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ACTIONS THAT FIBER AND VECTOR SEMIGROUPS

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Introduction. From [**2**], we can derive a criterion for determining when an action of a Lie group on a locally compact space leads to a fiber bundle. Here, we present an equivalent criterion which can be stated purely in the language of actions of groups on spaces. This is Theorem I. Using this result, we are able to give a version of a result of Horne [**1**] for dimensions greater than one. This is done in Theorem IV and Corollary IVA. In Theorem II, we show that if a vector semigroup V_n^- acts on a space X, then whenever the map $t \mapsto tx$ is $1 - 1$ from V_n^- onto V_n^-x, it is in fact a homeomorphism. Also, V_n^-x is a closed subset of X. This is also a version of a result in [**1**].

Preliminaries. We shall invariably use the words *semigroup* and *group* to mean topological semigroup and topological group, respectively. Furthermore, all topological spaces are to be non-empty Hausdorff spaces. An *action* of a semigroup S on a space X is a continuous function $\phi: S \times X \to X$ with $\phi(s, x)$ usually denoted by sx, such that for all $s, t \in S$ and $x \in X$, $s(tx) = (st)x$. If S has an identity, 1, we further require that $1x = x$, for all x in X. If the semigroup S acts on the space X, i.e., there is an action of S on X, then for each $x \in X$ the set $Sx = \{sx: s \in S\} \subset X$ is called the *orbit* of S through x. If S acts on X and $x \in X$, we define $\phi_x: S \to Sx$ by $\phi_x(s) = sx$, and see that ϕ_x maps S onto Sx continuously. The set $S_x = \{s \in S: sx = x\}$ is called the *isotropy subsemigroup* of S at x, if it is non-empty. It is known that if S_x is non-empty, it is a closed subsemigroup of S, and, furthermore, if S is in fact a group, then S_x is a closed subgroup of S. If the group G acts on the space X, then the collection $\{Gx: x \in X\}$ of orbits of G in X is a decomposition of X. We denote this collection with the decomposition topology by X/G, and call it the *orbit space* of G acting on X. The natural map $\Phi: X \to X/G$ is an open mapping, and furthermore, a subset K of X/G is closed in X/G if and only if $\Phi^{-1}(K)$ is a closed subset of X.

Actions that fiber. A group G is said to act *freely* on a space X if, whenever $gx = x$ for some $g \in G$ and $x \in X$, then $g = 1$. From [**2**], we derive the following criterion that the action of a Lie group G on a locally compact space X lead to a fiber bundle. Suppose that G acts freely on X, and $T = \{(x, gx): x \in X, g \in G\}$ in $X \times X$. Then X is a fiber bundle over X/G if and only if X/G is Hausdorff and the function $h(x, gx) = g$ from T onto G is continuous.

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Our first objective is to give a criterion equivalent to the above which may be stated purely in the language of actions of groups on spaces. The result is

THEOREM I. *Suppose that the Lie group G acts freely on a locally compact space X. Then X is a fiber bundle over X/G if and only if G is IP on X.*

If $\{x_\rho\}$ is a net in a space X, we say that $\{x_\rho\}$ is finally in Y, $Y \subset X$, if there is an index Γ such that, if $\rho \geqq \Gamma$, then $x_\rho \in Y$. We say that $x_\rho \to \infty$ if, whenever K is a compact subset of X, then $\{x_\rho\}$ is finally in $-K$. It is easy to see that if $\{x_\rho\}$ has no convergent subnets, then $x_\rho \to \infty$ in X, and that if X is locally compact, then the converse is also true. It is well-known that a net $\{x_\rho\}$ in a space X converges to a point x in X if and only if every subnet of $\{x_\rho\}$ converges to x. If the group G acts on the space X, we say G is IP on X (relative to this action) if, whenever $\{g_\rho\}$ and $\{x_\rho\}$ are nets in G and X, respectively, with $g_\rho \to \infty$ in G and $x_\rho \to x \in X$, then $g_\rho x_\rho \to \infty$ in X. Theorem I is a consequence of two results which are of independent interest.

