Elements of Algebraic Geometry and the Positive Theory of Partially Commutative Groups

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Abstract. The first main result of the paper is a criterion for a partially commutative group $\mathbb{G}$ to be a domain. It allows us to reduce the study of algebraic sets over $\mathbb{G}$ to the study of irreducible algebraic sets, and reduce the elementary theory of $\mathbb{G}$ (of a coordinate group over $\mathbb{G}$) to the elementary theories of the direct factors of $\mathbb{G}$ (to the elementary theory of coordinate groups of irreducible algebraic sets).

Then we establish normal forms for quantifier-free formulas over a non-abelian directly indecomposable partially commutative group $\mathbb{H}$. Analogously to the case of free groups, we introduce the notion of a generalised equation and prove that the positive theory of $\mathbb{H}$ has quantifier elimination and that arbitrary first-order formulas lift from $\mathbb{H}$ to $\mathbb{H} * F$, where F is a free group of finite rank. As a consequence, the positive theory of an arbitrary partially commutative group is decidable.

1 Introduction

This paper can be considered as a part of a project the aim of which is to construct algebraic (diophantine) geometry over partially commutative groups, and, more generally, to study the elementary theory of partially commutative groups.

Classical algebraic geometry is concerned with the study of the geometry of sets of solutions of systems of equations, *i.e.,* the geometry of algebraic sets. Taking the collection of all algebraic sets as a pre-base of closed sets one gets a topology, known as the Zarsiki topology. In the Zariski topology, every closed set is a union (maybe infinite) of algebraic sets. In the case that the ring of coefficients or, equivalently, the Zariski topology, is Noetherian, every closed set Y is a finite union of algebraic sets $Y = Y_1 \cup \cdots \cup Y_k$. In the case that $Y_i \nsubseteq Y_j$, $i \neq j$, and Y_i cannot be non-trivially presented as a union of algebraic sets, this decomposition is unique and the sets $Y_1, \ldots, Y_k$ are referred to as the irreducible components of Y. In general, however, a finite union of algebraic sets is not necessarily again an algebraic set. In classical algebraic geometry, it suffices to require that the ring of coefficients be a domain. Under these assumptions there exists a one-to-one correspondence between algebraic sets and closed sets. Thus, the study of algebraic sets reduces completely to the study of irreducible algebraic sets.

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notions from algebraic geometry over fields [1]. The counterpart to the notion of a Noetherian ring is the notion of an equationally Noetherian group: a group G is called *equationally Noetherian* if every system $S(X) = 1$ with coefficients from G is equivalent to a finite subsystem $S_0 = 1$, where $S_0 \subset S$, *i.e.,* the algebraic set defined by S coincides with the one defined by S_0. The notion of a domain carries over from rings to groups as follows: a group G is called a *domain* if for any $x, y \neq 1$ there exists $g \in G$ such that $[x, y^g] \neq 1$.

The notions of equationally Noetherian group and domain, play an analogous role (to their ring-theoretic counterparts) in algebraic geometry over groups (see [1]):

- a group G is equationally Noetherian if and only if the Zariski topology is Noetherian, in particular every closed set is a finite union of algebraic sets;
- if a group G is a domain, then the collection of all algebraic sets is a base for the Zariski topology.

Our main interest in this paper is algebraic geometry over (free) partially commutative groups. Partially commutative groups are widely studied in different branches of mathematics and computer science, which explains the variety of names they were given: *graph groups*, *right-angled Artin groups*, *semifree groups*, etc. Without trying to give an account of the literature and results in the field we refer the reader to a recent survey [4] and the introduction and references in [14].

Partially commutative groups are linear, see [17], hence, equationally Noetherian, see [1]. In [1] the authors gave several sufficient conditions for a group to be a domain. In particular, any CSA group is a domain and various group-theoretic constructions preserve the property of being a domain. However, none of the criteria obtained in [1] apply to the case of partially commutative groups. The major obstacle here is that a partially commutative group may contain a direct product of two free groups.

In Section 4.2 we give a criterion for a partially commutative group to be a domain.

***Theorem* 4.19**   *Let $\mathbb{G}$ be a partially commutative group. Then $\mathbb{G}$ is a domain if and only if $\mathbb{G}$ is non-abelian and directly indecomposable.*

Note that even if a partially commutative group is directly indecomposable, it still may contain a direct product of free groups.

The proof of this theorem is given in Section 4. It makes use of the technique of van Kampen diagrams over partially commutative groups, which we present in Section 3 and the description of centralisers in partially commutative groups (see Theorem 2.3).

The remaining part of the paper has a model-theoretic flavor. In Section 5, using results from [20], we prove that the elementary theory of $\mathbb{G}$ (of a coordinate group over $\mathbb{G}$) reduces to the elementary theories of the direct factors of $\mathbb{G}$ (to the elementary theory of coordinate groups of irreducible algebraic sets):

***Corollary* 5.2**   *Let $\mathbb{G}$ be a non-abelian directly indecomposable partially commutative group.*

(i) *If $Y = Y_1 \cup \cdots \cup Y_k$ is an algebraic set over $\mathbb{G}$, where $Y_1, \ldots, Y_k$ are the irreducible components of Y, then the elementary theory of the coordinate group $\Gamma(Y)$ of Y is decidable if and only if the elementary theory of $\Gamma(Y_i)$ is decidable for all $i = 1, \ldots, k$.*

(ii) *If $Y = Y_1 \cup \cdots \cup Y_k$ and $Z = Z_1 \cup \cdots \cup Z_l$ are two irreducible algebraic sets, where $Y_1, \ldots, Y_k$ and $Z_1, \ldots, Z_l$ are the irreducible components of Y and Z, respectively, then $\Gamma(Y)$ is elementarily equivalent to $\Gamma(Z)$ if and only if $k = l$ and, after a certain re-enumeration, $\Gamma(Y_i)$ is elementarily equivalent to $\Gamma(Z_i)$ for all $i = 1, \ldots, k$.*

It is known that coordinate groups of algebraic sets over $\mathbb{G}$ are separated by $\mathbb{G}$ (are residually $\mathbb{G}$); see [25]. If a coordinate group Γ is a coordinate group of an irreducible set, then Γ is discriminated by $\mathbb{G}$ (is fully residually $\mathbb{G}$), or equivalently, is universally equivalent to $\mathbb{G}$. Hence, the class of coordinate groups of irreducible algebraic sets is much narrower and admits a convenient logical description.

In his seminal work, Makanin introduced the notion of a generalised equation [21]. He used it in order to show that the existential theory (the compatibility problem) of free groups and monoids is decidable [22]. Since then this result has been generalised in various ways. Schulz generalised Makanin's result to the case of systems of equations over a free monoid with regular constraints [28], and Diekert, Gutierrez, and Hagenah showed the decidability of the compatibility problem for systems of equations over a free group with rational constraints [6]. Using the latter result, Diekert and Lohrey showed that the existential theory of a certain class of graph products of groups is decidable [8]. Furthermore in [7], the authors showed the decidability of the existential theory for an even wider class of groups. A common feature of the results mentioned above is that that they reduce the problem to the one for free groups with rational constraints.

One of the main applications of the decidability of the compatibility problem for free groups is the decidability of the positive theory of the respective group. In the case of free groups this is a very well-known result. Merzlyakov performed quantifier elimination for positive formulas over free groups by describing the Skolem functions [24]. Then, using the result of Makanin [22], one gets the decidability of the positive theory.

The aim of Sections 6 and 7 is to carry over the approach of Merzlyakov and Makanin to the case of partially commutative groups. In Section 6, we prove that any positive quantifier-free formula over a non-abelian directly indecomposable partially commutative groups is equivalent to a single equation. In order to do so we prove that

- for any finite system of equations $S_1(X) = 1, \ldots, S_k(X) = 1$ one can effectively find a single equation $S(X) = 1$ such that the algebraic set defined by the equations $S_1, \ldots, S_k$ and by S coincide for any non-abelian directly indecomposable partially commutative group $\mathbb{G}$;
- for any finite set of equations $S_1(X) = 1, \ldots, S_k(X) = 1$ one can effectively find a single equation $S(X) = 1$ such that the union of algebraic sets defined by the equations $S_1, \ldots, S_k$ coincides with the algebraic set defined by S for any non-abelian directly indecomposable partially commutative group $\mathbb{G}$.

In the case of free groups, the first result is due to Malcev [23], and Makanin attributes the second result to Gurevich [22]. These results hold in fact in a much more general setting (for groups that satisfy certain first-order formulas). For example, in [19] it is proved that this is the case for torsion-free, non-abelian, CSA groups that satisfy Vaught's conjecture, in particular, for all non-abelian fully residually free

groups and torsion-free hyperbolic groups. Note that a non-abelian directly inde-composable partially commutative group is almost never a CSA group. We generalise the results of Malcev and Gurevich to the case of partially commutative groups. The exposition in this section as well as in Section 7 is based on [19]. As an immediate consequence of these results we get a normal form for first order formulas over partially commutative groups (in fact, over a much wider class of groups).

In Section 7, we use the normal form for Van Kampen diagrams obtained in Lemma 3.3 to describe the finite number of all possible cancellation schemes for a given equation. This allows us to introduce the notion of a generalised equation for partially commutative groups. Then we introduce an analogue of the so-called Merzlyakov words and perform quantifier elimination for positive formulas over non-abelian directly indecomposable partially commutative groups.

Theorem 7.8   *If* $\mathbb{G} \models \forall x_1 \exists y_1 \cdots \forall x_k \exists y_k (S(X, Y, A) = 1)$, *then there exist words (with constants from* $\mathbb{G}$) $q_1(x_1), \ldots, q_k(x_1, \ldots, x_k) \in \mathbb{G}[X]$, *such that*

$$\mathbb{G}[X] \models S(x_1, q_1(x_1), \ldots, x_k, q_k(x_1, \ldots, x_k, A)) = 1,$$

i.e., *the equation* $S(x_1, y_1, \ldots, x_k, y_k, A) = 1$ *(in variables* Y*) has a solution in the group* $\mathbb{G}[X]$.

Our approach, therefore, is a natural analog of the classical approach of Merzlyakov and Makanin to the positive theory of free groups and avoids the technically involved language of constraints.

In particular, quantifier elimination gives a reduction of the decidability of the positive theory of non-abelian directly indecomposable partially commutative groups to the decidability of the compatibility problem of an equation, which is known to be decidable; see [9].

Finally, in order to prove that the positive theory of any partially commutative group is decidable, we need to study the positive theory of the direct product of groups. In folklore, it is known that if $G = H_1 \times \cdots \times H_k$, then the positive theory of G is decidable if the positive theories of $H_1, \ldots, H_k$ are decidable. However, we were unable to find a reference until M. Lohrey pointed out that in [8], the authors give a proof of this result. We present another proof of this fact in the Appendix. The proof is purely model-theoretic and makes use of the ideas of the proof of Theorem A.1, which is due to Feferman and Vaught [16].

2   Preliminaries

2.1   Partially Commutative Groups

We begin with the basic notions of the theory of free partially commutative groups. Recall that a (free) *partially commutative* group is defined as follows. Let Γ be a finite, undirected, simple graph. Let $A = V(\Gamma) = \{a_1, \ldots, a_n\}$ be the set of vertices of Γ and let $F(A)$ be the free group on A. Let

$$R = \{[a_i, a_j] \in F(A) \mid a_i, a_j \in A \text{ and there is an edge of } \Gamma \text{ joining } a_i \text{ to } a_j\}.$$

The partially commutative group corresponding to the (commutation) graph Γ is the group $\mathbb{G}(\Gamma)$ with presentation $\langle A \mid R \rangle$. This means that the only relations imposed on the generators are commutations of some of the generators. When the underlying graph is clear from the context we write simply $\mathbb{G}$.

From now on $A = \{a_1, \ldots, a_r\}$ always stands for a finite alphabet, its elements called *letters*. We reserve the term *occurrence* to denote an occurrence of a letter or of the formal inverse of a letter in a word. In a more formal way, an occurrence is a pair (letter, its placeholder in the word).

For a given word w denote $\alpha(w)$ the set of letters occurring in w. For a word $w \in \mathbb{G}$, we denote by $\overline{w}$ a geodesic of w. For a word $w \in \mathbb{G}$ define $\mathbb{A}(w)$ to be the subgroup of $\mathbb{G}$ generated by all letters that do not occur in $\overline{w}$ and commute with w. The subgroup $\mathbb{A}(w)$ is well defined (independent of the choice of a geodesic $\overline{w}$) (see [15]). An element $w \in \mathbb{G}$ is called *cyclically reduced* if the length of $\overline{w^2}$ is twice the length of $\overline{w}$.

For a partially commutative group $\mathbb{G}$ consider its non-commutation graph Δ. The vertex set V of Δ is a set of generators A of $\mathbb{G}$. There is an edge connecting a_i and a_j if and only if $[a_i, a_j] \neq 1$. The graph Δ is a union of its connected components $I_1, \ldots, I_k$. Then

$$(2.1) \qquad\qquad \mathbb{G} = \mathbb{G}(I_1) \times \cdots \times \mathbb{G}(I_k).$$

Consider $w \in \mathbb{G}$ and the set $\alpha(w)$. For this set, just as above, consider the graph $\Delta(\alpha(w))$ (it is a full subgraph of Δ with vertices $\alpha(w)$). This graph can be either connected or not. If it is connected we will call w a *block*. If $\Delta(\alpha(w))$ is not connected, then we can split w into the product of commuting words

$$(2.2) \qquad\qquad w = w_{j_1} \cdot w_{j_2} \cdots w_{j_t} \quad j_1, \ldots, j_t \in J,$$

where $|J|$ is the number of connected components of $\Delta(\alpha(w))$ and the word w_{j_i} involves letters from the j_i-th connected component. Clearly, the words $\{w_{j_1}, \ldots, w_{j_t}\}$ pairwise commute. Each word w_{j_i}, $i \in 1, \ldots, t$ is a block and so we refer to presentation (2.2) as the block decomposition of w.

An element $w \in \mathbb{G}$ is called a least root (or simply, root) of $v \in \mathbb{G}$ if there exists an integer $0 \neq m \in \mathbb{Z}$ such that $v = w^m$ and there do not exist $w' \in \mathbb{G}$ and $m' \in \mathbb{Z}$, $|m'| > 1$, such that $w = w'^{m'}$. In this case we write $w = \sqrt{v}$. By [11], partially commutative groups have least roots; that is, the root element of v is defined uniquely.

The following proposition reduces the conjugacy problem for arbitrary elements of a partially commutative group to the one for block elements.

Proposition 2.1 ([15, Proposition 5.7]) *Let $w = w_1 \cdot w_2 \cdots w_t$ and $v = v_1 \cdot v_2 \cdots v_s$ be cyclically reduced elements decomposed into the product of blocks. Then v and w are conjugate if and only if $s = t$ and, after some certain index re-enumeration, w_i is conjugate to v_i, $i = 1, \ldots, t$.*

Corollary 2.2 *Let $w = w_1^{r_1} \cdot w_2^{r_2} \cdots w_t^{r_t}$ and $v = v_1^{l_1} \cdot v_2^{l_2} \cdots v_s^{l_s}$ be cyclically reduced elements decomposed into the product of blocks, where w_i and v_j are root elements, $l_i, r_j \in \mathbb{Z}$, $i = 1, \ldots, t$, $j = 1, \ldots, s$. Then w and v are conjugate if and only if*

s = t and, after some certain index re-enumeration, $r_i = l_i$ and w_i is conjugate to v_i, $i = 1, \ldots, t$.

The next result describes centralisers of elements in partially commutative groups. As the definition of "being a domain" relies on the structure of centralisers, we shall make substantial use of the following theorem.

Theorem 2.3 (**Centraliser Theorem** [11, Theorem 3.10]) *Let $w \in \mathbb{G}$ be a cyclically reduced word, $w = v_1 \cdots v_k$ its block decomposition. Then the centraliser of w is the following subgroup of $\mathbb{G}$: $C(w) = \langle \sqrt{v_1} \rangle \times \cdots \times \langle \sqrt{v_k} \rangle \times \mathbb{A}(w)$.*

Corollary 2.4 *For any $w \in \mathbb{G}$ the centraliser $C(w)$ of w is an isolated subgroup of $\mathbb{G}$, i.e., $C(w) = C(\sqrt{w})$.*

2.2 Algebraic Geometry over Groups

In this section we recall basic notions of algebraic geometry over groups (see [1] for details). For the purposes of algebraic geometry over a group G, one has to consider the category of G-groups, *i.e.,* groups that contain a designated subgroup isomorphic to the group G. If H and K are G-groups, then a homomorphism $\varphi \colon H \to K$ is a G-homomorphism if $\varphi(g) = g$ for every $g \in G$. In the category of G-groups morphisms are G-homomorphisms; subgroups are G-subgroups, etc.

Let G be a group generated by a finite set A, $F(X)$ a free group with basis $X = \{x_1, x_2, \ldots, x_n\}$, and $G[X] = G * F(X)$, the free product of G and $F(X)$. A subset $S \subset G[X]$ is called *a system of equations* over G. As an element of the free product, the left side of every equation in $S = 1$ can be written as a product of some elements from $X \cup X^{-1}$ (which are called *variables*) and some elements from $A \subset G$ (*constants*).

A *solution* of the system $S(X) = 1$ over a G-group H is a tuple of elements $h_1, \ldots, h_n \in H$ such that every equation from S vanishes at $(h_1, \ldots, h_n)$, *i.e.,* $S(h_1, \ldots, h_n) = 1$ in H. Equivalently, a solution of the system $S = 1$ over H is a G-homomorphism $\phi \colon G[X] \to H$ such that $S \subseteq \ker(\phi)$. Denote by ncl(S) the normal closure of S in $G[X]$, and by G_S the quotient group $G[X]/\mathrm{ncl}(S)$. Then every solution of $S(X) = 1$ in H gives rise to a G-homomorphism $G_S \to H$, and vice versa. By $V_H(S)$ we denote the set of all solutions in H of the system $S = 1$ and call it the *algebraic set defined by S*.

