0 \neq 0, m_1,\ldots,m_n$ so that $|m_0 \prod_{i=1}^{n} (m_0x_i-m_i)| < \varepsilon$. In this paper we show this conjecture holds for all $\underline{\xi} = (\xi_1,\ldots,\xi_n)$ such that $1, \xi_1,\ldots,\xi_n$ is a rational basis of a real algebraic number field of degree $n+1$.

1. Introduction

In a paper by Cassels and Swinnerton-Dyer in 1955, [2] they show that if $1, \alpha_1, \alpha_2$ is a basis (over $\mathbb{Q}$) of a real cubic number field, then, for any $\varepsilon > 0$, there exist integers $m_0 \neq 0, m_1, m_2$ such that

$$|m_0(m_0\alpha_1-m_1)(m_0\alpha_2-m_2)| < \varepsilon$$

This result reinforces (but of course does not prove) for $n = 2$ a conjecture by Littlewood that for arbitrary $\underline{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$, $n \geq 2$, and any $\varepsilon > 0$ there exists $\underline{m} = (m_0,m_1,\ldots,m_n) \in \mathbb{Z}^{n+1}$ with $m_0 \neq 0$ such that

Received 26 March 1985,

$$|m_0 \prod_{i=1}^{n} (m_0 x_i - m_i)| < \varepsilon.$$

In this paper we extend the Cassels, Swinnerton-Dyer result from $n = 2$ to all $n \geq 2$. That is to say if $\underline{\xi} = (\xi_1, \ldots, \xi_n)$ with $1, \xi_1, \ldots, \xi_n$ a basis of F, a real number field of degree $n + 1$, then, for any $\varepsilon > 0$, there exists $\underline{m} \in \mathbb{Z}^{n+1}$, $m_0 \neq 0$ such that

$$|m_0 \prod_{i=1}^{n} (m_0 \xi_i - m_i)| < \varepsilon$$

If M is a full $\mathbb{Z}$-module in F, $\mathcal{R}(M)$ denotes the coefficient ring of M. Namely

$$\mathcal{R}(M) = \{\alpha \in F \ : \ \alpha M \subseteq M\}.$$

We will first need to prove the following lemma concerning units in $\mathcal{R}(M)$ which may be of independent interest.

LEMMA. *M is a full $\mathbb{Z}$-module in F a real (algebraic) number field of degree $n+1$ and $\mathcal{R}(M)$ the coefficient ring of M. For all $\alpha = \alpha_{[0]} \in F$, $\alpha_{[j]}$, $j = 0, \ldots, n$ denote the conjugates of α ordered so that $\alpha_{[j]} \in \mathbb{R}$, $j = 0, \ldots, n-2s$, $\alpha_{[j]} = \overline{\alpha}_{[s+j]} \in \mathbb{C}$, $j = n-2s+1, \ldots, n-s$ where s is the number of pairs of complex conjugates. Then there exists an infinite sequence $\Gamma = (\gamma_k \in \mathcal{R}(M),\ k = 1, 2, \ldots)$ with each γ_k a unit in $\mathcal{R}(M)$ such that*

(i) $\lim_{k \to \infty} \gamma_{k[j]}/\gamma_{k[n-s]} = 1$, $j = 1, \ldots, n-s-1$ (*with* $\gamma_{k[j]} = (\gamma_k)_{[j]}$)

(ii) $\lim_{k \to \infty} \gamma_k = \infty$.

2. Proof of Lemma

By a theorem of Dirichlet the (multiplicative) group of units in $\mathcal{R}(M)$ is generated by $n-s$ independent units, [1, p. 112]. Let β_i, $i = 1, \ldots, n-s$ be $n-s$ independent units in $\mathcal{R}(M)$. For $\varepsilon > 0$ it is clear that the system of inequalities

$$(2.1)\qquad \left|\sum_{i=1}^{n-s} x_i \log|\beta_{i[j]}/\beta_{i[n-s]}|\right| < \varepsilon,\ j = 1,\ldots,n-s-1.$$

has infinitely many (integer) solutions $\underline{x} = \underline{v} = (v_1,\ldots,v_{n-s}) \in \mathbb{Z}^{n-s}$.