LEMMA 1. *Suppose that the group G acts on a locally compact space X such that G is IP on X. The orbit space X/G is a locally compact Hausdorff space.*

Proof. We first show that if C is a compact subset of X, then

$$GC = \{gc: g \in G, c \in C\}$$

is closed in X. For, let $y \in (GC)^*$, the closure of GC in X. Then, there exist nets $\{g_\rho\}$ in G and $\{c_\rho\}$ in C such that $g_\rho c_\rho \to y$. Since C is compact, $\{c_\rho\}$ must have a convergent subnet and, by passing to this subnet, we may assume that $c_\rho \to c$ for some $c \in C$. If $\{g_\rho\}$ has no convergent subnets, then $g_\rho \to \infty$ in G so, since G is IP on X, $g_\rho c_\rho \to \infty$ in X. However, since $g_\rho c_\rho \to y$ and X is locally compact, we have arrived at a contradiction. Therefore, $\{g_\rho\}$ has a convergent subnet, and, by passing to this subnet, we may assume that $g_\rho \to g$ for some $g \in G$. Hence, $g_\rho c_\rho \to gc \in GC$, so, since X is Hausdorff and $g_\rho c_\rho \to y$, $y = gc \in GC$, and we conclude that GC is closed in X.

Letting $\Phi: X \to X/G$ be the natural map, we recall that a subset N of X/G is closed in X/G if and only if $\Phi^{-1}(N)$ is closed in X. It is also known that if $K \subset X$, then $\Phi^{-1}(\Phi(K)) = GK$. From this and from the above, we see that if K is a compact subset of X, then, since GK is closed in X, $\Phi(K)$ is a closed subset of X/G. Since $\{x\}$ is a compact subset of X, $Gx = \Phi(x)$ is closed in X/G. Therefore, points are closed in X/G, so X/G is a T_1-space.

Being a locally compact Hausdorff space, X is regular. Let $Gx \in X/G$ and U be a neighborhood of Gx in X/G. Then, $\Phi^{-1}(U)$ is a neighborhood of x in X, so there is a neighborhood V of x in X such that V^* is compact and $V^* \subset \Phi^{-1}(U)$. Since V^* is compact, $\Phi(V^*)$ is closed in X/G. Since Φ is an open mapping, $\Phi(V)$ is a neighborhood of Gx in X/G such that

$$\Phi(V) \subset \Phi(V)^* \subset \Phi(V^*)^* = \Phi(V^*) \subset \Phi(\Phi^{-1}(U)) = U.$$

Therefore, since Gx and U are arbitrary, X/G is a regular space. Being a regular T_1-space, X/G is a Hausdorff space.

Since X is locally compact and $\Phi: X \to X/G$ is an open mapping, X/G is a locally compact space.

Our next result is a generalization of one which appears in [**1**]. It is

LEMMA 2. *Let G be a locally compact group acting freely on a locally compact space X. G is IP on X if and only if X/G is Hausdorff and the function $h(x, tx) = t$, mentioned earlier, is continuous.*

Proof. Suppose first that X/G is Hausdorff and h is continuous. Let $\{g_\rho\}$ and $\{x_\rho\}$ be nets in G and X, respectively, with $g_\rho \to \infty$ in G and $x_\rho \to x \in X$. If G is not IP on X, we may as well assume that $g_\rho x_\rho \not\to \infty$ in X, and thereby conclude that $\{g_\rho x_\rho\}$ has a convergent subnet. By passing to this subnet, we may further assume that $g_\rho x_\rho \to k$ for some $k \in X$. Thus, $(x_\rho, g_\rho x_\rho) \to (x, k)$, which implies that $Gx = Gk$. For, letting U be a neighborhood of x and V a neighborhood of k, there is an index δ such that $x_\delta \in U$, and $g_\delta x_\delta \in V$, because $x_\rho \to x$ and $g_\rho x_\rho \to k$. Thus, $Gx_\delta \in \Phi(U) \cap \Phi(V)$, where $\Phi: X \to X/G$ is the natural map. Since X/G is Hausdorff and since U and V are arbitrarily chosen, we conclude that $Gx = Gk$.

Since $Gx = Gk$, there is a $g \in G$ such that $gx = k$. Then, $(x_\rho, g_\rho x_\rho) \to (x, gx)$ so, since h is continuous, $g_\rho \to g$ in G. Since G is locally compact and we know that $g_\rho \to \infty$, we have arrived at a contradiction. Therefore, we may conclude that $g_\rho x_\rho \to \infty$ in X and further that G is IP on X.

Conversely, suppose that G is IP on X. Then, by Lemma 1, X/G is Hausdorff. We need only show that h is continuous. Let $\{g_\rho\}$ be a net in G and $\{x_\rho\}$ a net in X such that for some $g \in G$ and $x \in X$, $(x_\rho, g_\rho x_\rho) \to (x, gx)$. We must show that $g_\rho \to g$.