The normal subgroup of $G[X]$ of the form

$$R(S) = \{T(X) \in G[X] \mid \forall A \in H^n(S(A) = 1 \to T(A) = 1)\}$$

is called the *radical of the system S*. Note that $S \subseteq R(S)$. There exists a one-to-one correspondence between algebraic sets $V_H(S)$ of systems of equations in $G[X]$ and radical subgroups.

The quotient group $G_{R(S)} = G[X]/R(S)$ is called the *coordinate group* of the algebraic set $V_H(S)$, and every solution of $S(X) = 1$ in H is a G-homomorphism $G_{R(S)} \to H$.

A G-group H is called *G-equationally Noetherian* if every system $S(X) = 1$ with coefficients from G is equivalent over G to a finite subsystem $S_0 = 1$, where $S_0 \subset S$,

i.e., the systems S and S_0 define the same algebraic set. If G is G-equationally Noetherian, then we say that G is equationally Noetherian. If a G-group H is equationally Noetherian, every algebraic set V in G^n is a finite union of *irreducible components* of V.

Let H and K be G-groups. We say that a family of G-homomorphisms $\mathcal{F} \subset \mathrm{Hom}_G(H, K)$ *G-separates* (*G-discriminates*) H into K if for every non-trivial element $h \in H$ (every finite set of non-trivial elements $H_0 \subset H$) there exists $\phi \in \mathcal{F}$ such that $h^\phi \neq 1$ ($h^\phi \neq 1$ for every $h \in H_0$). In this case we say that H is *G-separated* (*G-discriminated*) into K by $\mathcal{F}$. In the case that $G = 1$, we simply say that H is separated (discriminated) by K.

A G-group H is called a *G-domain* if for any $x, y \neq 1$ there exists $g \in G$ such that $[x, y^g] \neq 1$. In the case that G is G-domain, we say that G is a domain.

3 van Kampen Diagrams

In this section we present some preliminary results on van Kampen diagrams. We refer the reader to [3, 27] for a more detailed account on van Kampen diagrams. Our aim here is to review some basic notions and techniques and apply them to the particular case of partially commutative groups.

3.1 van Kampen Diagrams in Partially Commutative Groups

By van Kampen's lemma (see [3]) the word w represents the trivial element in a fixed group G given by the presentation $\langle A \mid R \rangle$ if and only if there exists a finite connected, oriented, based, labeled, planar graph $\mathcal{D}$ where each oriented edge is labeled by a letter in $A^{\pm 1}$, each bounded region (cell) of $\mathbb{R}^2 \setminus \mathcal{D}$ is labeled by a word in R (up to shifting cyclically or taking inverses) and w can be read on the boundary of the unbounded region of $\mathbb{R}^2 \setminus \mathcal{D}$ from the base vertex. Then we say that $\mathcal{D}$ is a *van Kampen diagram* for the boundary word w over the presentation $\langle A \mid R \rangle$. If $w = uv^{-1} =_\mathbb{G} 1$, we say that $\mathcal{D}$ is a *van Kampen diagram* realising the equality $u = v$. In the event that a van Kampen diagram $\mathcal{D}$ realises the equality $w = \overline{w}$ we say that $\mathcal{D}$ is a *geodesic van Kampen diagram* for w.

Any van Kampen diagram can also be viewed as a 2-complex, with a 2-cell attached for each bounded region (see Figure 1).

We shall further restrict our considerations to the case when G is a partially commutative group.

Following [27], if we complete the set of defining relations adding the trivial relations $1 \cdot a = a \cdot 1$ for all $a \in A$, then every van Kampen diagram can be transformed so that its boundary is a simple curve. In other words, as a 2-complex the van Kampen diagram is homeomorphic to a disc tiled by cells that are also homeomorphic to a disc (see Figure 1). We further assume that all van Kampen diagrams are of this form.

Let $\mathcal{D}$ be a van Kampen diagram for the boundary word w. Given an occurrence a in w, there is a cell C in the 2-complex $\mathcal{D}$ attached to a. Since every cell in a van Kampen diagram is either labelled by a relation of the form $a^{-1}b^{-1}ab$ or is a so-called 0-cell, *i.e.*, a cell labelled by $1 \cdot a = a \cdot 1$, there is just one occurrence of a and one
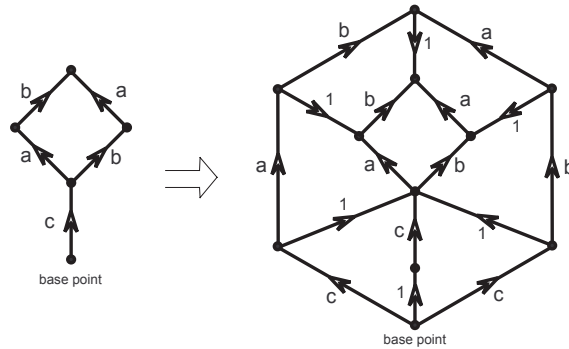
Figure 1: van Kampen diagram and non-singular van Kampen diagram for $w = caba^{-1}b^{-1}c^{-1}$ over $\langle a, b, c \mid [a, b] = 1 \rangle$.

occurrence of a^{-1} on the boundary of C.

Since $\mathcal{D}$ is homeomorphic to a disc, if the occurrence of a^{-1} on the boundary of C is not on the boundary of $\mathcal{D}$, there exists a unique cell $C' \neq C$ attached to this occurrence of a^{-1} in $\mathcal{D}$. Repeating this process, we obtain a unique *band* in $\mathcal{D}$.

Because of the structure of the cells and the fact that $\mathcal{D}$ is homeomorphic to a disc, a band never self-intersects; indeed, since $\mathcal{D}$ is homeomorphic to a disc, the only way a band can self-intersect is shown in Figure 2. But then the cell corresponding to the self-intersection of the band is labelled by the word $aaa^{-1}a^{-1}$.
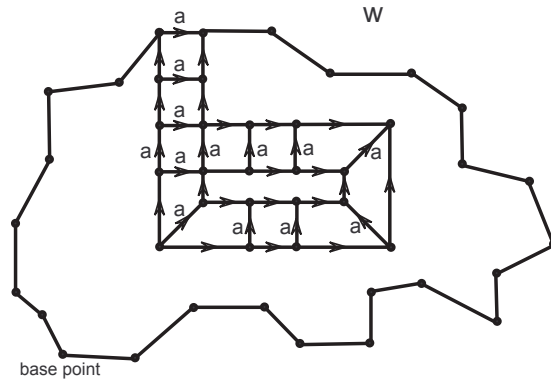

Figure 2: Bands do not self-intersect

Then since the number of cells in $\mathcal{D}$ is finite, in a finite number of steps the band will again meet the boundary in an occurrence of a^{-1} in w (see Figure 3).

We will use the notation L_a to indicate that a band begins (and thus ends) in an occurrence of a letter $a \in A^{\pm 1}$.
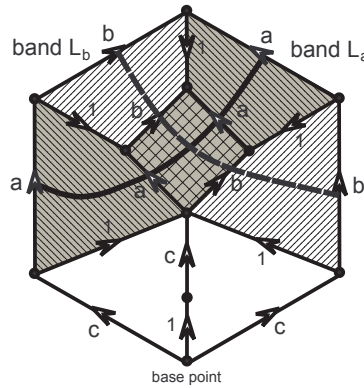
Figure 3: Bands in a van Kampen diagram

Remark 3.1 • If two bands L_a and L_b cross then the intersection cell realises the equality $a^{-1}b^{-1}ab = 1$ and so $a \neq b$ and $[a, b] = 1$ (see Figure 3).
• Every band L_a gives a decomposition of w in the following form $w = w_1 a w_2 a^{-1} w_3$, where $[a, \alpha(w_2)] = 1$ and $a \notin \alpha(w_2)$.

Lemma 3.2 *Let $\mathbb{G}$ be a partially commutative group. A word w in $\mathbb{G}$ is not geodesic if and only if w contains a subword aBa^{-1} such that $[a, \alpha(B)] = 1$, $a \in A^{\pm 1}$ if and only if there exists a geodesic van Kampen diagram for w that contains a band L_a with both ends in w.*

A proof of this lemma can be found in [29].

It is known (see [15]) that if a word w represents the trivial element in $\mathbb{G}$, it can be reduced to the empty word using commutation relations of letters and free cancellation. This reduction process of w to the empty word induces a pairing of occurrences in the word w that cancel. This pairing is independent of the order in which the letters are freely cancelled.

Lemma 3.2 reflects a consequence of a deeper fact: there exists a one-to-one correspondence between van Kampen diagrams for w and pairings induced by procedures of reductions of w to the empty word. Indeed, let $\mathcal{D}$ be a van Kampen diagram for the boundary word w. Every band L_a gives a decomposition of the form $w = w_{1,a} a w_{2,a} a^{-1} w_{3,a}$. Let L_a be a band such that the length of $w_{2,a}$ is minimal. Hence, every band L_b with an end in an occurrence b in $w_{2,a}$ cannot have the other end in an occurrence b^{-1} in $w_{2,a}$. Thus for every occurrence b in $w_{2,a}$ the band L_b crosses the band L_a and hence $[a, \alpha(w_{2,a})] = 1$, $a \notin \alpha(w_{2,a})$. This implies that $w = w_{1,a} a w_{2,a} a^{-1} w_{3,a} = w_{1,a} w_{2,a} a a^{-1} w_{3,a}$ and thus there exists a process of reduction of w to the empty word in which the occurrence a is cancelled with the occurrence a^{-1}. Collapsing the band L_a in $\mathcal{D}$, we get a van Kampen diagram $\mathcal{D}'$ for the boundary word $w' = w_{1,a} w_{2,a} w_{3,a}$; note that the number of cells in $\mathcal{D}'$ is lower than the number of cells in $\mathcal{D}$. The statement follows by induction.

Conversely, if w represents the trivial element in $\mathbb{G}$, w can be written in the form

$w = w_1 a w_2 a^{-1} w_3$, where $a \in A^{\pm 1}$ and $[a, \alpha(w_2)] = 1$ and $a \notin \alpha(w_2)$. Construct a $|w|$-polygon, designate a point, and orient and label its edges so that starting from the designated point and reading clockwise (or, counterclockwise) one reads w. To every edge labelled by an occurrence w_{2i} from w_2 we attach a cell labeled by $a w_{2i} a^{-1} w_{2i}^{-1}$. Identifying, as appropriate, the edges labeled by $a^{\pm 1}$ we get a band L_a with ends in a and a^{-1} (see Figure 4). We thereby get a $(|w| - 2)$-polygon with the boundary word $w' = w_1 w_2 w_3$ and thus, by induction, the van Kampen diagram is constructed.
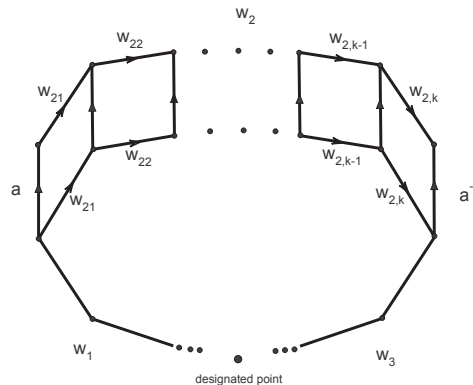

Figure 4: Constructing a van Kampen diagram by a process of reduction

If either in a geodesic van Kampen diagram for w both ends of a band L_a lie in w or equivalently, if the occurrences a and a^{-1} freely cancel each other in a reduction process of the word w to the empty word, we say that the occurrence a *cancels with* a^{-1}. Otherwise, if one of the ends of the band L_a is in an occurrence of a in w and the other is in an occurrence of a^{-1} in $\overline{w}$, we say that a *does not cancel*.

3.2 Cancellation in a Product of Elements

We now consider in detail van Kampen diagrams corresponding to a product of k geodesic words $w_1 \cdots w_k = 1$.

By Lemma 3.2 for any van Kampen diagram $\mathcal{D}$ of $w_1 \cdots w_k = 1$ every band with an end in w_i has its other end in w_j, $j \neq i$, $i, j = 1, \ldots, k$. Since every occurrence in w_1 cancels, there is a band with an end in a given occurrence a of w_1 and another end in w_i, $1 < i \leq k$. Then for any occurrence b in w_1 such that

- b is to the right of a, i.e., $w_1 = w_1' a w_1'' b w_1'''$ and
- the band L_b with an end in the occurrence b has its other end in w_j, $j > i$,

the bands L_a and L_b cross and thus $[a, b] = 1$ in $\mathbb{G}$ (see Figure 5). Therefore the word w_1 equals the following geodesic word $w_1 = w_1^k \cdots w_1^2$, where the band with an end in any occurrence of w_1^i has its other end in w_i.

A similar argument for w_l shows that w_l admits the following decomposition into a product of geodesic words (perhaps trivial) (see Figure 5):

$$w_l = w_l^{l-1} \cdots w_l^1 w_l^k \cdots w_l^{l+1},$$

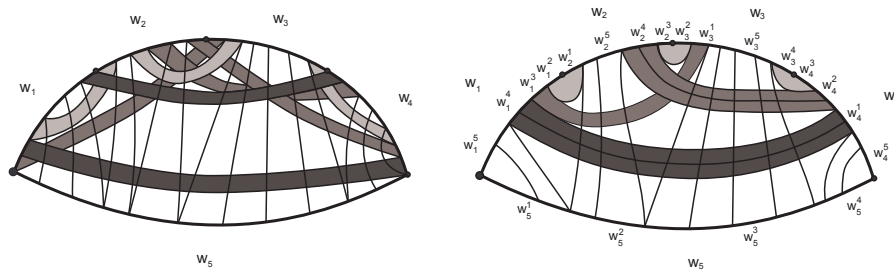where the band with an end in any occurrence of w_l^i has its other end in w_i.

Figure 5: Normal form of a van Kampen diagram, see Lemma 3.3

We summarise the above discussion in the following lemma.

Lemma 3.3 *Let $\mathbb{G}$ be a partially commutative group, let $w_1, \ldots w_k$ be geodesic words in $\mathbb{G}$ such that $w_1 \cdots w_k = 1$. Then there exist geodesic words w_i^j, $1 \le i, j \le k$ such that for any $1 \le l \le k$ there exists the following decomposition of w_l into a product of geodesic words: $w_l = w_l^{l-1} \cdots w_l^1 w_l^k \cdots w_l^{l+1}$, where $w_l^i = w_i^{l-1}$.*

Corollary 3.4 *Let $\mathbb{G}$ be a partially commutative group, let $w_1, \ldots w_k, v$ be geodesic words in $\mathbb{G}$ such that $w_1 \cdots w_k = v$. Then there exists geodesic words v_m, w_i^j, $1 \le i, j, m \le k$ such that for any $1 \le l \le k$ there exists the following decomposition of w_l into a product of geodesic words: $w_l = w_l^{l-1} \cdots w_l^1 v_l w_l^k \cdots w_l^{l+1}$, where $w_l^i = w_i^{l-1}$ and $v_1 \cdots v_k = v$.*

4 Partially Commutative Groups and Domains

It is well known that free groups are domains. The key point of the proof (which relies on the fact that free groups are CSA) is that for $a, x, y \in F$, $x \ne 1$ if $[x, y] = 1$, $[x, y^a] = 1$, then $y \in C(a)$. Therefore, to see that free groups are domains it suffices to apply the above argument for two elements a and b such that $C(a) \cap C(b) = 1$.

Although, directly indecomposable partially commutative groups are not CSA, using the description of centralisers, in Section 4.2 we prove that for $a, x, y \in \mathbb{G}$, such that $x \ne 1$ and $C(a)$ is cyclic if $[x, y] = 1$, $[x, y^a] = 1$, then either $y \in C(a)$ or $x \in \mathbb{A}(y^a)$.

The aim of Section 4.1 below is to find an element $A \in \mathbb{G}$ with cyclic centraliser for which $\mathbb{A}(y^A) = 1$. More precisely, we prove that for any $a \in \mathbb{G}$, such that $C(a)$ is cyclic, the element $A = a^{3\,\mathrm{cdim}(\mathbb{G})+4}$ possesses this property. Hence, for $a, x, y \in \mathbb{G}$, such that $x \ne 1$ and $C(a)$ is cyclic if $[x, y] = 1$, $[x, y^A] = 1$, then $y \in C(A)$.

4.1 Cancellation and Conjugation

Definition 4.1 We treat the graph Δ as a metric space with the metric d being the path metric. Let y be a vertex of Δ, define $\text{adj}(y)$ to be $\{v \in \Delta \mid d(v, y) \leq 1\}$, *i.e.*, the closed ball of radius 1 centered at y. For a subset $Y \subseteq A$, set

$$\text{adj}(Y) = \{v \in \Delta \mid d(v, y) \leq 1 \text{ for some } y \in Y\}.$$

We set

$$\text{adj}^n(y) = \underbrace{\text{adj}(\text{adj}(\ldots \text{adj}(y) \ldots))}_{n \text{ times}},$$

thus $\text{adj}^n(y) = \{v \in \Delta \mid d(v, y) \leq n\}$ is the closed ball of radius n centered at y. Similarly $\text{adj}^n(Y)$, $Y \subseteq A$ is just an n-neighbourhood of Y in Δ,

$$\text{adj}^n(Y) = \{v \in \Delta \mid d(v, y) \leq n \text{ for some } y \in Y\}.$$

Let Δ_1 be a subgraph of Δ. Then by $\text{adj}(Y)_{\Delta_1}$ we denote the following set $\text{adj}(Y)_{\Delta_1} = \text{adj}(Y) \cap \Delta_1$.

We shall further use the notion of centraliser dimension $\text{cdim}(G)$ of a group G (see Definition 4.2). An interested reader may consult [12, 13, 26] and the references therein for a detailed discussion of this notion.

Definition 4.2 If there exists an integer d such that the group G has a strictly descending chain of centralisers $C_0 > C_1 > \cdots > C_d$ of length d and no centraliser chain of length greater than d, then G is said to have *centraliser dimension* $\text{cdim}(G) = d$. If no such integer d exists, we say that the centraliser dimension of G is infinite, $\text{cdim}(G) = \infty$.

All partially commutative groups have finite centraliser dimension [13].