Let $\varepsilon_k > 0$, $k = 1,2,\ldots$ with $\lim\limits_{k \to 0} \varepsilon_k = 0$ and let

$\underline{v}_k = (v_{k1},\ldots,v_{k,n-s}) \in \mathbb{Z}^{n-s}$ be an integer solution of (2.1) for $\varepsilon = \varepsilon_k$, $k = 1,2,\ldots$. Then writing

$$(2.2)\qquad \psi_k = \prod_{i=1}^{n-s} \beta_i^{v_{ki}}\ ,\quad k = 1,2,3,\ldots$$

we have by construction

$$(2.3)\qquad \lim_{k \to \infty} |\psi_{k[j]}/\psi_{k[n-s]}| = 1\ ,\ j = 1,\ldots,n-s-1.$$

Now we will write for all k (with $e(x) = e^{\sqrt{-1}\,2\pi x}$)

$$\psi_{k[j]} = |\psi_{k[j]}|\, e(\theta_{kj}),\ -\tfrac{1}{2} < \theta_{kj} \le \tfrac{1}{2}\ ,\ j = 0,\ldots,n.$$

Since $\psi_{k[j]} \in \mathbb{R}$, $j = 0,\ldots,n-2s$, replacing β_i by β_i^2, $i = 1,\ldots,n$, if necessary, we may assume without loss of generality.

$$(2.4)\qquad \theta_{kj} = 0,\quad j = 0,\ldots,n-2s.$$

Now the infinite set of "amplitude" vectors

$$\Theta = \{\underline{\theta}_k = (\theta_{k,n-2s+1},\ldots,\theta_{k,n-s}) : k = 1,2,\ldots\}$$

must contain at least one limit point $\underline{\phi} = (\phi_{n-2s+1},\ldots,\phi_{n-s})$, say, with $-\tfrac{1}{2} \le \phi_j \le \tfrac{1}{2}$, $j = n-2s+1,\ldots,n-s$.

If $\underline{\phi} = \underline{0}$, then $\gamma_k = \psi_k$ satisfies (i) of the lemma. So we suppose $\underline{\phi} \ne \underline{0}$. We may then choose an infinite subsequence of the ψ_k

$$\rho_p = \psi_{k_p}\ ,\ p = 1,2,\ldots$$

such that

$$\lim_{p \to \infty} \rho_{p[j]}/|\rho_{p[j]}| = e(\phi_j),\quad j = n-2s+1,\ldots,n-s.$$

Finally we put

https://doi.org/10.1017/S0004972700002495 Published online by Cambridge University Press

$$\gamma_k = \rho_{k+1}/\rho_k \ .$$

Writing $\gamma_{k[j]} = |\gamma_{k[j]}| \ e(\phi_{kj})$ it is clear that

$$\lim_{k \to \infty} \phi_{kj} = 0 \ , \quad j = n - 2s + 1, \ldots, n-s.$$

Of course $\phi_{kj} = 0$, $j = 0, \ldots, n-2s$ by (2.4).

Then we only need observe that, for $j = 1, \ldots, n-s-1$,

$$|\gamma_{k[j]}/\gamma_{k[n-s]}| = |\rho_{k+1,[j]}/\rho_{k+1,[n-s]}| \ |\rho_{k[j]}/\rho_{k[n-s]}| \to 1, \text{ as } \ k \to \infty$$

to see that γ_k, $k = 1, 2, \ldots$ satisfies (i) of the lemma.

Now the (homogeneous) simultaneous equation system

$$\text{(2.5)} \qquad \sum_{i=0}^{n-s} x_i \ log|\beta_{i[j]}/\beta_{i[n-s]}| = 0 \ , \quad j = 1, \ldots, n-s-1$$

has at least one of the $(n-s-1) \times (n-s-1)$ submatrices of its coefficient matrix non-singular (since the regulator is non zero). So the solution set of (2.5) is the line, L, where

$$L = \{\lambda \underline{y} : \in \mathbb{R} \ , \ \text{and} \ \underline{y} \neq \underline{0}, \ \underline{y} \in \mathbb{R}^{n-s} \ \text{is a solution of (2.5)}\}.$$

The set

$$\{\underline{v}_k \in \mathbb{Z}^{n-s} : \gamma_k = \prod_{i=1}^{n-s} \beta_i^{v_{ki}} \ , \ k = 1, 2, \ldots\}$$

is a subset of solutions $\underline{x} \in \mathbb{Z}^{n-s}$ satisfying (2.1) and the elements are lattice points lying "near" the line L.

