# A REMARK ON CONTINUOUS BILINEAR MAPPINGS

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(Received 11th May 1970)

The main theorem of this paper is a little involved (though the proof is straightforward using a well-known idea) but the immediate corollaries are interesting. For example, take a complex normed vector space $A$ which is also a normed algebra with identity under each of two multiplications $*$ and $\circ$. Then these multiplications coincide if and only if there exists $\alpha$ such that $\| a \circ b \| \leqq \alpha \| a * b \|$ for $a$, $b$ in $A$. This is a condition for the two Arens multiplications on the second dual of a Banach algebra to be identical. By taking $*$ to be the multiplication of a Banach algebra and $\circ$ to be its opposite, we obtain the condition for commutativity given in (**3**). Other applications are concerned with conditions under which a bilinear mapping between two algebras is a homomorphism, when an element lies in the centre of an algebra, and a one-dimensional subspace of an algebra is a right ideal. An example shows that the theorem is false for algebras over the real field, but Theorem 2 gives the parallel result in this case.

Let $A$ be a normed algebra, and $M$ a normed vector space over the same scalar field. We shall call $M$ a *normed module over* $A$ if $M$ is a left module over $A$ for which the mapping $(a, m) \rightarrow am$ of $A \times M$ into $M$ is continuous. In that case, we can find a constant $k$ so that $\| am \| \leqq k \| a \| \| m \|$, for $a \in A$, $m \in M$. Suppose $A$ has a bounded approximate identity $\{e_\lambda : \lambda \in \Lambda\}$. We shall call $M$ a *unitary module over* $A$ if $\lim_\lambda e_\lambda m = m$, for $m \in M$; this condition is independent of the choice of approximate identity. As the method used in the proof of the following theorem is known—it is a direct generalization of arguments used in (**2**) and (**3**) for example—we shall only outline the proof.

**Theorem 1.** *Let $A$ be a complex normed algebra, with bounded approximate identity $\{e_\lambda : \lambda \in \Lambda\}$, let $M$ be a unitary normed module over $A$, and let $X$ be a complex normed vector space. If $h$ is a bilinear and continuous mapping of $A \times M$ into $X$, then $h(a, m) = \lim_\lambda h(e_\lambda, am)$, for $a \in A$, $m \in M$, if and only if there is a constant $\alpha$ such that $\| h(a, m) \| \leqq \alpha \| am \|$ for $a \in A$, $m \in M$.*

**Proof.** The necessity of the condition is clear. Suppose that the condition is satisfied. Since $h$ extends by continuity to a mapping involving the completions of the spaces concerned, we lose no generality in assuming that all three spaces are complete. If $A$ does not have an identity, denote by $A_e$ the algebra $A$ with an identity adjoined. Define $h_e : A_e \times M \rightarrow X$ by the equation $h(e, m) = \lim_\lambda h(e_\lambda, m)$ and linearity; the existence of the limit is guaranteed

https://doi.org/10.1017/S0013091500026961 Published online by Cambridge University Press

by the condition on $h$ and the fact that $M$ is unitary. From this construction it is clear that we may assume that $A$ has an identity element $e$, and prove that $h(a, m) = h(e, am)$.

For any complex number $z$, any $a \in A$, and any $m \in M$ we have

$$\| h(\exp(-za), \exp(za)m)\| \leqq \alpha \| \exp(-za) \exp(za)m \| \leqq \alpha k \| m \|.$$

Thus, by Liouville's Theorem, the power series

$$\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-z)^n}{n!} \frac{z^p}{p!} h(a^n, a^p m) = h(\exp(-za), \exp(za)m)$$

is constant. The coefficient of $z$ is therefore zero, i.e.

$$h(e, am) - h(a, m) = 0.$$

**Corollary 1.** *Let $A$ be a complex normed vector space which is a normed algebra with bounded approximate identity $\{e_\lambda: \lambda \in \Lambda\}$ for each of two multiplications $*$ and $\circ$. These multiplications coincide if and only if there exists $\alpha$ so that $\| a \circ b \| \leqq \alpha \| a * b \|$ for $a, b \in A$.*

**Proof.** Take $X = M = A$ where $A$ has multiplication $*$, and put

$$h(a, b) = a \circ b.$$

**Corollary 2.** ((2), (3)). *A complex normed algebra $A$ with bounded approximate identity is commutative if and only if there exists $\alpha$ so that $\| ba \| \leqq \alpha \| ab \|$ for $a, b \in A$. This holds in particular if $\| a \| \leqq \alpha\rho(a)$ for $a \in A$ (where $\rho$ denotes spectral radius).*

**Proof.** The first result is immediate from Corollary 1 on taking $*$ to be the multiplication of $A$ and $\circ$ its opposite. If the second inequality holds, then for $a, b \in A$,

$$\| ba \| \leqq \alpha\rho(ba) = \alpha\rho(ab) = \alpha \| ab \|.$$

**Corollary 3.** ((1)). *Let $f$ be a linear functional on a complex normed algebra $A$ with bounded approximate identity so that for some $\alpha$, $|f(a)| \leqq \alpha\rho(a)$ for $a \in A$ (where $\rho$ is the spectral radius). Then $f(ba) = f(ab)$ for $a, b \in A$.*

**Proof.** Take $h(a, b) = f(ba)$. The argument of Corollary 2 shows that $| h(a, b)| \leqq \alpha \| ab \|$ so that Theorem 1 applies.