Lemma 4.3 *Let $\mathbb{G}$ be a directly indecomposable partially commutative group and $y \in A$. Then $\text{adj}^{\text{cdim}(\mathbb{G})}(y) = A$, i.e., the diameter $\text{diam}(\Delta(\mathbb{G}))$ of $\Delta(\mathbb{G})$ is less than or equal to $\text{cdim}(\mathbb{G})$.*

Proof The group $\mathbb{G}$ is directly indecomposable, hence the non-commutation graph $\Delta(\mathbb{G})$ is connected. Therefore for any pair of vertices $g, h \in V(\Delta)$ there exists a path p of minimal length connecting them. We claim that the length of p is less than or equal to $\text{cdim}(\mathbb{G})$.

Let $p = (g_0 = g, g_1, \ldots, g_r = h)$. The path p gives rise to a strictly descending chain of centralisers of length r:

$$\mathbb{G} > C(g_0) > C(g_0, g_1) > \cdots > C(g_0, \ldots, g_{r-2}) > C(g_0, \ldots, g_{r-1}).$$

Indeed, to see that each of the inclusions above is strict we use the minimality of the path p. Suppose $C(g_0, \ldots, g_{i-1}) = C(g_0, \ldots, g_i)$ for some $1 \leq i \leq r - 1$.

Then since $g_{i+1} \notin C(g_0, \ldots, g_i)$, we also have $g_{i+1} \notin C(g_0, \ldots, g_{i-1})$. So there exists $0 \leq j \leq i-1$, such that g_j does not commute with g_{i+1}, thus the distance between them is 1. Then $(g_0, g_1, \ldots, g_j, g_{i+1}, \ldots, g_r)$ is a shorter path from g to h, contradicting the minimality of the path p.

As the length r of any strictly descending chain of centralisers is bounded by $\mathrm{cdim}(\mathbb{G})$, so is the distance between any two points in Δ, so $\mathrm{adj}^{\mathrm{cdim}(\mathbb{G})}(g) = A$. ∎

Remark 4.4 Note that the equality $\mathrm{diam}(\Delta(\mathbb{G})) = \mathrm{cdim}(\mathbb{G})$ can be attained. Set $\mathbb{G}$ to be, for example, the partially commutative group whose non-commutation graph is a path with an odd number of vertices.

Given two geodesic words $w, v \in \mathbb{G}$, if the product wv is again geodesic, we write $w \circ v$. Let $g \in \mathbb{G}$ be a geodesic word. We refer to the decomposition $g = g_1 \circ g_2 \circ g_1^{-1}$, where g_2 is cyclically reduced as a *cyclic decomposition of g*.

Given a word g^n, we write $g^{(i)}$ when we refer to the i-th factor g in the product $g^n = g \cdots g$. Similarly, given an occurrence a in g, we write $a^{(i)}$, $1 \leq i \leq n$ to indicate that the occurrence a is in $g^{(i)}$.

Lemma 4.5 *Let $g, z \in \mathbb{G}$ be geodesic and $g = g_1 a g_2$, $a \in A^{\pm 1}$. Let $\mathcal{D}$ be a geodesic van Kampen diagram for gz. If the occurrence a does not cancel, neither does any occurrence b in g_1 that belongs to $\mathrm{adj}(a)$.*

Proof If a does not cancel, the band L_a with an end in a has the other end in $\overline{gz}$. Then, for any $b \in g_1$ that cancels, corresponding band L_b has one end in this occurrence of b in $g^{(1)}$ and the other in an occurrence b^{-1} in z. Hence the band L_b crosses the band L_a. By Remark 3.1, $b \neq a$ and $[b, a] = 1$, so $b \notin \mathrm{adj}(a)$. ∎

Lemma 4.6 *Let $g, z \in \mathbb{G}$ be geodesic and let $g = g_1 a g_2$, $a \in A^{\pm 1}$. Let $\mathcal{D}$ be a geodesic van Kampen diagram for $z^g = gzg^{-1}$. If the occurrence of a in g and the corresponding occurrence a^{-1} in g^{-1} do not cancel, then for $b \in \mathrm{adj}(a)$, no occurrence of b in g_1 cancels with b^{-1} in g_1^{-1}.*

Proof The proof is analogous to the one of Lemma 4.5. ∎

Corollary 4.7 *Let $g, z \in \mathbb{G}$ be geodesic and let g be a cyclically reduced block. Let $\mathcal{D}$ be a geodesic van Kampen diagram for gz. If there exists an occurrence a in g that does not cancel, then $g^{\mathrm{cdim}(\mathbb{G})+1} z = g \circ z'$, i.e., no occurrence in $g^{(1)}$ cancels.*

Proof Let a be an occurrence of g that does not cancel in gz. Since $g^{\mathrm{cdim}(\mathbb{G})+1}$ is geodesic, the occurrence $a^{(\mathrm{cdim}(\mathbb{G})+1)}$ in $g^{(\mathrm{cdim}(\mathbb{G})+1)}$ does not cancel.

Since g is a block, by definition, the graph $\Delta(\alpha(g))$ is connected, *i.e.*, the subgroup generated by $\alpha(g)$ is a directly indecomposable partially commutative group. Thus, applying Lemma 4.3 to this subgroup and using the fact that the centraliser dimension of $\langle \alpha(g) \rangle$ is less than or equal to the centraliser dimension of $\mathbb{G}$ (see [12]) we get that $\mathrm{adj}(a)^{\mathrm{cdim}(\mathbb{G})}_{\Delta(\alpha(g))} = \alpha(g)$.

Recursively applying Lemma 4.5, we get that no occurrence in $g^{(i)}$ that belongs to $\mathrm{adj}^{((\mathrm{cdim}(\mathbb{G})+1)-i)}_{\Delta(\alpha(g))}(a)$ cancels. Therefore, no occurrence from $g^{(1)}$ cancels in $g^{\mathrm{cdim}(\mathbb{G})+1}z$. ∎

Corollary 4.8 Let $g, z \in \mathbb{G}$ be geodesic and let g be a cyclically reduced block. Let $\mathcal{D}$ be a geodesic van Kampen diagram for $z^g = gzg^{-1}$. If there exists an occurrence a in g, such that a and the corresponding occurrence a^{-1} in g^{-1} do not cancel, then $z^{(g^{\mathrm{cdim}(\mathbb{G})+1})} = g \circ z' \circ g^{-1}$, where $z' = z^{\mathrm{cdim}(\mathbb{G})}$.

Proof The proof is analogous to the one of Corollary 4.7   ∎

Definition 4.9 Let $z \in \mathbb{G}$ be a cyclically reduced word and $g, z_1 \in \mathbb{G}$ be so that $z = g^{-1} \circ z_1$ (in the terminology of [15], g^{-1} is called a left-divisor of z). We say that the word $gzg^{-1} = z_1 g^{-1}$ is a cyclic permutation of z.

Conjugating a cyclically reduced word z one gets a conjugation of a cyclic permutation of z. In particular, all letters of z appear in a geodesic $\overline{z^g}$ for any $g \in \mathbb{G}$. A more precise description is given in the following lemma.
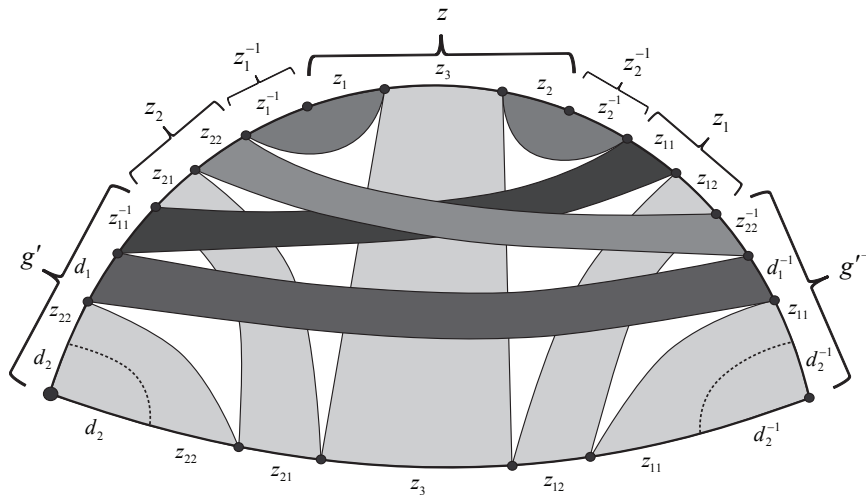

Figure 6: Lemma 4.10: cancellation in z^g.

Lemma 4.10 Let $z, g \in \mathbb{G}$, let z be cyclically reduced and g be a block. Then there exist decompositions $z = z_1 z_3 z_2$, $z_1 = z_{11} z_{12}$, $z_2 = z_{21} z_{22}$, $g = d_2 d_1 z_{22} z_{11}^{-1} z_2 z_1^{-1}$, where $z_1 \in \mathbb{A}(z_2)$, $z_{11} \in \mathbb{A}(z_3)$, $z_{22} \in \mathbb{A}(z_3)$, such that $\overline{z^g} = d_2 \circ z_{22} \circ z_{21} \circ z_3 \circ z_{12} \circ z_{11} \circ d_2^{-1}$. Furthermore, if $d_2 = 1$, then $d_1 = 1$. If $d_2 = 1$, then either $z_1 = 1$ or $z_2 = 1$, see Figure 6.

Proof Let z_1 be the maximal (where the maximum is taken over all geodesic words representing g and z) common initial subword of z and g^{-1}, i.e., $z = z_1 z_1'$, $g = g_1 z_1^{-1}$. Note that such z_1 exists and is well defined; in [15] the authors call it the left greatest common divisor of z and g^{-1}. Similarly, let z_2 be the right greatest common divisor of z and g, i.e., $z = z_2' z_2$, $g = g_2 z_2$. Then z_1^{-1} and z_2 are both right divisors of g.

Let e be the right greatest common divisor of z_1^{-1} and z_2. By [15, Proposition 3.18], $z_1^{-1} = e_1 e$, $z_2 = e_2 e$, and $g = g' e_2 e_1 e$, where $e_1 \in \mathbb{A}(e_2)$. Since z is cyclically reduced, it follows that $e = 1$. Hence, $z_1^{-1} = e_1$, $z_2 = e_2$, $g = g' z_2 z_1^{-1}$, $z_1 \in \mathbb{A}(z_2)$.

Apply Corollary 3.4 (the notation of which we use below) to the product of

$$g' z_2 z_1^{-1} z z_2^{-1} z_1 g'^{-1} = w_1 w_2 w_3 w_4 w_5 w_6 w_7 = v = \overline{g' z_2 z_1^{-1} z z_2^{-1} z_1 g'^{-1}}.$$

Since $g' z_2 z_1^{-1} = g$ is geodesic, $w_1^3 = w_2^1 = w_2^3 = w_5^7 = w_6^5 = w_6^7 = 1$. By definition of z_1 and z_2, we get that $w_1^4 = w_2^4 = w_6^4 = w_7^4 = 1$ and $w_3 = w_3^4$, $w_5 = w_5^4$, so $w_3^i = w_5^i = 1$, $i \neq 4$. As $\alpha(z_1) \cap \alpha(z_2) = 1$, so $w_2^6 = 1$.

It follows that $z_2 = w_2 = v_2 w_2^7 = z_{21} z_{22}$ and, analogously, $z_1 = w_6 = w_6^1 v_6 = z_{11} z_{12}$. Moreover, $g' = w_1 = v_1 w_1^7 w_1^6 = v_1 w_1^7 z_{11}^{-1}$, and $g'^{-1} = w_7 = w_7^2 w_7^1 v_7 = z_{22}^{-1} w_7^1 v_7$.

We obtain that $g' = v_1 w_1^7 z_{11}^{-1} = v_7^{-1} w_1^7 z_{22}$, thus, since $\alpha(z_{11}) \cap \alpha(z_{22}) = 1$, we have $v_1 = d_2 z_{22}$, $v_7^{-1} = d_2 z_{11}^{-1}$ and so $g' = d_2 z_{22} d_1 z_{11}^{-1} = d_2 z_{11}^{-1} d_1 z_{22}$. This implies that $[d_1, z_{11} z_{22}] = 1$, and, since $z_{11} \in \mathbb{A}(z_{22})$, we have $[d_1, z_{11}] = [d_1, z_{22}] = 1$.

By Corollary 3.4, $\overline{z^g} = v_1 \cdots v_7 = d_2 \circ z_{22} \circ z_{21} \circ z_3 \circ z_{12} \circ z_{11} \circ d_2^{-1}$.

Finally, since $g = d_2 d_1 z_{22} z_{11}^{-1} z_2 z_1^{-1}$, if $d_2 = 1$ and $d_1 \neq 1$, then as shown above $[\alpha(d_1), \alpha(z_{11})] = [\alpha(d_1), \alpha(z_{22})] = 1$, and $[\alpha(d_1), \alpha(z_{12})] = [\alpha(d_1), \alpha(z_{21})] = 1$, see Figure 6. Hence g is not a block. If $d_2 = 1$, and so $d_1 = 1$, then since g is a block and $[\alpha(z_1), \alpha(z_2)] = 1$, either $z_1 = 1$ or $z_2 = 1$. ∎

Lemma 4.11 *Let $z, g \in \mathbb{G}$, let z be cyclically reduced, g be a block and let $[z, g] \neq 1$. Furthermore, suppose that g^{-1} does not left-divide z and z^{-1}. Then $z^{g^2} = g^2 z g^{-2} = g_1 \circ z' \circ g_1^{-1}$, where $g_1 \neq 1$ is a left-divisor of g, i.e., there exist occurrences l in g^2 and, correspondingly, l^{-1} in g^{-2} that do not cancel.*

Proof By Lemma 4.10, the notation of which we now use, $\overline{z^g} = d_2 \circ z_{22} \circ z_{21} \circ z_3 \circ z_{12} \circ z_{11} \circ d_2^{-1}$. If $d_2 \neq 1$, the result follows by Lemma 4.6.

Suppose that $d_2 = 1$. Then by Lemma 4.10 $d_1 = 1$ and either z_1 or z_2 is trivial. Without loss of generality, we may assume that $z_1 = 1$. Then $z_2 \neq 1$ is a block and

$$z^{g^2} = g(z_{22} z_{21} z_3) g^{-1} = (z_{22} z_{21} z_{22})(z_{22} z_{21} z_3)(z_{22} z_{21} z_{22})^{-1}.$$

To prove the lemma it suffices to show that the word $(z_{22} z_{21} z_{22})^{-1}$ does not cancel completely in the product of geodesic words $(z_{22} z_{21} z_{22} z_{22} z_{21} z_3)(z_{22} z_{21} z_{22})^{-1}$.

If $\alpha(z_{21})$ is a subset of $\alpha(z_3)$, then since $z_{22} \in \mathbb{A}(z_3)$, we get that $z_{22} \in \mathbb{A}(z_{21})$, and z_2 is not a block: a contradiction. Therefore, we may write the word z_{21}^{-1} as $A^{-1} B^{-1}$, where A^{-1} is the left greatest divisor of z_{21}^{-1} such that $\alpha(A) \subseteq \alpha(z_3)$ and B is non-trivial. Note that $A \in \mathbb{A}(z_{22})$ and since $[z_{21}, z_{22}] \neq 1$, one has that $[B, z_{22}] \neq 1$.

The word z^{g^2} can be rewritten as follows:

$$z^{g^2} = (z_{22} z_{21} z_{22}) z_{22} B (A z_3 A^{-1}) z_{22}^{-1} B^{-1} z_{22}^{-1} = (z_{22} z_{21} z_{22}) z_{22} B z_3' z_{22}^{-1} B^{-1} z_{22}^{-1}.$$

By construction, $z_{22} \in \mathbb{A}(z_3')$ and no occurrence in B^{-1} cancels with an occurrence in z_3'.

If $[B, z_3'] \neq 1$, then there exists an occurrence in B^{-1} that does not cancel and the result follows. If $[B, z_3'] = 1$, then we have

$$z^{g^2} = (z_{22}z_{21}z_3'z_{22}z_{22}B)(z_{22}^{-1}B^{-1}z_{22}^{-1}).$$

If z_{22}^{-1} cancels completely, then we get that both B and z_{22} are right divisors of the word $z_{22}z_{21}z_3'z_{22}z_{22}B$. Then by [15, Proposition 3.18], $B = B'd$ and $z_{22} = z_{22}'d$, where $z_{22}' \in \mathbb{A}(B')$. Furthermore, since $[z_{22}, B] \neq 1$, then either $[d, z_{22}'] \neq 1$ or $[d, B'] \neq 1$.

Analogously, if $z_{22}^{-1}B^{-1}$ cancels completely, then we have that both $z_{22}B$ and Bz_{22} are right divisors of the word $z_{22}z_{21}z_3'z_{22}z_{22}B$. Applying again [15, Proposition 3.18], we get that $z_{22}B = U_1D$ and $Bz_{22} = U_2D$, where $U_1 \in \mathbb{A}(U_2)$.

Combining the above equalities, one has $z_{22}B = z_{22}'dB'd = U_1D$ and $Bz_{22} = B'dz_{22}'d = U_2D$. Since $\alpha(U_1) \cap \alpha(U_2) = \varnothing$, $\alpha(B') \cap \alpha(z_{22}') = \varnothing$ and either $[d, z_{22}'] \neq 1$ or $[d, B'] \neq 1$; the above equalities yield a contradiction. Thus $z_{22}^{-1}B^{-1}$ does not cancel completely in $z^{g^2} = (z_{22}z_{21}z_3'z_{22}z_{22}B)(z_{22}^{-1}B^{-1}z_{22}^{-1})$ and the result follows. ∎

We now record some basic properties of $\mathbb{A}(w)$ which we shall use later. Given $x, y \in \mathbb{G}$ the following hold:

(A) $x \in \mathbb{A}(y)$ if and only if $y \in \mathbb{A}(x)$;
(B) if $\alpha(x) \subset \alpha(y)$ then $\mathbb{A}(y) < \mathbb{A}(x)$;
(C) if the centraliser of x is cyclic then $\mathbb{A}(x) = 1$.