Let $\{g_{\rho\delta}\}$ be any subnet of $\{g_\rho\}$ and assume that this subnet has no convergent subnets. Then, $g_{\rho\delta} \to \infty$ in G so, since G is IP on X and $x_{\rho\delta} \to x$, $g_{\rho\delta} x_{\rho\delta} \to \infty$ in X. But, $\{g_{\rho\delta} x_{\rho\delta}\}$ is a subnet of $\{g_\rho x_\rho\}$ and $g_\rho x_\rho \to gx$, so $g_{\rho\delta} x_{\rho\delta} \to gx$. Therefore, since X is locally compact, we have arrived at a contradiction. Hence, every subnet of $\{g_\rho\}$ has a convergent subnet. In fact, every subnet of $\{g_\rho\}$ has a subnet converging to g. For, suppose that $\{g_{\rho\delta}\}$ has a subnet $\{g_{\rho\delta\sigma}\}$ converging to some $t \in G$, i.e., $g_{\rho\delta\sigma} \to t$. Then, since $x_{\rho\delta\sigma} \to x$, $g_{\rho\delta\sigma} x_{\rho\delta\sigma} \to tx$. But, $g_{\rho\delta\sigma} x_{\rho\delta\sigma} \to gx$, so, since X is Hausdorff, $gx = tx$. But, G acts freely on X, so $g = t$. Thus, every subnet of $\{g_\rho\}$ has a subnet which converges to g. But, one sees that this implies that $g_\rho \to g$. For, if not, let U be any neighborhood of g. Then, $\{g_{\rho\delta}: g_{\rho\delta} \notin U\}$ contains a subnet of $\{g_\rho\}$ which clearly has no subnets which converge to g. Therefore, $g_\rho \to g$ and h is continuous. This concludes the proof of Lemma 2.

Lemma 2 yields an immediate proof of Theorem I.

Proof of Theorem I. Being locally Euclidean, G is locally compact. By Lemma 2, G is IP on X if and only if X/G is Hausdorff and $h(x, tx) = t$ is continuous. However, as mentioned earlier, h is continuous and X/G is Hausdorff if and only if X is a fiber bundle over X/G.

Vector semigroups. We let P^- denote the semigroup of non-negative real numbers under multiplication and $P = P^- - \{0\}$ the multiplicative group. If n is a positive integer, the *n-dimensional vector semigroup* is $V_n^- = P^- \times \ldots \times P^-$ (n copies) under coordinatewise multiplication. $V_n = P \times \ldots \times P$ (n copies) is iseomorphic (topologically homeomorphic and algebraically isomorphic) to the n-dimensional vector group, and $V_n \subset V_n^-$. In fact, V_n^- is the topological closure, V_n^*, of V_n in E^n. We let L_n be the frontier of V_n^- in E^n and see that $L_n = V_n^- - V_n$. It is clear that $e = (0, \ldots, 0)$ is the zero for V_n^- and $1 = (1, \ldots, 1)$ is the identity for V_n^-. If n is a positive integer, then, for each j, $1 \leqq j \leqq n$, we define

(i) $P_j = \{(p_1, \ldots, p_n) \in V_n^-: p_i = 1, \text{ for } i \neq j\}$, and

(ii) $e_j = (1, \ldots, 1_{j-1}, 0, 1_{j+1}, \ldots, 1)$.

From this we see that in V_n^-, $P_j^- = P_j^* = P_j \cup \{e_j\}$. Furthermore, there is a natural iseomorphism from P_j^- onto P^- which maps e_j onto 0. In addition,

$$(*)\qquad V_n = \prod_{j=1}^{n} P_j, \qquad V_n^- = \prod_{j=1}^{n} P_j^-,$$

and

$$e = \prod_{j=1}^{n} e_j.$$

Set $\Omega = \{1, 2, \ldots, n\}$, $K = \{\text{non-empty, proper subsets of } \Omega\}$, and $K^* = K \cup \{\Omega\}$. If $T \in K^*$, we set

$$P_{(T)} = \prod_{j \in T} P_j,$$

$$P^-_{(T)} = \prod_{j \in T} P_j^-,$$

and

$$e_{(T)} = \prod_{j \in T} e_j.$$

Since V_n^- is abelian, we see that if T is in K, then $V_n = P_{(T)}P_{(-T)} = P_{(-T)}P_{(T)}$, etc. We also observe that $L_n = \bigcup \{P_{(-T)}e_{(T)}: T \in K\} \cup \{e\}$, and that $e_{(T)} = e$ if and only if $T = \Omega$.