**Corollary 4.** *Let $A$ and $B$ be complex normed algebras with identities ($e$ and $f$). Suppose that $T$ is a continuous linear mapping of $A$ into $B$ for which $T(e) = f$. Then $T$ is a homomorphism if and only if there exists $\alpha$ for which*

$$\| T(a)T(a')\| \leqq \alpha \| aa' \| \text{ for } a, a' \in A.$$

**Proof.** In Theorem 1 take $M = A$ and $X = B$, and put $h(a, a') = T(a)T(a')$.

**Corollary 5.** *Let $M_1$, $M_2$ be two unitary normed modules over a normed algebra $A$ with bounded approximate identity $\{e_\lambda: \lambda \in \Lambda\}$. Then a continuous linear mapping $T: M_1 \to M_2$ is $A$-linear ($T(am) = aT(m)$ for $a \in A$, $m \in M_1$) if and only if there is a constant $\alpha$ such that $\| aT(m)\| \leqq \alpha \| am \|$ for $a \in A$, $m \in M_1$.*

**Proof.** In Theorem 1 take $M = M_1$, $X = M_2$, and put $h(a, m) = aT(m)$. Then $aT(m) = h(a, m) = \lim_\lambda e_\lambda T(am) = T(am)$.

**Corollary 6.** *Let $A$ be a complex normed algebra, with identity $e$. Let $f$ be a continuous linear functional on $A$ with $f(e) \neq 0$. Suppose that $a \in A$ is such that $\| f(x)ay \| \leqq \alpha \| xy \|$ for $x, y \in A$. Then the subspace $\{za\colon z \in C\}$ is a right ideal of $A$.*

**Proof.** Take $M = X = A$, and put $h(x, y) = f(x)ay$. The theorem gives $f(e)axy = f(x)ay$; put $y = e$ and we have $f(e)ax = f(x)a$.

**Corollary 7.** *Let $A$ be a normed algebra with bounded approximate identity $\{e_\lambda\colon \lambda \in \Lambda\}$. An element $a$ of $A$ is in the centre of $A$ if and only if there exists $\alpha$ so that $\| xay \| \leqq \alpha \| xy \|$ for $x, y \in A$.*

**Proof.** In Theorem 1, take $M = X = A$, and put $h(x, y) = xay$ for $x, y \in A$. The theorem says that $xay = \lim_\lambda e_\lambda axy = axy$. Finally,

$$xa = \lim_\lambda xae_\lambda = \lim_\lambda axe_\lambda = ax.$$

Theorem 1, and more especially Corollary 1, fails if complex spaces are replaced by real spaces. For example, it is easy to provide $R^4$ with two multiplications $*$ and $\circ$ having the same identity and satisfying $\| a * b \| = \| a \circ b \|$ for $a, b \in R^4$. We may take $*$ to be the usual quaternion multiplication on $R^4$, and $\circ$ to be the multiplication derived from quaternion multiplication by regarding each element $(w, x, y, z)$ of $R^4$ as the quaternion $w+yi+xj+zk$. Then we have $\| a * b \| = \| a \| \| b \| = \| a \circ b \|$ for $a, b \in R^4$ and also $(1, 0, 0, 0)$ is an identity for both multiplications. The following result appears to be the best analogue of Theorem 1 for the real case.

**Theorem 2.** *Let $A$, $M$ and $X$ be as in Theorem 1, except that they are real, instead of complex, vector spaces. If $h$ is a bilinear and continuous mapping of $A \times M$ into $X$ then $h(a, m) = \lim_\lambda h(e_\lambda, am)$ for $a \in A$, $m \in M$, if and only if there exists $\alpha$ so that*

$$\| h(a, m) - h(a', m') \| \leqq \alpha \| am - a'm' \| \text{ for } a, a' \in A, \text{ and } m, m' \in M.$$

**Proof.** Let $A_C$, $M_C$, and $X_C$ be the complexifications of $A$, $M$, and $X$, respectively. Define $h_C\colon A_C \times M_C \to X_C$ by the equation

$$h_C((a, a'), (m, m')) = (h(a, m) - h(a', m'), h(a, m') + h(a', m)).$$

Then $h_C$ is clearly complex-bilinear and continuous with

$$\left\| h_C((a, a'), (m, m')) \right\| \leqq \alpha \left\| (a, a')(m, m') \right\|.$$

(It is clear that $M_C$ becomes a module over $A_C$.) Theorem 1 says that

$$h_C((a, a'), (m, m')) = \lim_\lambda h_C((e_\lambda, 0), (a, a')(m, m')).$$

Put $a' = m' = 0$. Then

$$(h(a, m), 0) = \lim_\lambda (h(e_\lambda, am), h(a, 0) + h(0, am)),$$

that is $h(a, m) = \lim_\lambda h(e_\lambda, am)$.

Corollaries similar to those for Theorem 1 can obviously be given. We offer an application to involutions; since these are usually conjugate linear, Theorem 1 will not apply.

**Corollary 8.** *Let $a \to a^*$ be a conjugate linear (i.e. $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$ for $a, b \in A$, $\lambda, \mu \in C$) mapping of a complex normed algebra A with identity e into itself. Suppose $e^* = e$. Then $(ab)^* = b^* a^*$ if and only if there exists $\alpha$ such that*

$$\| b^* a^* - d^* c^* \| \leqq \alpha \| ab - cd \|$$

*for $a, b, c, d \in A$.*

**Proof.** Consider $A$ as an algebra over the real field, and take $h(a, b) = b^* a^*$.

We would like to thank Professor Bonsall and the referee for suggesting some of these corollaries.

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