Lemma 4.12 (i) *Let $g \in \mathbb{G}$ be a cyclically reduced block and let $z \in \mathbb{G}$ be so that g^{-1} does not left-divide and right-divide z. Then one has $g^{\mathrm{cdim}(\mathbb{G})+1}zg^{\mathrm{cdim}(\mathbb{G})+1} = g \circ g^{\mathrm{cdim}(\mathbb{G})}zg^{\mathrm{cdim}(\mathbb{G})} \circ g$.*

(ii) *Let $g \in \mathbb{G}$ be a cyclically reduced block and let $z = z_1z_2z_1^{-1}$ be the cyclic decomposition of an element $z \in \mathbb{G}$. Suppose that g^{-1} does not left-divide z, z^{-1}, z_2, and z_2^{-1}, and $[g, z] \neq 1$. Then one has $z^{g^{3\,\mathrm{cdim}(\mathbb{G})+4}} = g \circ z^{g^{3\,\mathrm{cdim}(\mathbb{G})+3}} \circ g^{-1}$.*

Proof We first prove (i). We claim that in the product gzg there exist occurrences l_1 and l_2 in both subwords g of gzg that do not cancel in $\overline{gzg}$. Assume the contrary. Then one of the subwords g cancels completely. Without loss of generality we may assume that the second subword g of gzg cancels completely. By Lemma 3.3, we get that $g = g_1g_2$, where g_1 cancels with z and g_2 cancels with g. In other words, $z = z'g_1^{-1}$ and $g = g'g_2^{-1}$. Since g^{-1} does not right-divide z, we have that $g_2 \neq 1$. This yields a contradiction, as g is right-divisible by both g_2 and g_2^{-1}. The statement now follows from Corollary 4.7.

We now prove (i). Consider the product $g^{(\mathrm{cdim}(\mathbb{G})+1)}z_1$, then no occurrence in $g^{(1)}$ cancels. Indeed, since g does not left-divide z, there is an occurrence in g that does not cancel in gz_1. Applying Corollary 4.7, we get that no occurrence in $g^{(1)}$ cancels in $g^{(\mathrm{cdim}(\mathbb{G})+1)}z_1$. We thereby get $\overline{g^{(\mathrm{cdim}(\mathbb{G})+1)}z_1} = g \circ g' \circ z_1'$, where z_1' is a right-divisor (may be trivial) of z_1 and g' is a left-divisor of $g^{\mathrm{cdim}(\mathbb{G})}$.

Notice that since $\alpha(g') \subset \alpha(g)$ and g is a block, on one hand we get that gg' is a block and on the other that for any occurrence a in g' there exists an occurrence b in g such that $b \in \mathrm{adj}(a)$.

If $[z_1', g] \neq 1$ (or $[z_1', g'] \neq 1$), then there exists an occurrence a in g (or in g') that belongs to $\mathrm{adj}(\alpha(z_1'))$. Since no occurrence in z_1' and in $z_1'^{-1}$ cancels in $z_2^{gg'z_1'}$, by

Lemma 4.6 neither does the occurrence a in g (or in g'), and, correspondingly a^{-1} in g^{-1} (or in g'^{-1}). If a is an occurrence in g', then there exists an occurrence b in g that belongs to $\mathrm{adj}(a)$, and so by Lemma 4.6 this occurrence b in g and the corresponding occurrence b^{-1} in g^{-1} do not cancel. Hence, in any case, there exists an occurrence in g that does not cancel in $((z_2)^{g'z_1'})^g$. Therefore, by Corollary 4.8 we get that

$$((z_2)^{g'z_1'})^{(g^{\mathrm{cdim}(\mathbb{G})+1})} = g \circ z_2' \circ g^{-1}$$

and thus

$$z^{g^{(2\,\mathrm{cdim}(\mathbb{G})+1)}} = \left((z_2)^{g^{(\mathrm{cdim}(\mathbb{G})+1)}z_1}\right)^{(g^{\mathrm{cdim}(\mathbb{G})})} = g \circ z^{g^{2\,\mathrm{cdim}(\mathbb{G})}} \circ g^{-1}.$$

Assume now that $[z_1', g] = 1$ and $[z_1', g'] = 1$. If $[gg', z_2] \neq 1$ or $[g^2g', z_2] \neq 1$, since gg' (and g^2g') is a block, by Lemma 4.11, there exists an occurrence a in $(gg')^2$ (or in $(g^2g')^2$), and so an occurrence a in $g^{\mathrm{cdim}(\mathbb{G})+2}g'$ (or in $g^{\mathrm{cdim}(\mathbb{G})+4}g'$), such that a and the occurrence a^{-1} in $(g^{\mathrm{cdim}(\mathbb{G})+2}g')^{-1}$ (in $(g^{\mathrm{cdim}(\mathbb{G})+4}g')^{-1}$) do not cancel in $z_2^{g^{\mathrm{cdim}(\mathbb{G})+2}g'}$ (in $z_2^{(g^{\mathrm{cdim}(\mathbb{G})+4}g')}$). If a is an occurrence in g', then there exists an occurrence b in $g^{\mathrm{cdim}(\mathbb{G})+2}$ (in $g^{\mathrm{cdim}(\mathbb{G})+4}$) that belongs to $\mathrm{adj}(a)$, and thus by Lemma 4.6 this occurrence b in $g^{\mathrm{cdim}(\mathbb{G})+2}$ (in $g^{\mathrm{cdim}(\mathbb{G})+4}$) and the corresponding occurrence b^{-1} in $g^{-\,\mathrm{cdim}(\mathbb{G})-2}$ (in $g^{-\,\mathrm{cdim}(\mathbb{G})-4}$) do not cancel.

Now by Corollary 4.8 we get

$$(z_2^{g'})^{g^{(2\,\mathrm{cdim}(\mathbb{G})+4)}} = g \circ z_2' \circ g^{-1}$$

and thus

$$z^{g^{(3\,\mathrm{cdim}(\mathbb{G})+4)}} = \left((z_2)^{g^{(2\,\mathrm{cdim}(\mathbb{G})+4)}z_1}\right)^{(g^{\mathrm{cdim}(\mathbb{G})})} = \left((z_2)^{g^{\mathrm{cdim}(\mathbb{G})+4}g'z_1'}\right)^{(g^{\mathrm{cdim}(\mathbb{G})})}$$

$$= \left(((z_2)^{g'})^{(g^{2\,\mathrm{cdim}(\mathbb{G})+4})}\right)^{z_1'} = g \circ z^{g^{3\,\mathrm{cdim}(\mathbb{G})+3}} \circ g^{-1}.$$

Finally, suppose that $[z_1', g] = 1$, $[z_1', g'] = 1$, $[gg', z_2] = 1$, and $[g^2g', z_2] = 1$. We have $g^{\mathrm{cdim}(\mathbb{G})+2} = g^2 \circ g' \circ d$, $z_1 = d^{-1} \circ z_1'$. If $g' \in \langle\sqrt{g}\rangle$, then $d \in \langle\sqrt{g}\rangle$ and so $[d, z_1'] = [d, z_2] = 1$. This contradicts the assumption that $z = z_2^{d^{-1}z_1'}$ is geodesic. Thus, we may assume that $g' \notin \langle\sqrt{g}\rangle$. In this case, we have that $\alpha(g') \subseteq \alpha(g)$, $[g, g'] \neq 1$ and g is a block. Therefore, by Theorem 2.3, from $[g, z_1'] = [g', z_1'] = 1$ we get that $\alpha(g) \subset \mathbb{A}(z_1')$, and from $[gg', z_2] = [g^2g', z_2] = 1$ we get that $\alpha(g) \subset \mathbb{A}(z_2)$. Since $\alpha(d) \subseteq \alpha(g)$, we have $[d, z_1'] = [d, z_2] = 1$: a contradiction with the assumption that $z = z_2^{d^{-1}z_1'}$ is geodesic. ∎

Remark 4.13 A more subtle argument shows that the exponent $3\,\mathrm{cdim}(\mathbb{G}) + 4$ in Lemma 4.12 can be replaced by $\mathrm{cdim}(\mathbb{G})$.

Corollary 4.14 *Let $g \in \mathbb{G}$ have cyclic centraliser. Then for any element $z \in \mathbb{G}$ such that g^{-1} does not left-divide z, z^{-1}, z_2, and z_2^{-1}, one has $\mathbb{A}(z^{(g^{3\,\mathrm{cdim}(\mathbb{G})+4})}) = 1$.*

Proof Let $g = g_1 g_2 g_1^{-1}$ be the cyclic decomposition of g. We prove that no occurrence of $g_2^{(1)}$ cancels in $z^{(g^{3 \operatorname{cdim}(\mathbb{G})+4})}$, therefore, by properties (B) and (C) of $\mathbb{A}$, we get $\mathbb{A}(z^{(g^{3 \operatorname{cdim}(\mathbb{G})+4})}) \subseteq \mathbb{A}(g_2) = 1$.

By Lemma 4.12, no occurrence in $g_2^{(1)}$ cancels in $(z^{g_1^{-1}})^{(g_2^{3 \operatorname{cdim}(\mathbb{G})+4})}$. Then, since

$$z^{(g^{3 \operatorname{cdim}(\mathbb{G})+4})} = \left((z^{g_1^{-1}})^{g_2^{3 \operatorname{cdim}(\mathbb{G})+4}} \right)^{g_1},$$

no occurrence in $g_2^{(1)}$ cancels. ∎

Corollary 4.15 *Let $g \in \mathbb{G}$ have cyclic centraliser. Then for any element $z \in \mathbb{G}$ such that $\mathbb{A}(z) \neq 1$ one has $\mathbb{A}(z^{(g^{3 \operatorname{cdim}(\mathbb{G})+4})}) = 1$.*

Proof If g^{-1} left-divides $z^{\pm 1}$ or $z_2^{\pm 1}$, then by property (B) of $\mathbb{A}$ we get that $\mathbb{A}(z) \subset \mathbb{A}(g) = 1$. ∎

Remark 4.16 In Lemma 4.12, Corollary 4.14, and Corollary 4.15 we impose the condition that g^{-1} does not left-divide z, z^{-1}, z_2, and z_2^{-1}, because we seek the bound $3 \operatorname{cdim}(\mathbb{G}) + 4$ on the number of times one has to conjugate z by g. The reason for this is that the notion of centraliser dimension is axiomatisable using (existential) first-order formulas, [12] (we refer the reader to Section 6 for consequences of this result). If one does not impose this condition, Lemma 4.12 could be rephrased as follows.

Lemma *Let $g \in \mathbb{G}$ be a cyclically reduced block, then for any element $z \in \mathbb{G}$ there exists $N \in \mathbb{N}$ such that $z^{(g^N)} = g \circ z^{(g^{N-1})} \circ g^{-1}$.*

4.2 Criterion To Be a Domain

Proposition 4.17 *Let $\mathbb{G}$ be a non-abelian directly indecomposable partially commutative group. Let $g \in \mathbb{G}$ have cyclic centraliser, $x, y \in \mathbb{G}$, $x, y \neq 1$ be such that $[x, y] = 1$ and $[x, y^{(g^{3 \operatorname{cdim}(\mathbb{G})+4})}] = 1$. Then $C(x) = C(y) = C(g)$.*

Proof Let $x = w x_1^{r_1} \cdots x_k^{r_k} w^{-1}$, where $x_1, \ldots, x_k$ are cyclically reduced root elements such that $x_1^{r_1}, \ldots, x_k^{r_k}$ are the blocks of $x^{w^{-1}}$ and $r_1, \ldots, r_k \in \mathbb{Z}$. Since $[x, y] = 1$, by Theorem 2.3, after a certain re-enumeration of indices, we may assume

$$(4.1) \qquad\qquad y = w x_1^{s_1} \cdots x_l^{s_l} z w^{-1},$$

where $z \in \mathbb{A}(x_1 \cdots x_k)$, $0 \leq l \leq k$ and $s_1, \ldots, s_l \in \mathbb{Z}$. Thus,

$$(4.2) \qquad y^{(g^{3 \operatorname{cdim}(\mathbb{G})+4})} = g^{3 \operatorname{cdim}(\mathbb{G})+4} w x_1^{s_1} \cdots x_l^{s_l} z w^{-1} g^{-(3 \operatorname{cdim}(\mathbb{G})+4)}.$$

Since $[x, y^{(g^{3 \operatorname{cdim}(\mathbb{G})+4})}] = 1$, applying Theorem 2.3 once again, we get

$$(4.3) \qquad y^{(g^{3 \operatorname{cdim}(\mathbb{G})+4})} = w x_{i_1}^{t_1} \cdots x_{i_m}^{t_m} z' w^{-1},$$

where $z' \in \mathbb{A}(x_1 \cdots x_k)$, $0 \le m \le k$ and $t_1, \ldots, t_m \in \mathbb{Z}$. Equating (4.2) and (4.3), we get

$$(4.4) \qquad (x_1^{s_1} \cdots x_l^{s_l} z)^{w^{-1} g^{3 \operatorname{cdim}(\mathbb{G})+4} w} = x_{i_1}^{t_1} \cdots x_{i_m}^{t_m} z'.$$

Suppose that $l \ge 1$. Then by Corollary 2.2, $l = m$ and for any $q \in \{1, \ldots, l\}$ there exists $j \in \{1, \ldots, m\}$ such that $\alpha(x_q) = \alpha(x_{i_j})$, $s_q = t_j$, and x_q is conjugated to x_{i_j} by $w^{-1} g^{3 \operatorname{cdim}(\mathbb{G})+4} w$.

Since $\alpha(x_q) = \alpha(x_{i_j})$, and since x_q and x_{i_j} are cyclically reduced root elements whose powers $x_q^{s_q}$ and $x_{i_j}^{t_j}$ are blocks of the same word $x^{(w^{-1})}$, we get that $x_q = x_{i_j}$ for all $1 \le q \le l$, i.e., $y^{w^{-1}}$ and $\left(y^{(g^{3 \operatorname{cdim}(\mathbb{G})+4})}\right)^{w^{-1}}$ have the same blocks.

From the above it follows that

$$x_q^{w^{-1} g^{3 \operatorname{cdim}(\mathbb{G})+4} w} = x_{i_j} = x_q,$$

i.e., x_q commutes with $w^{-1} g^{3 \operatorname{cdim}(\mathbb{G})+4} w$. Since the centraliser of g is cyclic, so is the centraliser of $w^{-1} g^{3 \operatorname{cdim}(\mathbb{G})+4} w$ and thus, so is the centraliser of x_q. More precisely, $C(x_q) = C(g^{3 \operatorname{cdim}(\mathbb{G})+4})^{w^{-1}} = C(g)^{w^{-1}}$.

Since x_q has cyclic centraliser, $x^{w^{-1}}$ and $y^{w^{-1}}$ both have a unique block; furthermore, since $z \in \mathbb{A}(x_q)$ by property (A) of $\mathbb{A}$, z is trivial. Therefore, $x = y = (x_q^{r_q})^w$ and so $C(x) = C(y) = C(x_q)^w = C(g)$.

Next suppose that $l = 0$. We prove then that x is trivial, contradicting the assumption. We rewrite equations (4.1) and (4.4) as follows

$$y = w z w^{-1} \quad \text{and} \quad z' = z^{w^{-1} g^{3 \operatorname{cdim}(\mathbb{G})+4} w}.$$

Notice that since $z, z' \in \mathbb{A}(x_1 \cdots x_k)$ by property (C) of $\mathbb{A}$, we get that $x_1, \ldots, x_k \in \mathbb{A}(z) \cap \mathbb{A}(z')$. Therefore, if either $\mathbb{A}(z)$ or $\mathbb{A}(z')$ is trivial, so is x. Assume $\mathbb{A}(z)$ is non-trivial. Since the centraliser of $w^{-1} g w$ is cyclic, Corollary 4.15 applies to $z' = z^{((w^{-1} g w)^{3 \operatorname{cdim}(\mathbb{G})+4})}$, thus $\mathbb{A}(z') = 1$ and so $x = 1$. ∎

Corollary 4.18 *Let $\mathbb{G}$ be a non-abelian directly indecomposable partially commutative group. Let $a, b \in \mathbb{G}$ be elements with cyclic centralisers and such that $C(a) \cap C(b) = 1$. Then for any solution $x, y \in \mathbb{G}$ of the system*

$$[x, y] = 1, \ [x, y^{(a^{3 \operatorname{cdim}(\mathbb{G})+4})}] = 1, \quad [x, y^{(b^{3 \operatorname{cdim}(\mathbb{G})+4})}] = 1,$$

either $x = 1$ or $y = 1$.

Proof Applying Proposition 4.17 for the triples x, y, a and x, y, b we get that if $x \ne 1$ and $y \ne 1$, then $C(x) = C(y) = C(a)$ and $C(x) = C(y) = C(b)$, a contradiction with $C(a) \cap C(b) = 1$. Note that the elements a and b satisfying the assumption of the corollary exist (it suffices to take two distinct block elements such that $[a, b] \ne 1$ and $\alpha(a), \alpha(b) = A$). ∎

Theorem 4.19 (**Criterion for a partially commutative group to be a domain**) *A partially commutative group $\mathbb{G}$ is a domain if and if $\mathbb{G}$ is non-abelian and directly indecomposable.*

Proof Since the direct product of two non-abelian groups is never a domain (see [1, 20]), the result follows immediately from Corollary 4.18.   ∎

Note that Corollary 4.18 shows in fact that any non-abelian directly indecomposable partially commutative group is a domain with respect to only two elements $a^{3\,\mathrm{cdim}(\mathbb{G})+4}$ and $b^{3\,\mathrm{cdim}(\mathbb{G})+4}$ which are independent of the choice of x and y (in the notation of the definition of domain).

Lemma 4.20 *Let $a \in \mathbb{G}$ be a cyclically reduced block element and let $w_1, w_2 \in \mathbb{G}$ be geodesic words of the form $w_1 = a^{\delta_1} \circ g_1 \circ a^{\epsilon}, w_2 = a^{\epsilon} \circ g_2 \circ a^{\delta_2}$, where $\epsilon, \delta_1, \delta_2 = \pm 1$. Then the geodesic word $\overline{w_1 w_2}$ has the form $\overline{w_1 w_2} = a^{\delta_1} \circ w_3 \circ a^{\delta_2}$.*

Proof We claim that no occurrence from the subword a^{ϵ} of w_1 cancels in the product $w_1 w_2$. Indeed, assume the contrary. Let l be the right divisor of a^{ϵ} of length one that cancels in the product $w_1 w_2$, i.e., $a^{\epsilon} = a'l$. It follows that $w_2 = l^{-1} w_2'$ (see [15]). Since a^{ϵ} and l^{-1} are both left divisors of w_2, by [15, Proposition 3.18] one has that either l^{-1} is a left divisor of a^{ϵ} or $l^{-1} \in \mathbb{A}(a)$. As a^{ϵ} is cyclically reduced and l is a right divisor of a^{ϵ}, so l^{-1} is not a left divisor of a. On the other hand, since l is an occurrence of a^{ϵ}, we have that $l^{-1} \notin \mathbb{A}(a)$. Therefore, no occurrence in the subword a^{ϵ} of w_1 cancels in the product $w_1 w_2$.

