DISTRIBUTION OF CLOSED GEODESICS
WITH A PREASSIGNED HOMOLOGY CLASS
IN A NEGATIVELY CURVED MANIFOLD

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§ 1. Introduction

Let M be a compact Riemannian manifold whose geodesic flow φ_t: $UM \to UM$ on the unit tangent bundle is of Anosov type. In this paper we count the number of φ_t-closed orbits and study the distribution of prime closed geodesics in a given homology class in $H_1(M, Z)$. Here a prime closed geodesic means an (oriented) image of a φ_t-closed orbit by the projection $p \colon UM \to M$.

In the preceding paper [3] with Sunada, we see that there exist infinitely many prime closed geodesics in each homology class. In this context we are led to a natural problem of the distribution of closed geodesics in a given homology class. Concerning this problem, Bowen [4], [5], having introduced a measure associated with closed orbits of an Axiom A flow, showed that closed geodesics (without any restrictions on their homology classes) are uniformly equidistributed. Recently, Parry [8] provides an alternative proof for Bowen's result, based on the zeta functions of flows.

For a loop c in M, let $[c]$ be the homology class represented by c. Given a φ_t-periodic point $x \in UM$, we denote by $\mathfrak{p}(x)$ the closed orbit containing x.

THEOREM 1. *Let M be a compact Riemannian manifold whose geodesic flow φ_t is of Anosov type. For any $\alpha \in H_1(M, Z)$, the set of all φ_t-periodic points x with $[p(\mathfrak{p}(x))] = \alpha$ is dense in UM.*

In particular, prime closed geodesics in α form a dense subset in M.

To get more precise information on the distribution of prime closed

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geodesics, we introduce the following orbital measures on UM. Given a coset $\mathscr{A}$ of a finite index subgroup of $H_1(M, \mathbb{Z})$ and a positive T and ε, we define measures $\tilde{\mu}_{T,\varepsilon,\mathscr{A}}$ and $\tilde{\nu}_{T,\varepsilon,\mathscr{A}}$ by

$$\tilde{\mu}_{T,\varepsilon,\mathscr{A}}(g) = \sum_{\substack{\mathfrak{p}=\mathfrak{p}(x),\,[p(\mathfrak{p})]\in\mathscr{A} \\ T-\varepsilon<\tau(\mathfrak{p})<T+\varepsilon}} \int_0^{\tau(\mathfrak{p})} g(\varphi_t(x))dt/\tau(\mathfrak{p}),$$

$$\tilde{\nu}_{T,\varepsilon,\mathscr{A}}(g) = \sum_{\substack{\mathfrak{p}=\mathfrak{p}(x),\,[p(\mathfrak{p})]\in\mathscr{A} \\ T-\varepsilon<\tau(\mathfrak{p})<T+\varepsilon}} \int_0^{\tau(\mathfrak{p})} g(\varphi_t(x))dt,$$

for each smooth function g on UM, where $\tau(\mathfrak{p})$ denotes the period of $\mathfrak{p}$. Normalize them so that

$$\mu_{T,\varepsilon,\mathscr{A}}(g) = \tilde{\mu}_{T,\varepsilon,\mathscr{A}}(g)/\tilde{\mu}_{T,\varepsilon,\mathscr{A}}(1) \quad \text{and} \quad \nu_{T,\varepsilon,\mathscr{A}}(g) = \tilde{\nu}_{T,\varepsilon,\mathscr{A}}(g)/\tilde{\nu}_{T,\varepsilon,\mathscr{A}}(1).$$

THEOREM 2. *Under the same assumption as in Theorem* 1, *prime closed geodesics with a preassigned homology class are "nearly" equidistributed in the following sense*: *For any cosets $\mathscr{A}, \mathscr{B}$ of an arbitrary finite index subgroup of $H_1(M, \mathbb{Z})$, positive ε and a smooth function g,*

$$\nu_{T,\varepsilon,\mathscr{A}}(g)\longrightarrow\int_{UM} g\,d\mu, \quad \nu_{T,\varepsilon,\mathscr{A}}(g)/\nu_{T,\varepsilon,\mathscr{B}}(g)\longrightarrow 1$$

$$\mu_{T,\varepsilon,\mathscr{A}}(g)\longrightarrow\int_{UM} g\,d\mu, \quad \mu_{T,\varepsilon,\mathscr{A}}(g)/\mu_{T,\varepsilon,\mathscr{B}}(g)\longrightarrow 1$$

as $T\to\infty$, where μ is the φ_t-invariant maximal entropy measure.

In the special case that $H_1(M, \mathbb{Z})$ is finite, we can conclude that prime closed geodesics are equidistributed.