Suppose for the present $\prod_{i=1}^{n-s} \beta_i^{y_i} \neq 1$. Observe both $\underline{x} = \underline{y}$ and $\underline{x} = -\underline{y}$ satisfy (2.5). Hence we may choose $\underline{y}$ with $\prod_{i=1}^{n-s} \beta_i^{y_i} > 1$. Then for any $J > 0$ there exists $\underline{v}_k$ near $\lambda \underline{y}$ for some $\lambda > J$, establishing (ii) of the lemma. So we need only show

$$\text{(2.6)} \qquad \prod_{i=1}^{n-s} \beta_i^{y_i} \neq 1, \ \ any \ \ \underline{y} \neq \underline{0}, \ \underline{y} \ \ a \ solution \ to \ (2.5).$$

Suppose $\prod_{i=1}^{n-s} \beta_i^{y_i} = 1$. Then it follows easily that there exist solutions

https://doi.org/10.1017/S0004972700002495 Published online by Cambridge University Press

$\underline{x} = \underline{v} \in \mathbb{Z}^{n-s}$ to (1) such that $|\theta| = |\prod_{i=1}^{n-s} \beta_i^{v_i}| \approx 1$ *and* $|\theta_{[1]}| \approx \ldots \approx |\theta_{[n-s]}|$ where "$\approx$" denotes equality up to any arbitrarily small fixed error.

But $|\theta_{[n-2s+j]}| = |\theta_{[n-s+j]}|$, $j = 1,\ldots,s$ and $1 = \prod_{j=0}^{n} |\theta_{[j]}|$, so $\prod_{i=1}^{n-s} \beta_i^{y_i} = 1 \Rightarrow$ there exist irrational units θ with all conjugates arbitrarily near the unit circle. But this is impossible, see [3, p.137]. So (2.6) is established proving the lemma.

The following results follow trivially.

COROLLARY 1. *The lemma holds with (i) replaced by*

(i') $$\lim_{k \to \infty} \gamma_{k[j]}/\gamma_{k[n]} = 1,\ j = 1,\ldots,n.$$

COROLLARY 2. *Both the lemma and Corollary 1 hold with*

(ii) *replaced by*

(ii') $$\lim_{k \to \infty} \gamma_k = 0\ .$$

3. Theorem

Let $\underline{\xi} = (\xi_1,\ldots,\xi_n)$ *so that* $1, \underline{\xi}$ *is a basis of* F *a real number field of degree* $n+1$. *Then for any* $\varepsilon > 0$ *there exist integers* $m_0 \neq 0, m_1,\ldots,m_n$ *so that*

$$|m_0 \prod_{j=1}^{n} (m_0\ \xi_j - m_j)| < \varepsilon$$

Proof. Let $\xi_0 = 1$ and $\xi_{[i]j} = (\xi_j)_{[i]}$, $i,j = 0,\ldots,n$ with conjugates ordered by the convention in the lemma. A is the matrix

$$A = (\xi_{[i]j} : i,j = 0,\ldots,n)\ .$$

It is well-known that $\det A \neq 0$. So we may define

$$U = (u_{ij} : i,j = 0,\ldots,\ n) = A^{-1}.$$

By the row conjugate structure of A we have

https://doi.org/10.1017/S0004972700002495 Published online by Cambridge University Press

$$u_{i0} \in F \quad \text{and} \quad u_{ij} = (u_{i0})_{[j]} \quad , \quad i,j = 0,\ldots,n.$$

Thus $u_{00},\ldots,u_{n0}$ is a base of the full $\mathbb{Z}$-module

$$M = \{\underline{m}.\underline{u}_0 = \sum_{i=0}^{n} m_i u_{i0} : \underline{m} = (m_0,\ldots,m_n) \in \mathbb{Z}^{n+1}\} \quad .$$

We have used the notation

$$\underline{u}_j = (u_{0j},\ldots,u_{nj})^t = j\text{-}th \text{ column of } U, \; j = 0,\ldots,n.$$

By Corollary 1 there exists a sequence of units

$$\Gamma = (\gamma_k \in \mathcal{R}(M), \; k = 1,2,\ldots) \quad \text{such that}$$

(3.1) $$\lim_{k\to\infty} \gamma_{k[j]}/\gamma_{k[n]} = 1, \quad j = 1,\ldots,n; \quad and \quad \lim_{k\to\infty} \gamma_k = \infty \; .$$

For convenience, noting $u_{n0} \neq 0$, $u_{n0} \in M$, we write

$$|Norm\; u_{n0}| = |\prod_{j=0}^{n} u_{nj}| = \upsilon > 0.$$

Clearly $\gamma_k u_{n0} \in M$, with $|Norm\; \gamma_k u_{n0}| = \upsilon$, all $\gamma_k \in \Gamma$.