Since no occurrence in a^{ϵ} of w_1 cancels and a is a block, by Lemma 4.5, no occurrence in a^{δ_1} cancels in the product $w_1 w_2$.

An analogous argument shows that no occurrence in the subword a^{ϵ} of w_2 and in a^{δ_2} cancels and the statement follows.   ∎

Theorem 4.21 *Let $\mathbb{G}$ be a non-abelian directly indecomposable partially commutative group. Then $\mathbb{G}[X]$ is $\mathbb{G}$-discriminated by $\mathbb{G}$.*

Proof The group $\mathbb{G}[X]$ is a non-abelian directly indecomposable partially commutative $\mathbb{G}$-group, thus by Theorem 4.19, $\mathbb{G}[X]$ is a domain. By [2, Theorem C1], it suffices to prove that $\mathbb{G}[X]$ is $\mathbb{G}$-separated by $\mathbb{G}$.

Without loss of generality, we may assume that $X = \{x\}$. Take an element $w \in \mathbb{G}[X]$. Without loss of generality, we may assume that w is cyclically reduced, $w = x^{k_1} g_1 \cdots g_{l-1} x^{k_l} g_l$, where $g_i \in \mathbb{G}$, $g_1, \ldots, g_l \neq 1$. Take $a \in \mathbb{G}$ such that the centraliser of a is cyclic (such a exists since $\mathbb{G}$ is directly indecomposable) and satisfies the assumptions of Lemma 4.12 for every g_i, $i = 1, \ldots, l$, where here, in the notation of Lemma 4.12, a plays the role of g and g_i play the role of z.

Consider the homomorphism $\varphi_a \colon \mathbb{G}[X] \to \mathbb{G}$, defined by $x \mapsto a^{6\,\mathrm{cdim}(\mathbb{G})+8}$. Then

$$
\begin{aligned}
\varphi_a(w) = {} & a^{k_1(3\,\mathrm{cdim}(\mathbb{G})+4)} \left(a^{k_1(3\,\mathrm{cdim}(\mathbb{G})+4)} g_1 a^{k_2(3\,\mathrm{cdim}(\mathbb{G})+4)} \right) \\
& \times \left(a^{k_2(3\,\mathrm{cdim}(\mathbb{G})+4)} g_2 a^{k_2(3\,\mathrm{cdim}(\mathbb{G})+4)} \right) \cdots \left(a^{k_{l-1}(3\,\mathrm{cdim}(\mathbb{G})+4)} g_{l-1} a^{k_l(3\,\mathrm{cdim}(\mathbb{G})+4)} \right) \\
& \times a^{k_l(3\,\mathrm{cdim}(\mathbb{G})+4)} g_l.
\end{aligned}
$$

By Lemma 4.12, every factor of $\varphi_a(w)$ of the form $\left(a^{k_i(3\,\mathrm{cdim}(\mathbb{G})+4)} g_i a^{k_{i+1}(3\,\mathrm{cdim}(\mathbb{G})+4)} \right)$ has the form $a^{\mathrm{sign}(k_i)} \circ \tilde{g}_i \circ a^{\mathrm{sign}(k_{i+1})}$. The statement now follows from Lemma 4.20. ∎

Corollary 4.22 *Let $\mathbb{G}$ be a non-abelian directly indecomposable partially commutative group. Then the group $\mathbb{G}[X]$ is universally equivalent to $\mathbb{G}$ (both in the language of groups and in the language $L_{\mathbb{G}}$ enriched by constants from $\mathbb{G}$).*

Proof The proof follows from [1, Theorem C2]. ∎

Corollary 4.23 *Let $\mathbb{G}$ be a non-abelian directly indecomposable partially commutative group. Then $\mathbb{G} \models \forall X(U(X) = 1) \Leftrightarrow \mathbb{G}[X] \models U(X) = 1$, i.e., only the trivial equation has the whole set $\mathbb{G}^n$ as its solution.*

Proof Since $\mathbb{G}[X]$ is $\mathbb{G}$-discriminated by $\mathbb{G}$, if the word $U(X)$ is a non-trivial element of $\mathbb{G}[X]$, then there exists a $\mathbb{G}$-homomorphism $\phi\colon \mathbb{G}[X] \to \mathbb{G}$ such that $U^\phi \neq 1$. Then $U(X^\phi) \neq 1$ in $\mathbb{G}$, a contradiction. ∎

5 Applications to Algebraic Geometry

The results and exposition of this section rely on [20]. We recall here some necessary definitions and restate some results in the case of partially commutative groups. We refer the reader to [20] for details and omitted proofs.

A *group code C* is a set of formulas

$$(5.1) \qquad C = \{U(X, P),\ E(X, Y, P),\ \mathrm{Mult}(X, Y, Z, P),\ \mathrm{Inv}(X, Y, P)\}$$

where X, Y, Z, P are tuples of variables with $|X| = |Y| = |Z|$. If $P = \varnothing$, then C is called an *absolute code* or 0-*code*.

Let C be a group code, H be a group, and B be a $|P|$-tuple of elements in H. We say that C (with parameters B) *interprets* a group $C(H, B)$ in H if the following conditions hold:

(i)   the truth set $U(H, B)$ in H of the formula $U(X, B)$ (with parameters B) is non-empty;

(ii) the truth set of the formula $E(X, Y, B)$ (with parameters B) defines an equivalence relation $\sim_B$ on $U(H, B)$;

(iii) the formulas $\mathrm{Mult}(X, Y, Z, B)$ and $\mathrm{Inv}(X, Y, B)$ define, correspondingly, a binary operation $(Z = Z(X, Y))$ and a unary operation $(Y = Y(X))$ on the set $U(H, B)$ compatible with the equivalence relation $\sim_B$;

(iv) the group $C(H, B)$ consists of the set of equivalence classes $U(H, B)/_{\sim_B}$, which form a group with respect to the operations defined by $\mathrm{Mult}(X, Y, Z, B)$ and $\mathrm{Inv}(X, Y, B)$.

We say that a group G is *interpretable* (or *definable*) in a group H if there exists a group code C and a set of parameters $B \subset H$ such that $G \simeq C(H, B)$. If C is 0-code, then G is *absolutely* or 0-*interpretable* in H. The following two types of interpretations are crucial. Let G be a definable subgroup of a group H, *i.e.,*, there exists a formula $U(x, P)$ and a set of parameters $B \subset H$ such that $G = \{g \in H \mid H \models U(g, B)\}$. Then G is interpretable in H by the code $C_G = \{U(x, P), x = y, xy = z, y = x^{-1}\}$ with parameters B. If in addition G is a normal subgroup of H, then the code

$$C_{H/G} = \{x = x, \exists v(x = yv \wedge U(v, P)), z = xy, y = x^{-1}\}$$

interprets the factor-group $H/_G$ in H with parameters B. Every group code (5.1) determines a *translation* T_C which is a map from the set of all formulas $\mathcal{F}_L$ in the language L into itself. We define T_C by induction as follows:

(i) $T_C(x = y) = E(X, Y, P)$;

(ii) $T_C(xy = z) = \text{Mult}(X, Y, Z, P)$ and $T_C(x^{-1} = y) = \text{Inv}(X, Y, P)$;

(iii)   if $\phi, \psi \in \mathcal{F}_L$ and $\circ \in \{\wedge, \vee, \rightarrow\}$, then

$$T_C(\phi \circ \psi) = T_C(\phi) \circ T_c(\psi) \quad \text{and} \quad T_C(\neg\phi) = \neg T_C(\phi);$$

(iv)   if $\phi \in \mathcal{F}_L$, then

$$T_C(\exists x \phi(x)) = \exists X(U(X, P) \wedge T_C(\phi)), \quad T_C(\forall x \phi(x)) = \forall X(U(X, P) \rightarrow T_C(\phi)).$$

Observe that the formula $T_C(\phi)$ can be constructed effectively from ϕ.

We say that the elementary theory $\text{Th}(G)$ of a group G is *interpretable* in the group H if there exists a group code $C(H, B)$ of the type (5.1) and a formula $\Psi(P)$ such that $\text{Th}(G) = \text{Th}(C(H, B))$ for any set of parameters $B \subset H$ that satisfies the formula $\Psi(P)$ in H.

Any partially commutative group is a direct product of finitely many non-abelian directly indecomposable partially commutative groups and its centre $Z(\mathbb{G})$, $Z(\mathbb{G}) \simeq \mathbb{Z}^k$, $k \in \mathbb{N}$. This decomposition is unique up to a permutation of factors. We refer to them as (direct) components of $\mathbb{G}$.

The centre $Z(\mathbb{G})$ is a normal subgroup and a definable subset of $\mathbb{G}$. It is the truth set of the formula $\Phi_Z(x) : \forall y [x, y] = 1$. Thus $Z(\mathbb{G})$ is 0-interpretable in $\mathbb{G}$. Consequently, as shown above, the quotient $\mathbb{G}/_{Z(\mathbb{G})}$ is interpretable in $\mathbb{G}$.

Therefore, to study partially commutative groups from a model-theoretic viewpoint, it suffices to consider free partially commutative groups with trivial centre.

Let $\mathbb{G}$ be a partially commutative group without centre. As mentioned above, in this event $\mathbb{G}$ is a direct product of directly indecomposable partially-commutative groups, which in turn are domains by Theorem 4.19. Thus Theorem A and Corollary A of [20] apply and can be restated as follows.

Theorem A (**cf.** [**20**])   *Let $\mathbb{G}$ be a partially commutative group with trivial centre. Then for each component $\mathbb{G}_i$ of $\mathbb{G}$ its elementary theory $\text{Th}(\mathbb{G}_i)$ is interpretable in the group $\mathbb{G}$.*

Corollary A (**cf.** [**20**])   *Let $\mathbb{G}$ be a partially commutative group and let $\mathbb{G} = \mathbb{G}_1 \times \cdots \times \mathbb{G}_n \times \mathbb{Z}^r$, where $\mathbb{G}_i$ is a non-abelian directly indecomposable partially commutative group, $i = 1, \ldots, n$. Then the following hold:*

(i)   *If $\mathbb{G} \equiv H$, then*

 (a) *$H = H_1 \times \cdots \times H_n \times Z(H)$ is a finite direct product of domains and the centre $Z(H)$, with $H_i \equiv \mathbb{G}_i$ and $Z(H) \equiv Z(\mathbb{G})$.*

 (b) *any other decomposition of H as a direct product of domains and its centre has this form (after a suitable re-ordering of the factors);*

(ii)   *$\text{Th}(\mathbb{G})$ is decidable if and only if $\text{Th}(\mathbb{G}_i)$ is decidable for every $i = 1, \ldots, n$.*

Let $G = G_1 \times \cdots \times G_k$ be a direct product of groups G_i. A subgroup H of G is called a *subdirect product* of groups G_i if $\pi_i(H) = G_i$ for every $i = 1, \dots, k$, where $\pi_i \colon G \to G_i$ is the canonical projection. An embedding

$$(5.2) \qquad\qquad \lambda \colon H \hookrightarrow G_1 \times \cdots \times G_k$$

is called a *subdirect decomposition* of H if $\lambda(H)$ is a subdirect product of the groups G_i. The subdirect decomposition (5.2) is termed *minimal* if $H \cap G_i \neq \{1\}$ for every $i = 1, \dots, k$ (here G_i is viewed as a subgroup of G under the canonical embedding).

Theorem B ([20]) *Let H be a minimal subdirect product of domains. Then the elementary theory of each component of H is interpretable in the group H.*

Corollary B ([20]) *Let H be a minimal subdirect product of k domains and*

$$H \hookrightarrow G_1 \times \cdots \times G_k$$

be its minimal component decomposition. Then the following hold:

(i)   *if $\mathrm{Th}(H)$ is decidable, then $\mathrm{Th}(G_i)$ is decidable for every $i = 1, \dots, k$;*
(ii) *if $\mathrm{Th}(H)$ is λ-stable, then $\mathrm{Th}(G_i)$ is λ-stable for every $i = 1, \dots, k$.*

Theorem 5.1 *Let $\mathbb{G}$ be a directly indecomposable partially commutative group and Y an algebraic set over $\mathbb{G}$. Then the following conditions hold:*

(i)   *the coordinate group $\Gamma(Y_i)$ of each irreducible component Y_i of Y is interpretable in the group $\Gamma(Y)$;*
(ii) *the elementary theory $\mathrm{Th}(\Gamma(Y_i))$ of each irreducible component Y_i of Y is interpretable in the group $\Gamma(Y)$.*

Proof Partially commutative groups are linear (see [17, 18]), thus equationally Noetherian (see [1]). We can therefore decompose Y as a finite union of irreducible algebraic sets, $Y = Y_1 \cup \cdots \cup Y_k$ (see [1, Corollary 12]). By [1, Proposition 12] the coordinate group $\Gamma(Y)$ is a minimal subdirect product of the coordinate groups $\Gamma(Y_1), \dots, \Gamma(Y_k)$. Every group $\Gamma(Y_i)$, being a coordinate group of an irreducible algebraic set over a domain, is again a domain by [1, Theorem D2]. Now (i), (ii) follow from Theorem B. ∎

Corollary 5.2 *Let $\mathbb{G}$ be a directly indecomposable partially commutative group.*

(i)   *If $Y = Y_1 \cup \cdots \cup Y_k$ is an algebraic set over $\mathbb{G}$, where $Y_1, \dots, Y_k$ are the irreducible components of Y, then the elementary theory of $\Gamma(Y)$ is decidable if and only if the elementary theory of $\Gamma(Y_i)$ is decidable for all $i = 1, \dots, k$.*
(ii) *If $Y = Y_1 \cup \cdots \cup Y_k$ and $Z = Z_1 \cup \cdots \cup Z_l$ are two irreducible algebraic sets, where $Y_1, \dots, Y_k$ and $Z_1, \dots, Z_l$ are the irreducible components of Y and Z, respectively, then $\Gamma(Y)$ is elementary equivalent to $\Gamma(Z)$ if and only if $k = l$ and, after a certain re-enumeration, $\Gamma(Y_i)$ is elementary equivalent to $\Gamma(Z_i)$ for all $i = 1, \dots, k$.*

6 Normal Forms of First-Order Formulas

6.1 Conjunctions of Positive Formulas

Let $L_{a,b}$ be the language of groups enriched by two constants a and b, and let $\mathcal{S}$ be the class of all groups G satisfying the following universal sentences.

(I) $\forall x\,(([x,a] = 1 \wedge [x,b] = 1) \rightarrow x = 1)$;
(II) $\forall x\forall y\forall z(x^2y^2z^2 = 1 \rightarrow [x,y] = 1 \ \wedge \ [x,z] = 1 \ \wedge \ [y,z] = 1)$;
(III) $\forall x\forall y\,\big(x^2 = y^2 \rightarrow x = y\big)$;
(IV) $\forall x\,\big([x^2,a] = 1 \rightarrow [x,a] = 1\big)$.

Let GROUPS be a set of axioms for group theory. Denote by $A_{\mathcal{S}}$ the union of axioms (I),(II),(III), (IV), and GROUPS. Notice that axiom (II) is equivalent modulo GROUPS to the following quasi-identity:

$$\forall x\forall y\forall z(x^2y^2z^2 = 1 \rightarrow [x,y] = 1).$$

It follows that all axioms in $A_{\mathcal{S}}$ are quasi-identities.

Lemma 6.1 *The class $\mathcal{S}$ contains all partially commutative groups with trivial centres.*

Proof We first prove that in any partially commutative group $\mathbb{G}$ with trivial centre there exist two elements a and b such that $C(a) \cap C(b) = 1$. Indeed, let $\mathbb{G} = \mathbb{G}_1 \times \cdots \times \mathbb{G}_k$ be the decomposition of $\mathbb{G}$ in the form (2.1). Since $Z(\mathbb{G}) = 1$, each $\mathbb{G}_i$, $i = 1, \ldots, k$ is a non-abelian directly indecomposable partially commutative group. For each i choose a pair of block elements $a_i, b_i \in \mathbb{G}_i$ such that $C_{\mathbb{G}_i}(a_i) \cap C_{\mathbb{G}_i}(b_i) = 1$. By Theorem 2.3, it follows that

$$C_{\mathbb{G}}(a_1 \ldots a_k) = \langle \sqrt{a_1} \rangle \times \cdots \times \langle \sqrt{a_k} \rangle \quad \text{and} \quad C_{\mathbb{G}}(b_1 \ldots b_k) = \langle \sqrt{b_1} \rangle \times \cdots \times \langle \sqrt{b_k} \rangle$$

and so $C_{\mathbb{G}}(a_1 \cdots a_k) \cap C_{\mathbb{G}}(b_1 \cdots b_k) = 1$. This proves that $\mathbb{G}$ satisfies Axiom (I).

In [5] Crisp and Wiest proved the following theorem.

Let G be a partially commutative group. Then the equation $x^2 y^2 z^2 = 1$ has only commutative solutions.

So $\mathbb{G}$ satisfies Axiom (II).

By [11], partially commutative groups have least roots, and thus $\mathbb{G}$ satisfies Axiom (III).

By Corollary 2.4, $\mathbb{G}$ satisfies Axiom (IV). ∎

Lemma 6.2 *Let $G \in \mathcal{S}$. Then the equation*

(6.1) $$x^2 a x^2 a^{-1}(ybyb^{-1})^{-2} = 1$$

has only the trivial solution $x = 1$ and $y = 1$ in G.