Our approach to Theorem 1 is based on the technique of graphical approximations for flows, which was given in [3]. There is an irreducible oriented finite graph embedded in UM such that each φ_t-closed orbit is approximated by a closed path in the graph (see §2). Briefly speaking, what we do is to compute traces associated with this graph. Edges are weighted so that they have information not only on periods of closed orbits but also on places where they are embedded. Once expressed by matrices the method given in [3] is refined in Section 4, in particular, Theorem 1 is proved. To prove Theorem 2 we apply a number-theoretic argument; we define "weighted" L-functions for φ_t and analyze their analyticity and poles (cf. [2], [8], [9]).

§ 2. Graphical approximation of the geodesic flow

In this section we recall a graphical approximation of the geodesic flow φ_t given in [3], which is a consequence of Bowen's symbolic dynamics [6].

We first sketch some basic materials. An oriented finite graph consists of a finite set of vertices V and a set of edges $E \subset V \times V$. A path c (or more precisely a r-step path) in a graph (V, E) is an element of the form $c = (v_0, \cdots, v_r) \in V \times \cdots \times V$ ($r + 1$ factor), with $(v_i, v_{i+1}) \in E$ for any $0 \leq i \leq r - 1$. Put $o(c) = v_0$, $t(c) = v_r$ and $|c| = r$. We call c closed if $o(c) = t(c)$, and call prime if it is not a non-trivial tower of other closed path. Two r-step closed paths $c = (v_0, \cdots, v_r)$ and $c' = (w_0, \cdots, w_r)$ are said to be equivalent if $v_{i+k} = w_i$, $i \in Z/rZ$ for some k. A cycle $\langle c \rangle$ is the equivalence class containing c. If c is a r-step closed path, there are at most r different closed paths in $\langle c \rangle$. From now we only treat irreducible oriented finite graphs, that means for any two vertices there exists a path joining them.

Given a finite graph (V, E), a subshift of finite type $\Sigma(V, E)$ is a compact 0-dimensional space defined by

$$\{\xi = (\xi_i) \in \prod_Z V \,|\, (\xi_i, \xi_{i+1}) \in E \qquad \text{for any } i \in Z\}\,.$$

The shift operator $\sigma \colon \Sigma(V, E) \to \Sigma(V, E)$ is a continuous map defined by $\sigma(\xi)_i = \sigma(\xi)_{i+1}$. For a strictly positive function l on $\Sigma(V, E)$ we define a suspension $\Sigma(V, E, l)$ by

$$\{(\xi, s) \,|\, \xi \in \Sigma(V, E) \quad \text{and} \quad 0 \leq s \leq l(\xi)\}\,,$$

where $(\xi, l(\xi))$ and $(\sigma\xi, 0)$ are identified. On this space there is a vertical flow $\sigma(l)_t$ given by $\sigma(l)_t(\xi, s) = (\xi, t + s)$ with appropriate identifications.

Let $c = (v_0, \cdots, v_r)$ be a closed path in (V, E). Define $\xi(c) \in \Sigma(V, E)$ by $\xi(c)_j = v_i$, $j \equiv i$ (mod r), and put

$$\mathfrak{p}(c) = \langle \sigma(l)_t(\xi(c), 0) \,|\, 0 \leq t \leq \tau(c) \rangle\,,$$

$$\tau(c) = \sum_{i=0}^{r-1} l(\sigma^i \xi(c))\,.$$

We frequently identify a cycle $\langle c \rangle$ with the $\sigma(l)_t$-orbit cycle $\mathfrak{p}(c)$. Here to avoid any confusion we explain the meaning of orbit cycles. An orbit cycle $\mathfrak{p}$ is a k-th tower of some closed orbit $\mathfrak{p}'$, and the period of $\mathfrak{p}$ is k-times the minimal period of $\mathfrak{p}'$. A closed orbit is also called a prime orbit cycle.

Let us return to our geodesic flow. The technique of graphical approximations is based on a Markov family. Each Markov family V defines an irreducible non-circuit graph (V, E), which is naturally embedded in UM as a CW-complex. The importance of this graph is the following relationship with the flow φ_t: There are strictly positive Lipschitz continuous function l and a continuous surjective finite-to-one map $\rho: \Sigma(V, E, l) \to UM$ such that $\rho \circ \sigma(l)_t = \varphi_t \circ \rho$. We refer to this suspension as a principal suspension. Let ι denote the natural embedding of (V, E) into UM. It is known [3] that if we define a finite-to-one correspondence Φ of the set of cycles in (V, E) onto the set of φ_t-orbit cycles by $c \mapsto \rho(\mathfrak{p}(c))$ then $\iota(c)$ and $\Phi(\langle c \rangle)$ are free homotopic. Therefore we can reduce the problem of estimating the number of φ_t-closed orbits to that of counting prime closed paths in an irreducible finite graph associated with φ_t (see [3]).