If V_n^- acts on a space X, we set $F_j = \{x \in X: e_j x = x\}$, for $1 \leqq j \leqq n$, and $F = \{x \in X: ex = x\}$. As in [**1**], it is easy to see that for each j,

$$F_j = \{x: P_j x = x\} = e_j X,$$

and

$$F = \{x: V_n x = x\} = eX.$$

Furthermore, one readily observes that $F = \bigcap_{j=1}^{n} F_j$.

LEMMA 3. *Suppose that V_n^- acts on a space X, $x \in X$, and $t \in L_n$. Choose $M \in K^*$ such that $t \in P_{(-M)}e_{(M)}$ and let r be the cardinality of M. If $tx \in V_n x$, then $x \in \bigcap \{F_m: m \in M\}$ and* $\dim V_n x \leqq n - r$. *In particular,* $\dim V_n x \neq n$.

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Proof. Since $t \in P_{(-M)}e_{(M)}$, it is easy to see that for each $m \in M$, $e_m t = t$. If $tx \in V_n x$, there is a $g \in V_n$ such that $tx = gx$. Thus, $x = (g^{-1}t)x$, so if $m \in M$, we have

$$e_m x = e_m[(g^{-1}t)x] = [e_m(g^{-1}t)]x = [g^{-1}(e_m t)]x = (g^{-1}t)x = x.$$

Hence, $x \in \bigcap \{F_m: m \in M\}$.

If $y \in \bigcap \{F_m: m \in M\}$, then for each $m \in M$, $P_m y = y$ so $P_{(M)}y = y$. Since $V_n = P_{(-M)}P_{(M)}$, $V_n y = P_{(-M)}P_{(M)} = P_{(-M)}y$. Since we easily see that $P_{(-M)}$ is iseomorphic to V_{n-r}, we conclude that dim $P_{(-M)}y \leqq$ dim $P_{(-M)} = n - r$, so dim $V_n y \leqq n - r$, whenever $y \in \bigcap \{F_m: m \in M\}$.

Since $x \in \bigcap \{F_m: m \in M\}$, dim $V_n x \leqq n - r$, and, since $M \neq \phi$, $r \neq 0$ so dim $V_n x \neq n$.

In [**1**], it is shown that if $V_1^- = P^-$ acts on a space X, then $P^-x = (Px)^*$ for all x, and further that either $x \in F$ or $t \mapsto tx$ is a homeomorphism from P^- onto P^-x. Unfortunately, this is not generally true. For, define $(a, b)(x_1, x_2) = (abx_1, bx_2)$ for $(a, b) \in V_2^-$ and $(x_1, x_2) \in E^2$. Then we have an action of V_2^- on E^2 such that, setting $x = (1, 1) \in E^2$, $V_2^-x \neq (V_2x)^*$, $x \notin F$, and $t \mapsto tx$ is not a homeomorphism from V_2^- onto V_2^-x.

If V_n^- acts on a space X, we set $X' = \{x: \phi_x$ is $1 - 1$ from V_n^- onto $V_n^-x\}$. In spite of the above example we are able to prove

THEOREM II. *Suppose that V_n^- acts on a space X. Then for every $x \in X'$, ϕ_x is a homeomorphism from V_n^- onto V_n^-x and V_n onto V_nx. Furthermore, $V_n^-x = (V_nx)^*$ so V_nx is closed in X'.*

Proof. Let $x \in X'$. We first prove that if $\{g_\rho\}$ is a net in V_n^- such that $g_\rho x_\rho \to y$ for some $y \in X$, then there is a $g \in V_n^-$ such that $g_\rho \to g$ in V_n^-.