Proof Let x, y be a solution of equation (6.1) in G. Then we can rewrite (6.1) as follows:

(6.2) $$(x^2 a)^2 a^{-2} = ((yb)^2 b^{-2})^2.$$

Since G satisfies (II), from (6.2) we deduce that $[x^2 a, a^{-1}] = 1$, hence $[x^2, a^{-1}] = 1$. Since G satisfies (IV), it follows that $[x, a] = 1$. Now, we can rewrite 6.2 in the form $(x^2)^2 = ((yb)^2 b^{-2})^2$, and then, since G satisfies (III) we get

(6.3) $$x^2 = (yb)^2 b^{-2}.$$

Again, since G satisfies (II) it follows that $[x, b] = 1$ and $[y, b] = 1$. This implies that $[x, a] = 1$ and $[x, b] = 1$. Therefore, applying (I), we get $x = 1$. In this event, (6.3) reduces to $y^2 = 1$, so $y = 1$, as desired. ∎

Corollary 6.3 *For any finite system of equations $S_1(X) = 1, \ldots, S_k(X) = 1$ one can effectively find a single equation $S(X) = 1$ such that given a group $G \in \mathcal{S}$, the following holds:*

$$V_G(S_1, \ldots, S_n) = V_G(S).$$

Proof By induction it suffices to prove the result for $k = 2$. In this case, by the lemma above, $S_1(X)^2 a S_1(X)^2 a^{-1} (S_2(X) b S_2(X) b^{-1})^{-2} = 1$ can be chosen as the equation $S(X) = 1$. ∎

Corollary 6.4 *For any finite system of atomic formulas $S_1(X) = 1, \ldots, S_k(X) = 1$ in $L_{a,b}$, one can effectively find an atomic formula $S(X) = 1$ in $L_{a,b}$ such that*

$$(\bigwedge_{i=1}^{k} S_i(X) = 1) \quad \text{is } A_{\mathcal{S}}\text{-equivalent to} \quad S(X) = 1,$$

written

$$(\bigwedge_{i=1}^{k} S_i(X) = 1) \sim_{A_{\mathcal{S}}} S(X) = 1.$$

6.2 Disjunctions of Positive Formulas

Our next aim is to be able to rewrite finite disjunctions of equations into conjunctions of equations.

Let $\mathcal{T}_D$ be the elementary theory (in the language $L_{a,b}$ of groups enriched by two constants) of non-abelian directly indecomposable partially commutative groups whose centraliser dimension is lower than a fixed number D, *i.e.*, the set of all first order sentences in the language of groups enriched by two constants a and b which are true in all non-abelian directly indecomposable partially commutative groups of centraliser dimension lower than D, together with the following two formulas.

- The intersection of centralisers of a and b is trivial:

$$\forall x([x, a] = 1 \wedge [x, b] = 1) \rightarrow x = 1$$

- The centralisers of a and b are cyclic. An interested reader may verify that this condition can indeed be written using the first order language.

Remark 6.5 Note that any model of $\mathcal{T}_D$ lies in $\mathcal{S}$.

In Section 7, we consider models in the language $L_{\mathbb{G}}$, that is, the language of $\mathbb{G}$-groups enriched by all constants from $\mathbb{G}$ (where $\mathbb{G}$ is a directly indecomposable, non-abelian partially commutative group). We denote the elementary theory in the language $L_{\mathbb{G}}$ of directly indecomposable partially commutative $\mathbb{G}$-groups whose centraliser dimension is lower than a fixed number D by the same symbol $\mathcal{T}_D$. The results of this and the next sections hold for both definitions of $\mathcal{T}_D$.

Proposition 6.6 *Let G be a model of $\mathcal{T}_D$. Let $a, b \in G$ be elements with cyclic centralisers and such that $C(a) \cap C(b) = 1$. Then for any solution $x, y \in G$ of the system*

$$[x, y] = 1, \quad [x, y^{(a^{3\,\mathrm{cdim}(G)+4})}] = 1, \quad [x, y^{(b^{3\,\mathrm{cdim}(G)+4})}] = 1,$$

either $x = 1$ or $y = 1$.

Proof By Corollary 4.18, any directly indecomposable partially commutative group satisfies the following sentence in $L_{a,b}$:

$$\forall x \forall y \big([x, y] = 1 \wedge [x, y^{(a^{3\,\mathrm{cdim}(\mathbb{G})+4})}] = 1 \wedge [x, y^{(b^{3\,\mathrm{cdim}(\mathbb{G})+4})}] = 1 \big) \rightarrow (x = 1 \vee y = 1).$$

Since the class of all groups that have centraliser dimension D is universally axiomatisable (see [12]), any model G of the theory $\mathcal{T}_D$ satisfies the above sentence and the statement follows. ∎

Combining Proposition 6.6 and Lemmas 6.1 and 6.2 yields an algorithm to encode an arbitrary finite disjunction of equations into a single equation.

Corollary 6.7 *For any finite set of equations $S_1(X) = 1, \ldots, S_k(X) = 1$ one can effectively find a single equation $S(X) = 1$ such that given any model G of $\mathcal{T}_D$, the following holds: $V_G(S_1) \cup \cdots \cup V_G(S_k) = V_G(S)$.*

Corollary 6.8 *For any finite set of atomic formulas $S_1(X) = 1, \ldots, S_k(X) = 1$ one can effectively find a single atomic formula $S(X) = 1$ such that*

$$\big(\bigvee_{i=1}^{k} S_i(X) = 1 \big) \sim_{\mathcal{T}_D} S(X) = 1.$$

Corollary 6.9 *Every positive quantifier-free formula $\Phi(X)$ is equivalent modulo $\mathcal{T}_D$ to a single equation $S(X) = 1$.*

6.3 Conjunctions and Disjunctions of Inequations

The next result shows that one can effectively encode finite conjunctions and finite disjunctions of *inequations* (negations of atomic formulas) into a single inequation modulo $\mathcal{T}_D$.

Lemma 6.10 *For any finite set of inequations $S_1(X) \neq 1, \ldots, S_k(X) \neq 1$, one can effectively find an inequation $R(X) \neq 1$ and an inequation $T(X) \neq 1$ such that*

$$\big(\bigwedge_{i=1}^{k} S_i(X) \neq 1 \big) \sim_{\mathcal{T}_D} R(X) \neq 1 \quad and \quad \big(\bigvee_{i=1}^{k} S_i(X) \neq 1 \big) \sim_{\mathcal{T}_D} T(X) \neq 1.$$

Proof By Corollary 6.8 there exists an equation $R(X) = 1$ such that

$$\bigvee_{i=1}^{k} (S_i(X) = 1) \sim_{\mathfrak{T}_D} R(X) = 1.$$

Hence

$$\Big(\bigwedge_{i=1}^{k} S_i(X) \neq 1\Big) \sim_{\mathfrak{T}_D} \neg\Big(\bigvee_{i=1}^{k} S_i(X) = 1\Big) \sim_{\mathfrak{T}_D} \neg(R(X) = 1) \sim_{\mathfrak{T}_D} R(X) \neq 1.$$

This proves the first part of the result. Similarly, by Corollary 6.4 there exists an equation $T(X) = 1$ such that $(\bigwedge_{i=1}^{k} S_i(X) = 1) \sim_{\mathfrak{T}_D} T(X) = 1$. Hence

$$\Big(\bigvee_{i=1}^{k} S_i(X) \neq 1\Big) \sim_{\mathfrak{T}_D} \neg\Big(\bigwedge_{i=1}^{k} S_i(X) = 1\Big) \sim_{\mathfrak{T}_D} \neg(T(X) = 1) \sim_{\mathfrak{T}_D} T(X) \neq 1. \quad \blacksquare$$

Corollary 6.11 *For every quantifier-free formula $\Phi(X)$, one can effectively find a formula $\Psi(X) = \bigvee_{i=1}^{n}(S_i(X) = 1 \wedge T_i(X) \neq 1)$ which is equivalent to $\Phi(X)$ modulo $\mathfrak{T}_D$. In particular, if G is a model of $\mathfrak{T}_D$, then every quantifier-free formula $\Phi(X)$ is equivalent over G to a formula $\Psi(X)$ as above.*

7 Positive Theory of Partially Commutative Groups

In this section we present a procedure of quantifier elimination for positive formulas over partially commutative groups (an analog of Merzlyakov's theorem for free groups). Our approach to the positive theory of partially commutative groups is based on the proof of Merzlyakov's theorem given in [19].

7.1 Generalised Equations

Let $A = \{a_1, \ldots, a_m\}$ be a set of constants and $X = \{x_1, \ldots, x_n\}$ be a set of variables. Set $\mathbb{G} = G(A)$ to be a partially commutative group generated by A and $\mathbb{G}[X] = \mathbb{G} * F(X)$.

Definition 7.1 A combinatorial generalised equation Ω (with constants from $A^{\pm 1}$) consists of the following objects:

(i) A finite set of *bases* $BS = BS(\Omega)$. Every base is either a constant base or a variable base. Each constant base is associated with exactly one letter from $A^{\pm 1}$. The set of variable bases $\mathcal{M}$ consists of $2n$ elements $\mathcal{M} = \{\mu_1, \ldots, \mu_{2n}\}$. The set $\mathcal{M}$ comes equipped with two functions: a function $\varepsilon\colon \mathcal{M} \to \{1, -1\}$ and an involution $\Delta\colon \mathcal{M} \to \mathcal{M}$ (*i.e.*, Δ is a bijection such that Δ^2 is an identity on $\mathcal{M}$). Bases μ and $\Delta(\mu)$ (or $\bar{\mu}$) are called *dual bases*. We denote variable bases by $\mu, \lambda, \ldots$.

(ii) A set of *boundaries* $BD = BD(\Omega)$, where BD is a finite initial segment of the set of positive integers $BD = \{1, 2, \ldots, \rho + 1\}$. We use letters $i, j, \ldots$ for boundaries.

(iii) Two functions $\alpha \colon BS \to BD$ and $\beta \colon BS \to BD$. We call $\alpha(\mu)$ and $\beta(\mu)$ the initial and terminal boundaries of the base μ (or endpoints of μ). These functions satisfy the following conditions: $\alpha(b) < \beta(b)$ for every base $b \in BS$; if b is a constant base, then $\beta(b) = \alpha(b) + 1$.

(iv) A subset C of $BD(\Omega) \times BD(\Omega)$.

To a combinatorial generalised equation Ω one can associate a system of equations in *variables* $h_1, \ldots, h_\rho$ over $G(A)$ (variables h_i are sometimes called *items*). This system is called a *generalised equation*, and, abusing the notation, we denote it by the same symbol Ω. The generalised equation Ω consists of the following three types of equations.

(i) Each pair of dual variable bases $(\lambda, \Delta(\lambda))$ provides an equation over a partially commutative group $\mathbb{G}$

$$[h_{\alpha(\lambda)} h_{\alpha(\lambda)+1} \cdots h_{\beta(\lambda)-1}]^{\varepsilon(\lambda)} = [h_{\alpha(\Delta(\lambda))} h_{\alpha(\Delta(\lambda))+1} \cdots h_{\beta(\Delta(\lambda))-1}]^{\varepsilon(\Delta(\lambda))}.$$

These equations are called *basic equations*. In the case when $\beta(\lambda) = \alpha(\lambda) + 1$ and $\beta(\Delta(\lambda)) = \alpha(\Delta(\lambda)) + 1$, *i.e.*, the corresponding basic equation takes the form $[h_{\alpha(\lambda)}]^{\varepsilon(\lambda)} = [h_{\alpha(\Delta(\lambda))}]^{\varepsilon(\Delta(\lambda))}$, without loss of generality, we shall assume that the equality above is graphical.

(ii) For each constant base b we write down a *coefficient equation* $h_{\alpha(b)} = a$, where $a \in A^{\pm 1}$ is the constant associated with b.

(iii) For every element $c = (i, j) \in C$ we write the following equation: $[h_i, h_j] = 1$.

Remark 7.2 We assume that every generalised equation comes associated with a combinatorial one.

Let $\mathbb{G} = G(A)$. Then the monoid given by the presentation

$$\mathbb{T} = \mathbb{T}(A^{\pm 1}) = \langle A \cup A^{-1} \mid [a_i^{\pm 1}, a_j^{\pm 1}] = 1 \rangle, \text{ where } \left\{ \left[a_i^{\pm 1}, a_j^{\pm 1} \right] = 1 \text{ in } \mathbb{G} \right\}$$

is called *partially commutative monoid associated with* $\mathbb{G}$. Partially commutative monoids are also known as *trace* monoids and are extensively studied; see [10] and the references therein.

Definition 7.3 Let $\Omega(h) = \{L_1(h) = R_1(h), \ldots, L_s(h) = R_s(h)\}$ be a generalised equation in variables $h = (h_1, \ldots, h_\rho)$ with constants from $A^{\pm 1}$. A sequence of reduced nonempty words $U = (U_1(A), \ldots, U_\rho(A))$ in the alphabet $A^{\pm 1}$ is a *solution* of Ω if:

- all words $L_i(U), R_i(U)$ are geodesic (treated as elements of $\mathbb{G}$);
- $L_i(U) = R_i(U), i = 1, \ldots s$ in the partially commutative monoid $\mathbb{T}(A^{\pm 1})$.

The notation (Ω, U) means that U is a solution of the generalised equation Ω.

Remark 7.4 Notice that a solution U of a generalised equation Ω can be viewed as a solution of Ω in the partially commutative monoid $\mathbb{T}(A^{\pm 1})$ (*i.e.*, $L_i(U) = R_i(U)$ modulo commutation) which satisfies an additional condition: $U \in \mathbb{T}(A^{\pm 1})^\rho$ and U is a tuple of geodesic words in $\mathbb{G}$.

Obviously, each solution U of Ω gives rise to a solution of Ω in the partially commutative group $G(A)$. The converse does not hold in general, *i.e.*, it might happen that U is a solution of Ω in $G(A)$ but not in $\mathbb{T}(A^{\pm 1})$, *i.e.*, some equalities $L_i(U) = R_i(U)$ hold only after a reduction in $\mathbb{G}$. We introduce the following notation which will allow us to distinguish in which structure ($\mathbb{T}(A^{\pm 1})$ or $G(A)$) we are resolving Ω.

If $S = \{L_1(h) = R_1(h), \ldots, L_s(h) = R_s(h)\}$ is an arbitrary system of equations with constants from $A^{\pm 1}$, then by S^* we denote the system of equations

$$S^* = \{L_1(h)R_1(h)^{-1} = 1, \ldots, L_s(h)R_s(h)^{-1} = 1\}$$

over the group $G(A)$.

7.2 Reduction to Generalised Equations

Similarly to the case of free groups, we now show how for a given finite system of equations $S(X, A) = 1$ over a partially commutative group $\mathbb{G}$ one can associate a finite collection of generalised equations $\mathcal{GE}(S)$ with constants from $A^{\pm 1}$. The collection $\mathcal{GE}(S)$ to some extent describes all solutions of the system $S(X, A) = 1$.

Informally, Lemma 3.3 describes all possible cancellation schemes for the set of all solutions of the system $S(X, A)$ in the following way: the cancellation scheme corresponding to a particular solution can be obtained from the one described in Lemma 3.3 by setting some of the words w_i^j's (and the corresponding bands) to be trivial. Therefore, every partition table (to be defined below) corresponds to one of the cancellation schemes obtained from the general one by setting some of the words w_i^j's to be trivial. Every non-trivial word w_i^j corresponds to a variable z_k and the word w_j^i to the variable z_k^{-1}. If a variable x that occurs in the system $S(X, A) = 1$ is subdivided as a product of some words w_i^j's, *i.e.*, is a word in the w_i^j's, then the word V_{ij} from the definition of a partition table is this word in the corresponding z_k's. If the bands corresponding to the words w_i^j and w_k^l cross, then the corresponding variables z_r and z_s commute in the group Γ. We refer the reader to the construction of a partition table by a solution of the system $S(X, A)$ given in the end of this section to gain an intuition of the definition of a partition table.

Write $\{S(X, A) = 1\} = \{S_1 = 1, \ldots, S_m = 1\}$ in the form

$$r_{11}r_{12} \ldots r_{1l_1} = 1,$$
$$r_{21}r_{22} \ldots r_{2l_2} = 1,$$

(7.1)

$$\vdots$$

$$r_{m1}r_{m2} \ldots r_{ml_m} = 1,$$

where r_{ij} are letters of the alphabet $X^{\pm 1} \cup A^{\pm 1}$.

A pair (a set of geodesic words, a $\mathbb{G}$-partially commutative group), $T = (V, \Gamma)$ of the form:

$$V = \{V_{ij}(z_1, \ldots, z_p)\} \subset \mathbb{G} * F(Z) = \mathbb{G}[Z] \ (1 \leq i \leq m, 1 \leq j \leq l_i), \quad \Gamma = \mathbb{G}(A \cup Z),$$

is called a *partition table* of the system $S(X, A)$ if the following conditions are satisfied:

(i)   Every letter from $Z \cup Z^{-1}$ occurs in the words V_{ij}, moreover it occurs only once;

(ii)   The equality $V_{i1}V_{i2} \cdots V_{il_i} = 1, 1 \leq i \leq m$, holds in Γ;

(iii) $|V_{ij}| \leq l_i - 1$;

(iv)   if $r_{ij} = a \in A^{\pm 1}$, then $|V_{ij}| = 1$.

Since $|V_{ij}| \leq l_i - 1$, then at most $|S(X, A)| = \sum_{i=1}^{m}(l_i - 1)l_i$ different letters z_i can occur in a partition table of $S(X, A) = 1$. Therefore we will always assume that $p \leq |S|$. We call the number $|S(X, A)|$ the *size* of the system S.

Each partition table encodes a particular type of cancellation that happens when one substitutes a particular solution $W(A) \in G(A)$ into $S(X, A) = 1$ and then reduces (in a certain way) the words in $S(W(A), A)$ into the empty word.

Lemma 7.5   *Let $S(X, A) = 1$ be a finite system of equations over $G(A)$. Then*

(i)   *the set* $\mathrm{PT}(S)$ *of all partition tables of $S(X, A) = 1$ is finite, and its cardinality is bounded by a number which depends only on $|S(X, A)|$;*

(ii)   *one can effectively enumerate the set* $\mathrm{PT}(S)$.