Among various features of geodesic flows we pay attention to the reversible property; $\varphi_t(-x) = -\varphi_{-t}(x)$ for any $x \in UM$. One can construct a Markov family and the principal suspension so that they inherit this property (see [1] and [14]).

PROPOSITION 3.   *There exists a Markov family (V, E) for φ_t of arbitrarily small size and with orientation-reversing involution $\kappa: V \to V$ such that for any closed path $c = (v_0, \cdots, v_r)$, $p(\Phi(\langle c \rangle))$ and $p(\Phi(\langle \kappa c \rangle))$ represent the same closed geodesic in M with different orientations, where $\kappa c = (\kappa v_r, \cdots, \kappa v_0)$.*

§ 3.   Twisted Perron-Frobenius theorem

In counting closed paths in a graph, we shall make use of "twisted" aperiodic matrices associated to this graph. We summarize in this section some of results about eigenvalues of "twisted" non-negative matrices. See [3] for proofs.

Let (V, E) be an oriented irreducible finite graph. Regarding (V, E) as a 1-dimensional CW-complex, we denote by $\pi: (\hat{V}, \hat{E}) \to (V, E)$ the universal covering and by $\pi_1(V, E)$ the fundamental group, which is generated by homotopy classes of closed paths in (V, E). Decompose V into primitive parts $V_1, \cdots, V_\nu$ (see [2] for example), put $\hat{V}_1 = \pi^{-1}(V_1)$ and choose a subset $\mathscr{D}$ in $\hat{V}_1$ so that $\hat{V}_1 = \cup_{\gamma \in \pi_1} \gamma \mathscr{D}$ and $\gamma \mathscr{D} \cap \mathscr{D} = \varnothing$ if $\gamma \neq \mathrm{Id}$.

Given a character $\chi: \pi_1(V, E) \to \{z \in C \,|\, |z| = 1\}$, let $\mathscr{G}_\chi$ be the vector space of complex-valued functions h on $\hat{V}_1$ satisfying $h \circ \gamma = \chi(\gamma)h$ for any $\gamma \in \pi_1(V, E)$. If one defines $\delta_{\chi, w}: \hat{V}_1 \to C$ for $w \in \mathscr{D}$ by

$$\delta_{\chi, w}(v) = \begin{cases} \chi(\gamma) & \text{if } v = \gamma w \\ 0 & \text{otherwise}, \end{cases}$$

$\{\delta_{\chi, w}\}_{w \in \mathscr{D}}$ forms a basis of $\mathscr{G}_\chi$.

Let F be a non-negative function on E. We always suppose $F \neq 0$. Define an operator $\mathscr{L}_{F, \chi} \colon \mathscr{G}_\chi \to \mathscr{G}_\chi$ by

$$\mathscr{L}_{F, \chi} h(v) = \sum_c F(\pi(v_0), \pi(v_1)) \cdots F(\pi(v_{\nu-1}), \pi(v_\nu)) h(v_\nu),$$

where $c = (v_0, \cdots, v_\nu)$ runs over all ν-step closed paths in $(\hat{V}, \hat{E})$ with $o(c) = v$. By using the basis $\{\delta_{\chi, w}\}_{w \in \mathscr{D}}$, we represent $\mathscr{L}_{F, \chi}$ by a d-dimensional matrix $L_{F, \chi}$, where $d = \sharp(V_1)$, the cardinality of V_1. These matrices $L_{F, \chi}$ varies smoothly with respect to F and continuously with respect to χ.

When $\chi = \mathbf{1}$, the trivial character, $L_{F, \mathbf{1}}$ is an aperiodic non-negative matrix. For a general χ we can view $L_{F, \chi}$ something like a matrix obtained by "twisting" $L_{F, \mathbf{1}}$ with the character χ. Using Perron-Frobenius theorem we can prove

PROPOSITION 4 ([3]). 1) $L_{F, \mathbf{1}}$ has a simple positive eigenvalue $\lambda(F)$ and an associated positive eigenvector,

2) other eigenvalue μ of $L_{F, \mathbf{1}}$ than $\lambda(F)$ satisfies $|\mu| < \lambda(F)$,

3) $\lambda(F) = \max_{h \in \mathbf{R}^d} \{ \min_{1 \leq j \leq d, h_j \neq 0} (L_{F, \mathbf{1}} h)_j / h_j \mid h \geq 0, \ h \neq 0 \}$,

4) for each character χ, any eigenvalue μ of $L_{F, \chi}$ satisfies $|\mu| \leq \lambda(F)$,

5) $L_{F, \chi}$ has an eigenvalue of the form $\lambda(F) \exp \sqrt{-1} \psi$ if and only if $\chi([\![c]\!]) = \exp(\sqrt{-1} \psi |c| / \nu)$ for any closed path c in (V, E), where $[\![c]\!]$ is the conjugacy class in $\pi_1(V, E)$ represented by c.