We start by showing that $\{g_\rho\}$ has a subnet which converges to some element of V_n^-. From (*), we see that for every ρ, $g_\rho = \prod_{j=1}^n p_{j\rho}$ with $p_{j\rho} \in P_j^-$. Thus, for each j, $\{p_{j\rho}\}$ is a net in P_j^-. If for each j, $\{p_{j\rho}\}$ has no subnets which converge in P_j^-, then $p_{j\rho} \to \infty_j$ for every j. From this we see that there is an index Γ such that if $\rho \geqq \Gamma$, the $p_{j\rho} \in P_j$. Thus, if $\rho \geqq \Gamma$, $g_\rho \in V_n$ so we may form $g_\rho^{-1} = \prod_{j=1}^n p_{j\rho}^{-1}$. Also, since $p_{j\rho} \to \infty_j$, $p_{j\rho}^{-1} \to e_j$ and $g_\rho^{-1} = \prod_{j=1}^n p_{j\rho}^{-1} \to \prod_{j=1}^n e_j = e$. Hence, since $g_\rho x_\rho \to y$, $x = g_\rho^{-1}(g_\rho x) \to ey \in eX = F$, which is impossible because $x \in X'$. Therefore, there is at least one j such that $\{p_{j\rho}\}$ has a subnet which converges to some $p_j \in P_j^-$. Passing to this subnet, if necessary, we may assume that $p_{j\rho} \to p_j$.

Choose $M \in K$ and assume that for each $i \in M$, $p_{i\rho} \to p_i$ for some $p_i \in P_i^-$. Suppose that there is no $j \in -M$ such that $\{p_{j\rho}\}$ has a subnet which converges to a $p_j \in P_j^-$. Then, as above, we see that $\prod_{j \notin M} p_j^{-1} \to e_{(-M)}$. Thus, $(\prod_{i \in M} p_{i\rho})x = (\prod_{j \notin M} p_{j\rho}^{-1})(g_\rho x) \to e_{(-M)}y$. But, $\prod_{i \in M} p_{i\rho} \to \prod_{i \in M} p_i$ so $(\prod_{i \in M} p_{i\rho})x \to (\prod_{i \in M} p_i)x$. Thus, since X is Hausdorff, $(\prod_{i \in M} p_i)x = e_{(-M)}y$. Then, $e_{(M)}x = e_{(M)}([\prod_{i \in M} p_i]x) = e_{(M)}[e_{(-M)}y] = ey$. Thus, $e_{(M)}x = ey = ex$, which is impossible because $M \in K$ and $x \in X'$. Thus, there is a $j \in -M$ such that $\{p_{j\rho}\}$ has a subnet which converges to some $p_j \in P_j^-$. By passing to subnets, if necessary, we may assume that $p_{j\rho} \to p_j \in P_j^-$.

The above has shown firstly that there is a $T \in K$ such that for all $j \in T$, $p_{j\rho} \to p_j \in P_j^-$, and secondly that, if $M \in K$ such that $p_{i\rho} \to p_i \in P_i^-$ for all $i \in M$, then there is a $k \notin M$ such that $p_{k\rho} \to p_k \in P_k^-$. Combining these two, we see that for every j, $1 \leqq j \leqq n$, there is a $p_j \in P_j^-$ such that $p_{j\rho} \to p_j$. Thus, $g_\rho = \prod_{j=1}^n p_{j\rho} \to \prod_{j=1}^n p_j = g \in V_n^-$.

Our passing to subnets above actually only shows that $\{g_\rho\}$ has a convergent subnet. The method used can be applied to give us the fact that every subnet of $\{g_\rho\}$ has a convergent subnet. However, if $\{g_{\rho\delta}\}$ and $\{g_{\rho\sigma}\}$ are two subnets of $\{g_\rho\}$ converging to t and t', respectively, then $g_{\rho\delta}x \to tx$, $g_{\rho\sigma}x \to t'x$, $g_{\rho\delta}x \to y$, and $g_{\rho\sigma}x \to y$. Thus, $tx = y = t'x$, so, since $x \in X'$, $t = t'$. Hence, we conclude that there is a $g \in V_n^-$ such that every subnet of $\{g_\rho\}$ has a subnet which converges to g. As in the proof of Lemma 2, this implies that we indeed have $g_\rho \to g \in V_n^-$.

To show that the desired maps are homeomorphisms, we need only show that if $\{g_\rho\}$ is a net in V_n^-, respectively V_n, with $g_\rho x \to gx$ with $g \in V_n^-$, respectively V_n, then $g_\rho \to g$. But, from the above, there is a $t \in V_n^-$ such that $g_\rho \to t$. Then, $g_\rho x \to tx$ so $tx = gx$. Therefore, since $x \in X'$, $t = g$.