Let

$$\mathbb{Z}(\Gamma) = \{\underline{m} = \underline{m}_k \in \mathbb{Z}^{n+1} : \underline{m}_k.\underline{u}_0 = \gamma_k u_{n0}, \; \gamma_k \in \Gamma\}$$

By (3.1) and this definition

(3.2') $$\begin{cases} \lim_{k\to\infty} \underline{m}_k.\underline{u}_j/\underline{m}_k.\underline{u}_n = u_{nj}/u_{nn} \; , \; j = 1.\ldots,n \\ \lim_{k\to\infty} |\underline{m}_k.\underline{u}_0| = \infty \; . \end{cases}$$

Thus for any $\varepsilon > 0$ and $K > 1$ we have for all sufficiently large k

(3.2') $$\begin{cases} \underline{m}_k.\underline{u}_j/\underline{m}_k.\underline{u}_n - u_{nj}/u_{nn} = \varepsilon_{kj}, \; |\varepsilon_{kj}| < \varepsilon \; , \; j = 1,\ldots,n \\ |\underline{m}_k.\underline{u}_0| > K \end{cases}$$

Since $\upsilon = |\prod_{j=0}^{n} \underline{m}_k.\underline{u}_j| = |\underline{m}_k.\underline{u}_0|\,|\underline{m}_k.\underline{u}_m|^n \prod_{j=1}^{n-1} |\underline{m}_k.\underline{u}_j/\underline{m}_k.\underline{u}_k|$

it follows from (3.2') for all $\underline{m}_k \in \mathbb{Z}(\Gamma)$ with k sufficiently large

$$\upsilon > K|\prod_{j=1}^{n-1} \tfrac{1}{2}\, u_{nj}/u_{nn}|\,|\underline{m}_k.\underline{u}_n|^n \; .$$

https://doi.org/10.1017/S0004972700002495 Published online by Cambridge University Press

So $\underline{m}_k \cdot \underline{u}_n = O(K^{-1/n})$ and then by (3.2')

(3.3) $\underline{m}_k \cdot \underline{u}_j = O(K^{-1/n}),\ j = 1,\dots,n,\ \underline{m}_k \in \mathbb{Z}(\Gamma)$ (sufficiently large) k.

We note, and it is easily shown, that there are only finitely many $\underline{m} \in \mathbb{Z}(\Gamma)$ with $m_0 = 0$. So without loss of generality we suppose

(3.4) $$\underline{m} \in \mathbb{Z}(\Gamma) \Rightarrow m_0 > 0$$

as (3.2) holds if we replace $\underline{m}_k$ by $-\underline{m}_k$.

Now suppose $\lim_{k \to \infty} \underline{m}_k \cdot \underline{u}_0 / m_{k0} = 0$. Then writing

$$\underline{w}_k = (w_{k0},\dots,w_{kn}),\quad w_{kj} = \underline{m}_k\, \underline{u}_j / m_{k0},\quad j = 0,\dots,n$$

we have by this assumption together with (3.3) and (3.4)

$$\underline{0} \neq \lim_{k \to \infty} \underline{m}_k / m_{k0} = \lim_{k \to \infty} \underline{w}_k\, U^{-1} = 0 .$$

By this contradiction we have shown there exists $w > 0$ so that

(3.5) $|\underline{m}_k \cdot \underline{u}_0 / m_{k0}| > w,\ \underline{m}_k \in \mathbb{Z}(\Gamma)$, all (sufficiently large) k.

$$\upsilon = |\underline{m}_k \cdot \underline{u}_0 / m_{k0}|\, |m_{k0}^{1/n}\ \underline{m}_k \cdot \underline{u}_n|^n \prod_{j=1}^{n-1} |\underline{m}_k \cdot \underline{u}_j / \underline{m}_k \cdot \underline{u}_n| .$$

$$\Rightarrow \upsilon > w \left| \prod_{j=1}^{n-1} \tfrac{1}{2}\, u_{nj}/u_{nn} \right| |m_{k0}^{1/n}\ \underline{m}_k \cdot \underline{u}_n|^n .$$

So by the above result and the first part of (3.2) there exists $J > 0$ so that

(3.6) $|m_{k0}^{1/n}\ \underline{m}_k \cdot \underline{u}_j| < J,\ j = 1,\dots,n,\ \underline{m}_k \in \mathbb{Z}(\Gamma)$, (sufficiently large) k.