Proof   Since the words V_{ij} have bounded length, one can effectively enumerate the finite set of all collections of words $\{V_{ij}\}$ in $\mathbb{G}[Z]$ which satisfy the conditions (i), (iii), (iv) above. Now for each such collection $\{V_{ij}\}$ one can effectively check whether or not the equalities $V_{i1}V_{i2} \cdots V_{il_i} = 1, 1 \leq i \leq m$ hold in one of the finitely many (since $|Z| < \infty$) partially commutative groups Γ. This allows one to list effectively all partition tables for $S(X, A) = 1$. ∎

To each partition table $T = (\{V_{ij}\}, \Gamma)$ one can assign a generalised equation Ω_T in the following way (below we use $\doteq$ for graphical equality, *i.e.,* equality in the free monoid). Consider the following word V in $M(A^{\pm 1} \cup Z^{\pm 1})$:

$$V \doteq V_{11}V_{12} \cdots V_{1l_1} \cdots V_{m1}V_{m2} \cdots V_{ml_m} = y_1 \cdots y_\rho,$$

where $y_i \in A^{\pm 1} \cup Z^{\pm 1}$ and $\rho = l(V)$ is the length of V. Then the generalised equation $\Omega_T = \Omega_T(h)$ has $\rho + 1$ boundaries and ρ variables $h_1, \ldots, h_\rho$ which are denoted by $h = (h_1, \ldots, h_\rho)$.

Now we define bases of Ω_T and the functions $\alpha, \beta, \varepsilon$.

Let $z \in Z$. For the (only) pair of occurrences of z in V

$$y_i = z^{\epsilon_i}, \quad y_j = z^{\epsilon_j} \quad (\epsilon_i, \epsilon_j \in \{1, -1\})$$

we introduce a pair of dual variable bases $\mu_{z,i}, \mu_{z,j}$ such that $\Delta(\mu_{z,i}) = \mu_{z,j}$. Put

$$\alpha(\mu_{z,i}) = i, \quad \beta(\mu_{z,i}) = i + 1, \quad \varepsilon(\mu_{z,i}) = \epsilon_i,$$

$$\alpha(\mu_{z,j}) = j, \quad \beta(\mu_{z,j}) = j + 1, \quad \varepsilon(\mu_{z,j}) = \epsilon_j.$$

The basic equation that corresponds to this pair of dual bases is $h_i^{\epsilon_i} \doteq h_j^{\epsilon_j}$.

Let $x \in X$. For any two distinct occurrences of x in $S(X, A) = 1$

$$r_{i,j} = x^{\epsilon_{ij}}, \quad r_{s,t} = x^{\epsilon_{st}} \quad (\epsilon_{ij}, \epsilon_{st} \in \{1, -1\}),$$

so that (i, j) precedes (s, t) in left-lexicographical order, we introduce a pair of dual bases $\mu_{x,i,j,s,t}$ and $\mu_{x,s,t,i,j}$ such that $\Delta(\mu_{x,i,j,s,t}) = \mu_{x,s,t,i,j}$. Let V_{ij} occur in the word V as a subword $V_{ij} = y_{c_1} \cdots y_{d_1}$, and $V_{st} = y_{c_2} \cdots y_{d_2}$. Then we put

$$\alpha(\mu_{x,i,j,s,t}) = c_1, \quad \beta(\mu_{x,i,j,s,t}) = d_1 + 1, \quad \varepsilon(\mu_{x,i,j,s,t}) = \epsilon_{ij},$$
$$\alpha(\mu_{x,s,t,i,j}) = c_2, \quad \beta(\mu_{x,s,t,i,j}) = d_2 + 1, \quad \varepsilon(\mu_{x,s,t,i,j}) = \epsilon_{st}.$$

The basic equation which corresponds to these dual bases can be written in the form

$$[h_{\alpha(\mu_{x,i,j,s,t})} \cdots h_{\beta(\mu_{x,i,j,s,t})-1}]^{\epsilon_{ij}} =_{\mathbb{T}} [h_{\alpha(\mu_{x,s,t,i,j})} \cdots h_{\beta(\mu_{x,s,t,i,j})-1}]^{\epsilon_{st}}.$$

Let $r_{ij} = a \in A^{\pm 1}$. In this case we introduce a constant base μ_{ij} with the label a. If V_{ij} occurs in V as $V_{ij} = y_c$, then we put $\alpha(\mu_{ij}) = c$, $\beta(\mu_{ij}) = c + 1$. The corresponding coefficient equation is written as $h_c = a$.

For any two distinct occurrences $z_1, z_2 \in V$ such that $[z_1, z_2] = 1$ in Γ:

$$y_i = z_1{}^{\epsilon}, \quad y_j = z_2{}^{\epsilon_j}, \quad (\epsilon_i, \epsilon_j \in \{1, -1\})$$

we set $(i, j) \in C$. The corresponding equation is $[h_i, h_j] = 1$. This defines the generalised equation Ω_T. Put $\mathcal{GE}(S) = \{\Omega_T \mid T \text{ is a partition table for } S(X, A) = 1\}$. Then $\mathcal{GE}(S)$ is a finite collection of generalised equations which can be effectively constructed for a given $S(X, A) = 1$.

By $\mathbb{G}_{R(\Omega)}$ we denote the coordinate group of Ω^*, $\mathbb{G}_{R(\Omega)} = \mathbb{G}[h]/R(\Omega^*)$ (recall that by Ω^* we denote the system of equation over the group $G(A)$). Now we explain relations between the coordinate groups of $S(X, A) = 1$ and Ω_T^*.

For a letter x in X we choose an arbitrary occurrence of x in $S(X, A) = 1$ as $r_{ij} = x^{\epsilon_{ij}}$. Let $\mu = \mu_{x,i,j,s,t}$ be a base that corresponds to this occurrence of x. Then V_{ij} occurs in V as the subword $V_{ij} = y_{\alpha(\mu)} \cdots y_{\beta(\mu)-1}$. Notice that the word V_{ij} does not depend on the choice of the base $\mu_{x,i,j,s,t}$ corresponding to the occurrence r_{ij}.

Define a word $P_x(h) \in \mathbb{G}[h]$ (where $h = \{h_1, \ldots, h_\rho\}$) as follows $P_x(h, A) = (h_{\alpha(\mu)} \cdots h_{\beta(\mu)-1})^{\epsilon_{ij}}$, and put $P(h) = (P_{x_1}, \ldots, P_{x_n})$.

The tuple of words $P(h)$ depends only on the choice of occurrences of letters from X in V. It follows from the construction above that the map $X \to \mathbb{G}[h]$ defined by $x \to P_x(h, A)$ gives rise to a $\mathbb{G}$-homomorphism $\pi \colon \mathbb{G}_{R(S)} \to \mathbb{G}_{R(\Omega_T)}$. Indeed, if $f(X) \in R(S)$, then $\pi(f(X)) = f(P(h))$. Then given a solution of Ω_T, it follows from condition (ii) of the definition of partition table that $f(P(h)) = 1$, thus $R(f(S)) \subseteq R(\Omega_T^*)$ and π is a homomorphism.

Observe that the image $\pi(x)$ in $\mathbb{G}_{R(\Omega_T)}$ does not depend on a particular choice of the occurrence of x in $S(X, A)$ (the basic equations of Ω_T make these images equal). Hence π depends only on Ω_T.

To relate solutions of $S(X, A) = 1$ to solutions of generalised equations from $\mathcal{GE}(S)$ we need the technique developed in Section 3.2.

Let $W(A)$ be a solution of $S(X, A) = 1$ in $G(A)$. If in the system (7.1) we make the substitution $\sigma \colon X \to W(A)$, then $(r_{i1} r_{i2} \cdots r_{il_i})^{\sigma} = r_{i1}^{\sigma} r_{i2}^{\sigma} \cdots r_{il_i}^{\sigma} = 1$ in $G(A)$ for every $i = 1, \ldots, m$.

Since every product $R_i = r_{i1}^\sigma r_{i2}^\sigma \ldots r_{il_i}^\sigma$ is trivial, we can choose a van Kampen diagram $\mathcal{D}_{R_i}$ for R_i. Denote by $\tilde{z}_{i,1}, \ldots, \tilde{z}_{i,p_i}$ the subwords w_j^k, $1 \leq j < k \leq l_i$ of r_{ij}^σ, where here w_j^k are defined as in Lemma 3.3. Since, by Lemma 3.3 $w_j^k = w_k^{j-1}$, the word r_{ij} can be written as a word in $\tilde{z}_{i,1}, \ldots, \tilde{z}_{i,p_i}$, $r_{ij}^\sigma = V_{ij}(\tilde{z}_{i,1}, \ldots, \tilde{z}_{i,p_i})$ for some freely reduced words $V_{ij}(Z_i)$ in variables $Z_i = \{z_{i,1}, \ldots, z_{i,p_i}\}$. Observe that if $r_{ij} = a \in A^{\pm 1}$, then $r_{ij}^\sigma = a$ and we have $|V_{ij}| = 1$. By Lemma 3.3, r_{ij}^σ is a product of at most $l_i - 1$ words w_j^k, we have $|V_{ij}| \leq l_i - 1$. Denote by $Z = \bigcup_{i=1}^m Z_i = \{z_1, \ldots, z_p\}$. Take a partially commutative group $\Gamma = G(A \cup Z)$ whose underlying commutation graph is defined as follows:

- two elements a_i, a_j in $A^{\pm 1}$ commute whenever they commute in $\mathbb{G}$;
- an element $a \in A^{\pm 1}$ commutes with z_i if and only if a commutes with the word w_j^k corresponding to z_i;
- two elements $z_i, z_j \in Z$ commute whenever the corresponding words w_j^k do.

In the above notation, the set $T = (\{V_{ij}\}, \Gamma)$ is a partition table for $S(X, A) = 1$. We define $U(A) = (\tilde{z}_1, \ldots, \tilde{z}_p)$ to be the solution of the generalised equation Ω_T induced by $W(A)$. From the construction of the map $P(h)$ we deduce that $W(A) = P(U(A))$.
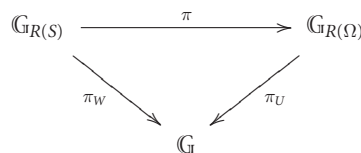
The converse is also true: if $U(A)$ is an arbitrary solution of the generalised equation Ω_T, then $P(U(A))$ is a solution of $S(X, A) = 1$.

We summarize the discussion above in the following lemma.

Lemma 7.6 *For a given system of equations $S(X, A) = 1$ over $\mathbb{G}$, one can effectively construct a finite set $\mathcal{GE}(S) = \{\Omega_T \mid T$ is a partition table for $S(X, A) = 1\}$ of generalised equations such that*

(i)   *if the set $\mathcal{GE}(S)$ is empty, then $S(X, A) = 1$ has no solutions in $\mathbb{G}$;*

(ii) *for each $\Omega(h) \in \mathcal{GE}(S)$ and for each $x \in X$ one can effectively find a word $P_x(h, A) \in \mathbb{G}[h]$ of length at most $|h|$ such that the map $x \mapsto P_x(h, A)$ ($x \in X$) gives rise to a $\mathbb{G}$-homomorphism $\pi_\Omega \colon \mathbb{G}_{R(S)} \to \mathbb{G}_{R(\Omega)}$;*

(iii) *for any solution $W(A) \in \mathbb{G}^n$ of the system $S(X, A) = 1$ there exists $\Omega(h) \in \mathcal{GE}(S)$ and a solution $U(A)$ of $\Omega(h)$ such that $W(A) = P(U(A))$, where $P(h) = (P_{x_1}, \ldots, P_{x_n})$, and this equality holds in the partially commutative monoid $\mathbb{T}(A^{\pm 1})$;*

Corollary 7.7 *In the notation of Lemma 7.6 for any solution $W(A) \in \mathbb{G}^n = G(A)^n$ of the system $S(X, A) = 1$ there exists $\Omega(h) \in \mathcal{GE}(S)$ and a solution $U(A)$ of $\Omega(h)$ such that the following diagram commutes*

$$
\begin{array}{ccc}
\mathbb{G}_{R(S)} & \xrightarrow{\ \pi\ } & \mathbb{G}_{R(\Omega)} \\
& \searrow{\scriptstyle \pi_W} \quad \swarrow{\scriptstyle \pi_U} & \\
& \mathbb{G} &
\end{array}
$$

7.3 Positive Theory of Partially Commutative Groups and Direct Products of Groups

In this section we first prove a result on elimination of quantifiers for positive sentences over a non-abelian directly indecomposable partially commutative group $\mathbb{G} = G(A)$. This proof is based on the notion of a generalised equation. Combining this result with a theorem of V. Diekert and A. Muscholl on decidability of equations over partially commutative groups (see [9]) we get that the positive theory of free partially commutative groups in the language of group theory L and the language $L_{\mathbb{G}}$ enriched by constants is decidable. Note that V. Diekert and M. Lohrey, using a different method, prove a similar result in [8]. Furthermore, we apply the techniques developed for the proof of quantifier elimination to obtain a result on lifting arbitrary formulas from $\mathbb{G}$ to $\mathbb{G} * F$, where F is a free group of finite rank (see Theorem 7.12). In order to prove that the positive theory of any partially commutative group is decidable, we need to study the positive theory of the direct product of groups. In the appendix of the paper we prove that if $G = H_1 \times \cdots \times H_k$, then the positive theory of G in both languages L and L_G is decidable if the positive theories of $H_1, \ldots, H_k$ are decidable in L and L_{H_i}, correspondingly.

Recall that every positive formula $\Psi(Z)$ in the language L_G is equivalent modulo $\mathcal{T}_D$ to a formula of the type

$$\forall x_1 \exists y_1 \cdots \forall x_k \exists y_k (S(X, Y, Z, A) = 1),$$

where $S(X, Y, Z, A) = 1$ is an equation with constants from $A^{\pm 1}$, $X = (x_1, \ldots, x_k)$, $Y = (y_1, \ldots, y_k)$, $Z = (z_1, \ldots, z_m)$. Indeed, one can insert auxiliary quantifiers to ensure the direct alternation of quantifiers in the prefix. In particular, every positive sentence in L_G is equivalent modulo $\mathcal{T}_D$ to a formula of the type

$$\forall x_1 \exists y_1 \cdots \forall x_k \exists y_k (S(X, Y, A) = 1).$$

***Theorem 7.8* (Elimination of Quantifiers)** *If*

$$\mathbb{G} \models \forall x_1 \exists y_1 \cdots \forall x_k \exists y_k (S(X, Y, A) = 1),$$

then there exist words (with constants from $\mathbb{G}$) $q_1(x_1), \ldots, q_k(x_1, \ldots, x_k) \in \mathbb{G}[X]$, such that $\mathbb{G}[X] \models S(x_1, q_1(x_1), \ldots, x_k, q_k(x_1, \ldots, x_k), A) = 1$, i.e., the equation

$$S(x_1, y_1, \ldots, x_k, y_k, A) = 1$$

(in variables Y) has a solution in the group $\mathbb{G}[X]$.

Proof Let $\mathcal{GE}(S) = \{\Omega_1(Z_1), \ldots, \Omega_r(Z_r)\}$ be generalised equations associated with the equation $S(X, Y, A) = 1$ in Lemma 7.6. Denote by $\rho_i = |Z_i|$ the number of variables in Ω_i.

Since the group $\mathbb{G}$ is directly indecomposable, there exists a path p in the non-commutation graph Δ of $\mathbb{G}$ beginning in a vertex b_1 which goes through every vertex of Δ at least once. Denote by $b_1 \cdots b_n$ the label of the path p. Set

$$b = b_1 b_2 \cdots b_{n-1} b_n b_{n-1} \cdots b_1, \quad a = b_2 b b_2 = b_2 b_1 b_2 \cdots b_{n-1} b_n b_{n-1} \cdots b_2 b_1 b_2,$$

$$g_1 = b^m a^{m_{1,1}} b^m a^{m_{1,2}} b \cdots a^{m_{1,n_1}} b^m,$$

where $0 < m_{1,1} < m_{1,2} < \cdots < m_{1,n_1}$, $\max\{\rho_1, \ldots, \rho_r\}|S(X,A)| < n_1$ (recall that by $|S(X,A)|$ we denote the size of the system S; see Section 7.2) and $m \in \mathbb{N}$ is a constant which depends only on the generalised equation.

For the word g_1 there exists h_1 such that

$$\mathbb{G} \models \forall x_2 \exists y_2 \cdots \forall x_k \exists y_k (S(g_1, h_1, x_2, y_2, \ldots, x_k, y_k) = 1).$$

Suppose now that elements $g_1, h_1, \ldots, g_{i-1}, h_{i-1} \in \mathbb{G}$ are such that

$$\mathbb{G} \models \forall x_i \exists y_i \cdots \forall x_k \exists y_k (S(g_1, h_1, \ldots, g_{i-1}, h_{i-1}, x_i, y_i, \ldots, x_k, y_k) = 1).$$

We define

$$(7.2) \qquad\qquad g_i = b^m a^{m_{i1}} b^m a^{m_{i2}} b^m \cdots a^{m_{in_i}} b^m$$

such that

(i) $0 < m_{i1} < m_{i2} < \cdots < m_{in_i}$;
(ii) $\max\{\rho_1, \ldots, \rho_r\}|S(X,A)| < n_i$;
(iii)   no subword of the type $b^m a^{m_{ij}} b^m$ occurs in any of the words g_l, $l < i$ and in any of the (finitely many) words h_l' such that $h_l' = h_l$ in $\mathbb{T}(A^{\pm 1})$, $l < i$.

Then there exists an element $h_i \in \mathbb{G}$ such that

$$\mathbb{G} \models \forall x_{i+1} \exists y_{i+1} \cdots \forall x_k \exists y_k (S(g_1, h_1, \ldots, g_i, h_i, x_{i+1}, y_{i+1}, \ldots, x_k, y_k) = 1).$$

By induction we have constructed elements $g_1, h_1, \ldots, g_k, h_k \in \mathbb{G}$ such that

$$S(g_1, h_1, \ldots, g_k, h_k) = 1$$

and each g_i has the form (7.2) and satisfies conditions (i)–(iii)

By Lemma 7.6, there exists a generalised equation $\Omega(Z) \in \mathcal{GE}(S)$, words $P_i(Z, A)$, $Q_i(Z, A) \in \mathbb{G}[Z]$ $(i = 1, \ldots, k)$ of length lower than $\rho = |Z|$, and a solution $U = (u_1, \ldots, u_\rho)$ of $\Omega(Z)$ in $\mathbb{G}$ such that the following words are equal in $\mathbb{T}(A^{\pm 1})$:

$$g_i = P_i(U), \quad h_i = Q_i(U) \quad (i = 1, \ldots, k).$$

Notice that from the definition of a and b it follows that no two consecutive letters in a and b, and thus in g_i commute. Therefore, the equality $g_i = P_i(U)$ is graphical, *i.e.*, $g_i = P_i(U)$ in the free monoid.