Finally, suppose that $y \in (V_n)^*$ so that there is a net $\{g_\rho\}$ in $V_n \subset V_n^-$ such that $g_\rho x \to y$. From the above, $g_\rho \to t$ for some $t \in V_n^-$ so $g_\rho x \to tx$ and, hence, $y = tx \in V_n^-x$. Thus, $(V_nx)^* \subset V_n^-x$ so, since $V_n^-x \subset (V_nx)^*$ by the continuity of the action, $V_n^-x = (V_nx)^*$. From this, it is easy to show that V_nx is closed in X'. First, $V_nx \subset X'$; for, let $g \in V_n$ and suppose that t, $t' \in V_n^-$ such that $t(gx) = t'(gx)$. Then, $tx = g^{-1}(gtx) = g^{-1}[t(gx)] = g^{-1}[t'(gx)] = g^{-1}(gt'x) = t'x$ so, since $x \in X'$, $t = t'$. Therefore, $V_nx \subset X'$. Now, if $y \in (V_nx)^* - V_nx$ then, since $(V_nx)^* = V_n^-x$, $y \in L_nx$. A quick investigation shows that $L_nt \cap X' = \phi$ for all $t \in X$. Therefore, if $y \in (V_nx)^* \cap X'$, $y \in V_nx$, and we see that V_nx is closed in X'.

A semigroup S is said to be *absolutely closed* if whenever T is a semigroup and $S \subset T$, then S is a closed subset of T. With this notion, Theorem II yields

COROLLARY IIA. *For each n, V_n^- is an absolutely closed semigroup.*

Proof. Suppose that T is a semigroup such that $V_n^- \subset T$. Then clearly $1 \in T'$, where the action of V_n^- on T is left multiplication in T. By Theorem II, $V_n^- = V_n^-1$ is closed in T. Therefore, since T is arbitrary, we conclude that V_n^- is an absolutely closed semigroup.

If V_n^- acts on a space X, then for each $M \in K$, we define

$$Y_M = \{y \in X: e_{(M)}y = ey\}.$$

We then set $Y = \bigcup \{Y_M: M \in K\}$ and $X'' = X - Y$.

LEMMA 4. *If V_n^- acts on X, then each Y_M is closed. Thus, Y is closed, so X'' is open.*

Proof. If $y \in Y_M^*$, there is a net $\{y_\rho\}$ in Y_M such that $y_\rho \to y$. Hence, $ey_\rho \to ey$ and $e_{(M)}y_\rho \to e_{(M)}y$. But, each $y_\rho \in Y_M$ so $ey_\rho = e_{(M)}y_\rho$ so, since X is Hausdorff, $e_{(M)}y = ey$ and $y \in Y_M$. Therefore, Y_M is closed. Since K is finite, we can now conclude that Y is closed, and therefore that X'' is open.

We now prove two results which permit us to prove a fibering theorem for actions of V_n^- on locally compact spaces. The first of these is

THEOREM III. *If V_n^- acts on a space X, then V_n acts on X'' and is IP on X''.*

Proof. We first show that V_n acts on X''. To do this we need only show that if $t \in V_n$ and $x \in X''$, then $tx \in X''$. Suppose that $t \in V_n$ and $x \in X''$. If $tx \notin X''$, there is an $M \in K$ such that $e_{(M)}(tx) = e(tx)$. Then, since $t \in V_n$, we have

$$e_{(M)}x = t^{-1}[e_{(M)}(tx)] = t^{-1}[e(tx)] = ex,$$

so $x \notin X''$, and V_n acts on X''.

Suppose that $\{g_\rho\} = \{\prod_{j=1}^n p_{j\rho}\}$ is a net in V_n, $P_{j\rho} \in P_j$, with $g_\rho \to \infty$ in V_n and suppose that $\{x_\rho\}$ is a net in X'' such that $x_\rho \to x$ for some $x \in X''$. If $g_\rho x_\rho \not\to \infty$ in X'', we know that $\{g_\rho x_\rho\}$ has a subnet which converges and, by passing to this subnet, we may assume that $g_\rho x_\rho \to y$ for some $y \in X''$.