We now define an $n \times n$ submatrix of $U = A^{-1}$ by

$$U_* = (u_{ij} : i,j = 1,\dots,n) .$$

We note (and it is easily shown) that

(3.7) $$\det U_* = \det U \neq 0.$$

We define, for all $\underline{m} \in \mathbb{Z}^{n+1}$,

$$h(\underline{m}) = |m_0|^{1/n} (m_0 \xi_1 - m_1,\dots, m_0\, \xi_n - m_n)$$

https://doi.org/10.1017/S0004972700002495 Published online by Cambridge University Press

and note the identity

(3.8) $$\underline{h}(\underline{m})\,U_* = -\,|m_0|^{1/n}\,(\underline{m}.\underline{u}_1\ ,\ \ldots,\ \underline{m}.\underline{u}_n)\ .$$

For $\underline{m}_k \in \mathbb{Z}(\Gamma)$ we write

$$\underline{\rho}_k = (\rho_{k1}\ ,\ldots,\rho_{kn}),\quad \rho_{kj} = |m_{k0}|^{1/n}\ \underline{m}_k.\underline{u}_j\ ,\ j = 1,\ldots,n\ .$$

Then by (3.8)

$$\underline{h}(\underline{m}_k)\ U_* = -\ \underline{\rho}_k = -\rho_{kn}(\rho_{k1}/\rho_{kn},\ldots,\rho_{k,n-1}/\rho_{kn},\ 1\ )\ .$$

By (3.2') $\rho_{kj}/\rho_{kn} = u_{nj}/u_{nn} + \varepsilon_{kj},\ |\varepsilon_{kj}| < \varepsilon,\ j=1,\ldots,n(\varepsilon_{kn} = 0).$

So

$$\underline{h}(\underline{m}_k)\ U_* = -\ (\rho_{kn}/u_{nn})(u_{n1},\ldots,u_{nn}) - \rho_{kn}(\varepsilon_{k1},\ldots,\varepsilon_{kn})$$

and

$$\underline{h}(\underline{m}_k) = -(\rho_{kn}/u_{nn})(0,\ldots,0,1) - (\delta_{k1},\ldots,\delta_{kn})$$

since $(u_{n1},\ldots,u_{nn})\ U_*^{-1} = (0,\ldots,0,1)$ and we write

$$(\delta_{k1},\ldots,\delta_{kn}) = \rho_{kn}(\varepsilon_{k1},\ldots,\varepsilon_{kn})\ U_*^{-1}\ .$$

Finally writing $\underline{h}(\underline{m}_k) = \underline{h}_k = (h_{k1},\ldots,h_{kn})$ we observe

$$\Big|\prod_{j=1}^{n} h_{kj}\Big| = \Big|\prod_{j=1}^{n-1} \delta_{kj}\Big|\,|\rho_{kn}/u_{nn}+\delta_{kn}| \to 0 \quad \text{as} \quad k \to \infty$$

since by (3.6) $\rho_{kn}/u_{nn} = O(1)$ and by (3.2') and (3.6)

$$\delta_{kj} \to 0\ ,\quad j = 1,\ldots,n \quad \text{as} \quad k \to \infty\ .$$

This completes the proof of the theorem.

References

[1] Borevich, Z.I. and Shafarevich, I.R., *Number Theory*, (Academic Press, 1966).

[2] Cassels, J.W.S. and Swinnerton-Dyer, H.P.F., "On the product of three homogeneous linear forms and indefinite ternary quadratic forms", *Philos. Trans. Roy. Soc. London, Ser. A*, 248 (1955), 73-96.

https://doi.org/10.1017/S0004972700002495 Published online by Cambridge University Press

[3] Pollard, H., *The theory of algebraic numbers*, (The Mathematical Association of America, 1950).

School of Mathematics,
University of New South Wales,
Kensington,
N.S.W. 2033.

https://doi.org/10.1017/S0004972700002495 Published online by Cambridge University Press