Since $n_i > \rho|S(X,A)|$ (by condition (ii)) and $P_i(U) = y_1 \cdots y_q$, where $y_i \in U^{\pm 1}$, $q \le \rho$, the graphical equalities

$$(7.3) \qquad g_i = b^m a^{m_{i1}} b^m a^{m_{i2}} b^m \cdots a^{m_{in_i}} b^m = P_i(U) \quad (i = 1, \ldots, k)$$

show that there exists a subword $v_i = b^m a^{m_{il}} b^m$ of g_i such that every occurrence of this subword in (7.3) is an occurrence inside some $u_j^{\pm 1}$. For each i fix such a subword $v_i = b^m a^{m_{il}} b^m$ in g_i. In view of condition (iii), the word v_i does not occur in any of

the words g_j ($j \neq i$), h_s ($s < i$), moreover, in g_i it occurs precisely once. Denote by $j(i)$ the unique index such that v_i occurs inside $u_{j(i)}^{\pm 1}$ in $P_i(U)$ from (7.3) (and v_i occurs in it precisely once).

The argument above shows that the variable $z_{j(i)}$ does not occur in words $P_t(Z, A)$ ($t \neq i$), $Q_s(Z, A)$ ($s < i$). Moreover, in $P_i(Z)$ it occurs precisely once. It follows that the variable $z_{j(i)}$ in the generalised equation $\Omega(Z)$ does not occur in either the coefficient equations or in the basic equations corresponding to the dual bases related to x_t ($t \neq i$), y_s ($s < i$).

We "mark" (or select) the unique occurrence of v_i (as $v_i^{\pm 1}$) in $u_{j(i)}$ $i = 1, \ldots, k$. Now we are going to mark some other occurrences of v_i in the words $u_1, \ldots, u_\rho$ as follows. Suppose that some u_d has a marked occurrence of some v_i. If Ω contains an equation of the type $z_d^\epsilon = z_r^\delta$, then $u_d^\epsilon = u_r^\delta$ graphically. Hence u_r has an occurrence of the subword $v_i^{\pm 1}$ which corresponds to the marked occurrence of $v_i^{\pm 1}$ in u_d. We mark this occurrence of $v_i^{\pm 1}$ in u_r.

Suppose Ω contains an equation of the type $[z_{\alpha_1} \cdots z_{\beta_1 - 1}]^{\epsilon_1} = [z_{\alpha_2} \cdots z_{\beta_2 - 1}]^{\epsilon_2}$ such that z_d occurs in it, say in the left-hand side of the equality. Then

$$[u_{\alpha_1} \cdots u_{\beta_1 - 1}]^{\epsilon_1} = [u_{\alpha_2} \cdots u_{\beta_2 - 1}]^{\epsilon_2}$$

in the partially commutative monoid $\mathbb{T}(A^{\pm 1})$. Since $v_i^{\pm 1}$ is a subword of u_d, a subword $v_{i,1} = b^{m-1} a^{m_{il}} b^{m-1}$ occurs also in the right-hand side of the above equality, say in u_r. Indeed, let $w_1 b^m a^{m_{il}} b^m w_2 = w$ in the monoid $\mathbb{T}(A^{\pm 1})$. Since for any letter ℓ in w_k, $k = 1, 2$ there exists a letter ℓ' in b such that $[\ell, \ell'] \neq 1$ and since, by the definition of a and b, no two consecutive occurrences in $b^{m-1} a^{m_{il}} b^{m-1}$ commute, the statement follows. We mark this occurrence of $v_{i,1}^{\pm 1}$ in u_r and in all the previously marked occurrences of $v_i^{\pm 1} = b v_{i,1} b$. We continue the marking process, but now, instead of v_i we mark the occurrences of $v_{i,1}$. The marking process stops in finitely many steps and all the occurrences of the subword $v_{i,k} = b^{m-k} a^{m_{il}} b^{m-k}$ are marked. For the above argument, it suffices to choose $m > k$, which depends on the generalised equation only.

Now in all words $u_1, \ldots, u_\rho$ we replace every marked occurrence of $v_{i,k} = b^{m-k} a^{m_{il}} b^{m-k}$ with a new word $b^{m-k} a^{m_{il}} x_i b^{m-k}$ from the group $\mathbb{G}[X]$. Denote the resulting words from $\mathbb{G}[X]$ by $\tilde{u}_1, \ldots, \tilde{u}_\rho$. It follows from the description of the marking process that the tuple $\tilde{U} = (\tilde{u}_1, \ldots, \tilde{u}_\rho)$ is a solution of the generalised equation Ω in $\mathbb{G}[X]$. Indeed, by construction, all basic and coefficient equations in Ω hold in the partially commutative monoid if we substitute z_i by $\tilde{u}_i$. Furthermore, since any word that contains the word of the form $b^{m-k} a^{m_{il}} x_i b^{m-k}$ has cyclic centraliser, it follows that $\tilde{U}$ satisfies the same commutation equations as U and thus the commutation equations of Ω. Now, by Lemma 7.6, $X = P(\tilde{U})$, $Y = Q(\tilde{U})$ is a solution of the equation $S(X, A) = 1$ over $\mathbb{G}[X]$ as desired. ∎

Corollary 7.9 *There is an algorithm which for a given positive sentence*

$$\forall x_1 \exists y_1 \cdots \forall x_k \exists y_k (S(X, Y, A) = 1)$$

in $L_\mathbb{G}$ determines whether or not this formula holds in $\mathbb{G}$, and if it does, the algorithm finds words $q_1(x_1), \ldots, q_k(x_1, \ldots, x_k) \in \mathbb{G}[X]$ such that

$$\mathbb{G}[X] \models S(x_1, q_1(x_1), \ldots, x_k, q_k(x_1, \ldots, x_k), A) = 1,$$

i.e., *the positive theory of any directly-indecomposable partially commutative group is decidable.*

Proof The proof follows from Theorem 7.8 and decidability of equations over free partially commutative groups. Indeed, the compatibility problem for a system of equations over a partially commutative group $\mathbb{G}$ reduces to the compatibility of a system of equations S' over a free group with rational constraints C, see [9]. In order to prove the corollary it suffices to check the compatibility of S' over a free group with constraints $C \cup \{y_i \in F[X_i]\}$, where $X_i = \{x_1, \ldots, x_i\}$. ∎

The next result follows directly from Corollary A.2.

Corollary 7.10 *Let $\mathbb{G}$ be an arbitrary partially commutative group. Then the positive theory of $\mathbb{G}$ is decidable.*

Definition 7.11 Let ϕ be a sentence in the language $L_\mathbb{G}$ written in the standard form

$$\phi = \forall x_1 \exists y_1 \cdots \forall x_k \exists y_k \, \phi_0(x_1, y_1, \ldots, x_k, y_k),$$

where ϕ_0 is a quantifier-free formula in $L_\mathbb{G}$. We say that $\mathbb{G}$ *freely lifts* ϕ if there exist words (with constants from $\mathbb{G}$) $q_1(x_1), \ldots, q_k(x_1, \ldots, x_k) \in \mathbb{G}[X]$, such that

$$\mathbb{G}[X] \models \phi_0(x_1, q_1(x_1), \ldots, x_k, q_k(x_1, \ldots, x_k), A) = 1.$$

Theorem 7.12 *Let $\mathbb{G}$ be a non-abelian directly indecomposable partially commutative group. Then $\mathbb{G}$ freely lifts every sentence in $L_\mathbb{G}$ that is true in $\mathbb{G}$.*

Proof Suppose a sentence

(7.4) $\phi = \forall x_1 \exists y_1 \cdots \forall x_k \exists y_k (U(x_1, y_1, \ldots, x_k, y_k) = 1 \wedge V(x_1, y_1, \ldots, x_k, y_k) \neq 1),$

is true in $\mathbb{G}$. We choose $x_1 = g_1, y_1 = h_1, \ldots, x_k = g_k, y_k = h_k$ precisely as in Theorem 7.8. Then the formula

$$U(g_1, h_1, \ldots, g_k, h_k) = 1 \wedge V(g_1, h_1, \ldots, g_k, h_k) \neq 1$$

holds in $\mathbb{G}$. In particular, $U(g_1, h_1, \ldots, g_k, h_k) = 1$ in $\mathbb{G}$. It follows from Corollary 7.9 that there are words $q_1(x_1) \in \mathbb{G}[x_1], \ldots, q_k(x_1, \ldots, x_k) \in \mathbb{G}[x_1, \ldots, x_k]$ such that

$$\mathbb{G}[X] \models U(x_1, q_1(x_1, \ldots, x_k), \ldots, x_k, q_k(x_1, \ldots, x_k)) = 1.$$

Moreover, it follows from the construction that $h_1 = q_1(g_1), \ldots, h_k = q_k(g_1, \ldots, g_k)$. We claim that

$$\mathbb{G}[X] \models V(x_1, q_1(x_1, \ldots, x_k), \ldots, x_k, q_k(x_1, \ldots, x_k)) \neq 1.$$

Indeed, if $V(x_1, q_1(x_1, \ldots, x_k), \ldots, x_k, q_k(x_1, \ldots, x_k)) = 1$ in $\mathbb{G}[X]$, then its image in $\mathbb{G}$ under any specialization $X \to \mathbb{G}$ is also trivial, but this is not the case for the specialization $x_1 \to g_1, \ldots, x_k \to g_k$, which is a contradiction. This proves the theorem for sentences ϕ of the form (7.4). A similar argument works for formulas of the type

$$\phi = \forall x_1 \exists y_1 \cdots \forall x_k \exists y_k \bigvee_{i=1}^{n} (U_i(x_1, y_1, \ldots, x_k, y_k) = 1 \wedge V_i(x_1, y_1, \ldots, x_k, y_k) \neq 1),$$

which is actually the general case, by Corollary 6.11. ∎

A   Appendix: Positive Theory of the Direct Product of Groups

In this section we prove that if $G = H_1 \times \cdots \times H_k$, then the positive theory of G in the language L_G (and in L) is decidable if the positive theories of $H_1, \ldots, H_k$ are decidable. Perhaps this result is known; nevertheless, we were not able to find a reference.

The following theorem is due to Feferman and Vaught [16].

Theorem A.1   *Let $G = H_1 \times \cdots \times H_k$. Then the elementary theory of G in the language L_G is decidable, provided that the elementary theory of H_i is decidable in the language L_{H_i}, $i = 1, \ldots, k$.*

Proof   Without loss of generality we may assume that $G = A \times B$.

We use induction on the complexity of the formula to prove the following statement. Given a formula $\varphi(x_1, \ldots, x_n)$ in the language L_G, one can effectively construct a finite family of formulas $\langle \phi \rangle = \{(\psi_i(y_1, \ldots, y_n), \psi_i'(z_1, \ldots, z_n)) \mid i \in I\}$ such that for all $a_1, \ldots, a_n, b_1, \ldots, b_n$ we have $A \times B \models \varphi((a_1, b_1), \ldots, (a_n, b_n))$ if and only if there exists $i \in I$ such that $A \models \psi_i(a_1, \ldots, a_n)$ and $B \models \psi_i'(b_1, \ldots, b_n)$.

- Let $\varphi = (x_i = x_j)$, set $\langle \varphi \rangle = \{(y_i = y_j, z_i = z_j)\}$.
- Let $\varphi = (x_i = c)$, where $c \in G$, $c = (c_1, c_2)$, set $\langle \varphi \rangle = \{(y_i = c_1, z_i = c_2)\}$.
- Let $\varphi = \varphi_1 \vee \varphi_2$ and set $\langle \varphi \rangle = \langle \varphi_1 \rangle \cup \langle \varphi_2 \rangle$.
- Let $\varphi = \neg \varphi_0$ and set

$$\langle \varphi \rangle = \Big\{ \big(\bigwedge_{j \in J} \neg \psi_j, \bigwedge_{i \in I \setminus J} \neg \psi_i' \big) \ \big| \ J \in \mathcal{P}(I) \Big\},$$

  where $\mathcal{P}(I)$ is the power set of I and $\langle \phi_0 \rangle = \{(\psi_i, \psi_i') \mid i \in I\}$.
- Let $\varphi = \exists x_0 \varphi_0(x_0, x_1, \ldots, x_n)$ and set

$$\langle \varphi \rangle = \{(\exists y_0 \psi_i(y_0, y_1, \ldots, y_n), \exists z_0 \psi_i'(z_0, z_1, \ldots, z_n)) \mid i \in I\},$$

  where $\langle \phi_0 \rangle = \{(\psi_i, \psi_i') \mid i \in I\}$.

∎

Corollary A.2   *Let $G = H_1 \times \cdots \times H_k$. Then the positive theory of G in the language L_G (in the language L) is decidable, provided that the positive theories of $H_1, \ldots, H_k$ are decidable.*

Proof   In the notation of Theorem A.1, we are left to show that if φ is a positive formula in L_G, then for all $i \in I$ the formulas ψ_i and ψ_i' are also positive.

By construction of $\langle \varphi \rangle$ it follows that ψ_i and ψ_i' are positive when $\varphi = (x_i = x_j)$, $\varphi = (x_i = c)$, $\varphi = P(x_1, \ldots, x_n)$, $\varphi = \varphi_1 \vee \varphi_2$, and $\varphi = \exists x_0 \varphi_0(x_0, x_1, \ldots, x_n)$. We are left to consider the two following cases: $\varphi = \varphi_1 \wedge \varphi_2$ and $\varphi = \forall x_0 \varphi_0(x_0, x_1, \ldots, x_n)$.

Let $\varphi = \forall x_0 \varphi_0$. Then φ is equivalent to $\neg(\exists x_0 \neg \varphi_0(x_0, x_1, \dots, x_n))$. Thus,

$$\langle \varphi \rangle = \neg \Big\{ \big(\exists x_0 \bigwedge_{j \in J} \neg \psi_j, \exists x_0 \bigwedge_{i \in I \setminus J} \neg \psi_i' \big) \mid J \in \mathcal{P}(I) \Big\}$$

$$= \Big\{ \big(\bigwedge_{J \in J'} \neg \big(\exists x_0 \bigwedge_{j \in J} \neg \psi_j \big), \bigwedge_{I \in \mathcal{P} \setminus J'} \neg \big(\exists x_0 \bigwedge_{i \in I} \neg \psi_i' \big) \big) \mid J' \in \mathcal{P}(\mathcal{P}(I)) \Big\}$$

$$= \Big\{ \big(\bigwedge_{J \in J'} \forall x_0 \bigvee_{j \in J} \psi_j, \bigwedge_{I \in \mathcal{P} \setminus J'} \forall x_0 \bigvee_{i \in I} \psi_i' \big) \mid J' \in \mathcal{P}(\mathcal{P}(I)) \Big\}.$$

Now let $\varphi = \varphi_1 \wedge \varphi_2$ and $\langle \varphi_l \rangle = \{ (\psi_{l,i}, \psi_{l,i}') \mid i \in I_l \}$, $l = 1, 2$. Then φ is equivalent to $\neg(\neg \varphi_1 \vee \neg \varphi_2)$. Thus,

$$\langle \varphi \rangle = \neg \Big\{ \big\{ \big(\bigwedge_{j \in J} \neg \psi_{1,j}, \bigwedge_{i \in I_1 \setminus J} \neg \psi_{1,i}' \big) \mid J \in \mathcal{P}(I_1) \big\}$$

$$\cup \big\{ \big(\bigwedge_{j \in J} \neg \psi_{2,j}, \bigwedge_{i \in I_2 \setminus J} \neg \psi_{2,i}' \big) \mid J \in \mathcal{P}(I_2) \big\} \Big\}$$

$$= \neg \Big\{ \big(\bigwedge_{j \in J} \neg \psi_{1,j}, \bigwedge_{i \in I_1 \setminus J} \neg \psi_{1,i}' \big) \mid J \in \mathcal{P}(I_1) \Big\}$$

$$\cap \neg \Big\{ \big(\bigwedge_{j \in J} \neg \psi_{2,j}, \bigwedge_{i \in I_2 \setminus J} \neg \psi_{2,i}' \big) \mid J \in \mathcal{P}(I_2) \Big\}$$

$$= \Big\{ \big(\bigwedge_{J \in J'} \neg \big(\bigwedge_{j \in J} \neg \psi_{1,j} \big), \bigwedge_{I \in \mathcal{P}(I_1) \setminus J} \neg \big(\bigwedge_{i \in I_1 \setminus J} \neg \psi_{1,i}' \big) \big) \mid J' \in \mathcal{P}(\mathcal{P}(I_1)) \Big\}$$

$$\cap \Big\{ \big(\bigwedge_{J \in J'} \neg \big(\bigwedge_{j \in J} \neg \psi_{2,j} \big), \bigwedge_{I \in \mathcal{P}(I_2) \setminus J} \neg \big(\bigwedge_{i \in I_2 \setminus J} \neg \psi_{2,i}' \big) \big) \mid J' \in \mathcal{P}(\mathcal{P}(I_2)) \Big\}$$

$$= \Big\{ \big(\bigwedge_{J \in J'} \bigvee_{j \in J} \psi_{1,j}, \bigwedge_{I \in \mathcal{P}(I_1) \setminus J} \bigvee_{i \in I_1 \setminus J} \psi_{1,i}' \big) \mid J' \in \mathcal{P}(\mathcal{P}(I_1)) \Big\}$$

$$\cup \Big\{ \big(\bigwedge_{J \in J'} \bigvee_{j \in J} \psi_{2,j}, \bigwedge_{I \in \mathcal{P}(I_2) \setminus J} \bigvee_{i \in I_2 \setminus J} \psi_{2,i}' \big) \mid J' \in \mathcal{P}(\mathcal{P}(I_2)) \Big\}$$

∎

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