Assume that for some j, $1 \leqq j \leqq n$, $p_{j\rho} \to \infty_j$ so that $p_{j\rho}^{-1} \to e_j$. Then, $(\prod_{i\neq j} p_{i\rho})x_\rho = p_{j\rho}^{-1}(g_\rho x_\rho) \to e_j y$. For each $i \neq j$, $e_i p_i = e_i$ for every ρ so, if $M = \Omega - \{j\}$, $e_{(M)}x = e_{(M)}[(\prod_{i\neq j} p_{i\rho})x_\rho] \to e_{(M)}(e_j y) = ey$ and, since $x_\rho \to x$, $ex_\rho \to ex$, so $ex = ey$ because X is Hausdorff. Thus, $e_{(M)}x = ex$ which is impossible since $x \in X''$. Thus, for each j, $1 \leqq j \leqq n$, $p_{j\rho} \not\to \infty_j$.

Next, assume that for some j, $1 \leqq j \leqq n$, $p_{j\rho} \to e_j$. Then, setting $M = \Omega - \{j\}$, $e_{(M)}(g_\rho x_\rho) \to e_{(M)}y$. But, $e_{(M)}(g_\rho x_\rho) = e_{(M)}[(\prod_{i\neq j} p_{i\rho})p_{j\rho}]x_\rho = e_{(M)}p_{j\rho}x_\rho$ so, since $p_{j\rho} \to e_j$ and $x_\rho \to x$, $e_{(M)}(g_\rho x_\rho) = e_{(M)}(p_{j\rho}x_\rho) \to e_{(M)}(e_j x) = ex$. Then, since X is Hausdorff, $e_{(M)}y = ex = ey$, which contradicts $y \in X''$. Hence, if $1 \leqq j \leqq n$, $p_{j\rho} \not\to e_j$.

Hence, for every j, $1 \leqq j \leqq n$, $\{p_{j\rho}\}$ has a subnet which converges to some member of P_j. But, this implies that we may find a subnet of $\{g_\rho\}$ which converges to some $g \in V_n$. However, this contradicts the fact that $g_\rho \to \infty$ in V_n. Therefore, we conclude that $g_\rho x_\rho \to \infty$ in X'' and from this see that V_n is *IP* on X''.

LEMMA 5. *If V_n^- acts on a locally compact space X, then $X' = X''$.*

Proof. From the way each of X' and X'' is defined, it is easy to see that $X' \subset X''$.

Conversely, let $y \in X''$. We shall first prove that the isotropy group, $(V_n)_y$, is trivial. If $g \neq 1$ is in V_n, then $\{g^t\}_{t=1}^\infty$ is a sequence in V_n such that $g^t \to \infty$. Hence, if $(V_n)_y$ is non-trivial, there is a net $\{g_\rho\}$ in $(V_n)_y$ such that $g_\rho \to \infty$ in V_n. But, for each ρ, $g_\rho \in (V_n)_y$, so $g_\rho y = y$ and hence $g_\rho y \to y$. But, by Lemma 4, X'' is open in the locally compact space X and is therefore locally compact. Hence, $g_\rho y \to \infty$ and $g_\rho y \to y$ is a contradiction. Therefore, we conclude that $(V_n)_y$ is trivial, so ϕ_y is a $1 - 1$ map from V_n onto $V_n x$. From this we see that dim $V_n y =$ dim $V_n = n$.

Suppose that $t \in L_n$ such that $ty \in V_n y$. Then, by Lemma 3, dim $V_n y \neq n$, which contradicts the above. Therefore, if $t \in L_n$, $ty \notin V_n y$. Thus, to prove that $y \in X'$, it is now sufficient to show that ϕ_y is $1 - 1$ on L_n.

Now, $L_n = \bigcup \{P_{(-M)}e_{(M)}\colon M \in K\} \cup \{e\}$, so let $t \in L_n - \{e\}$ and pick $M \in K$ such that $t \in P_{(-M)}e_{(M)}$. Hence, there are $p_j \in P_j$, $j \in -M$, such that, setting $g = \prod_{j \notin M} p_j$, $t = ge_{(M)}$ and $g \in V_n$. If $ty = ey$, then $g(e_{(M)}y) = ey$ so $e_{(M)}y = g^{-1}(ey) = ey$. This implies that $y \in Y_M \subset Y$, which contradicts $y \in X''$.

Assume that $N \in K$, with $N \neq M$, and that for some $t' = g'e_{(N)}$ in $P_{(-N)}e_{(N)}$, $g' \in P_{(-N)} \subset V_n$, we have $ty = t'y$. Then, $(ge_{(M)})y = (g'e_{(N)})y$. Since $N \neq M$, there is a $T \in K$ such that either $T \cup N = \Omega$ or $T \cup M = \Omega$, but not both. We may as well assume that $T \cup N = \Omega$ so that $T \cup M \in K$. Now, since $ty = t'y$, it follows that $e_{(M)}y = [(g^{-1}g')e_{(N)}]y$ so

$$\begin{aligned} e_{(T\cup M)}y &= e_{(T)}[e_{(M)}y] \\ &= e_{(T)}([(g^{-1}g')e_{(N)}]y) \\ &= [e_{(T)}(g^{-1}g')e_{(N)}]y \\ &= [(g^{-1}g')e_{(T)}e_{(T)}]y \\ &= [(g^{-1}g')e_{(T\cup N)}]y \\ &= [(g^{-1}g')e]y = ey. \end{aligned}$$

But, $T \cup M \in K$ so this implies that $y \in Y$, which is a contradiction. Therefore, if $N \neq M$ and $t' \in P_{(-N)}e_{(N)}$, then $ty \neq t'y$.

Finally, suppose that $t' = (\prod_{j \notin M} q_j)e_{(M)} \in P_{(-M)}e_{(M)}$ such that $ty = t'y$. If $t \neq t'$, there is a $k \in -M$ such that $p_k \neq q_k$. Setting $N = \Omega - \{k\}$, we see, since $ty = t'y$, that $p_k[e_{(N)}y] = q_k[e_{(N)}y]$ so $e_{(N)}y = p_k^{-1}q_k[e_{(N)}y]$. But, this implies [**1**] that $P_k[e_{(N)}y] = e_{(N)}y$. Hence, $e_{(N)}y \in F_k$ so $ey = e_k[e_{(N)}y] = e_{(N)}y$. Since this implies that $y \in Y$, which is a contradiction, $t = t'$.

Hence, we see that if $t' \in L_n$ such that $ty = t'y$, then $t = t'$ so ϕ_y is $1 - 1$ on L_n. Since we have shown that ϕ_y is $1 - 1$ on V_n and that, if $t \in L_n$ then $ty \notin V_n y$, we see that ϕ_y is $1 - 1$ on V_n^-. Therefore, $y \in X'$, so $X'' \subset X'$.

Since $X' \subset X'' \subset X'$, $X' = X''$, as was to be shown.

We are now in position to prove our fibering theorem for action of vector semigroups on locally compact spaces. It is

THEOREM IV. *Suppose that the vector semigroup V_n^- acts on a locally compact space X. Then X' is a fiber bundle over X'/V_n with fiber (orbit) homeomorphic to V_n.*

Proof. By Lemma 5, $X' = X''$ and, by Lemma 4, X'' is open in X. Being open in the locally compact space X, X'' is locally compact. From the proof of Theorem II, we see that if $x \in X'$, then $V_n x \subset X'$ so V_n acts on X'. Also, by the definition of X', we see that V_n acts freely on X'.

Hence, the Lie group V_n acts freely on the locally compact space X'. Furthermore, by Theorem III, V_n is IP on $X' = X''$. Thus, by Theorem I, X' is a fiber bundle over X'/V_n. The fiber over a point $V_n x \in X'/V_n$ is the orbit $V_n x \subset X'$ which, by Theorem II, is homeomorphic to V_n. This completes the proof of the Theorem.

This yields

COROLLARY IVA. *Suppose that V_n^- acts on a locally compact space X. If X'/V_n is normal and Lindelöf, then X' has a complete cross-section to the orbits of V_n in X'. In particular, there is a set $C \subset X'$ homeomorphic to X'/V_n such that $(v, c) \mapsto vc$ maps $V_n \times C$ homeomorphically onto X'.*

Proof. By Theorem IV, X' is a fiber bundle over X'/V_n with fiber homeomorphic to V_n. Since X' is locally compact and V_n is IP on X', it follows from Lemma 1 that X'/V_n is a locally compact Hausdorff space. Hence [**2**], a local cross-section to the orbits of V_n in X' exists at each point of X'/V_n because V_n is a Lie group. This, together with the facts that X'/V_n is normal and Lindelöf and that the bundle has fiber homeomorphic to V_n, implies the existence of a complete cross-section C to the orbits of V_n in X' [**3**, p. 55]. From this, we have that X'/V_n is homeomorphic to C and that the map $(v, c) \mapsto vc$ is a homeomorphism from $V_n \times C$ onto X'. This concludes the proof of Corollary IVA.

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