Remark. The maximal eigenvalue $\lambda(F)$ does not depend on the choice of primitive part V_1.

§4. Existence of infinitely many closed paths

We shall prove the following proposition which is a key of this paper.

PROPOSITION 5. *Let (V, E) be a irreducible finite graph of a Markov family for the geodesic flow φ_t with associated involution κ. For each $\alpha \in H_1(M, \mathbf{Z})$ and $v \in V$, there exist infinitely many prime closed paths c in (V, E) with $o(c) = v$ and with prime $\Phi(\langle c \rangle)$ satisfying $[p(\Phi(\langle c \rangle))] = \alpha$.*

To prove this we utilize the irreducible graph (V^r, E^r) of r-step paths in (V, E), which is defined by

$$V^r = \{ \boldsymbol{v} = (v_0, \cdots, v_r) \,|\, r\text{-step paths in } (V, E) \} \,,$$
$$E^r = \{ (\boldsymbol{v}, \boldsymbol{w}) \in V^r \times V^r \,|\, v_i = w_{i+1}, \ 0 \le i \le r-1 \} \,.$$

Note that this graph also admits an orientation reversing involution κ: $V^r \to V^r$ given by $\kappa(v_0, \cdots, v_r) = (\kappa(v_r), \cdots, (v_0))$. The natural projection $\omega \colon V^r \to V$ defined by $\omega(\boldsymbol{v}) = v_0$, though it also reverses the orientation, gives a connection between (V^r, E^r) and (V, E). For example, a closed path $c = (\boldsymbol{v}_0, \cdots, \boldsymbol{v}_m)$ in (V^r, E^r) can be identified with the closed path $\omega(c) = (\omega(\boldsymbol{v}_m), \cdots, \omega(\boldsymbol{v}_0))$ in (V, E). Define a continuous map of (V^r, E^r) into UM by $\iota \circ \omega$ and a correspondence of the set of cycles in (V^r, E^r) onto the set of φ_t-orbit cycles by $\varPhi \circ \omega$. By Proposition 3, for any closed path c, $\iota \circ \omega(c)$ is free homotopic to $\varPhi(\langle \omega c \rangle)$, and $p(\varPhi(\langle \omega c \rangle))$ and $p(\varPhi(\omega(\kappa c)))$ $= p(\varPhi(\langle \kappa(\omega c) \rangle))$ represent the same closed geodesic with different orientations.

Approximating the princpal suspending function ℓ we have

PROPOSITION 6 ([3]). *Given a positive ε with $\varepsilon < h_{\mathrm{top}}$, there exist a positive integer r and a strictly positive function $\tilde{f}$ on E^r such that*

1) $\lambda(\exp(\varepsilon - h_{\mathrm{top}}(\tilde{f}))) > 1$
2) $\min(\ell) \le \tilde{f} \le \max(\ell)$
3) $\tilde{f}(\kappa c) = \tilde{f}(c)$ *for any closed path c in (V^r, E^r), where*

$$\tilde{f}(c) = \sum_{i=0}^{m-1} \tilde{f}(\boldsymbol{v}_i, \boldsymbol{v}_{i+1}) \qquad \text{if } c = (\boldsymbol{v}_0, \cdots, \boldsymbol{v}_m) \,.$$

Given $v \in V$, let V_1^r be the primitive part of V^r containing an element $\boldsymbol{v} \in V^r$ with $\omega(\boldsymbol{v}) = v$. For simplicity we set $f = (\varepsilon - h_{\mathrm{top}})\tilde{f}$ and $\lambda = \lambda(\exp f)$. We define a non-negative function g_v on E^r by

$$g_v(\boldsymbol{v}, \boldsymbol{w}) = \begin{cases} 1 & \text{if } \omega(\boldsymbol{w}) = v \\ 0 & \text{otherwise} \,. \end{cases}$$

Let $\hat{H}_1 = \hat{H}_1(M, \boldsymbol{Z})$ denote the character group of $H_1(M, \boldsymbol{Z})$. Regarding $\chi \in \hat{H}_1$ as a character of $\pi_1(V^r, E^r)$ through the homomorphisms

$$\pi_1(V^r, E^r) \xrightarrow{\iota \circ \omega_*} \pi_1(UM) \xrightarrow{p_*} \pi_1(M) \longrightarrow H_1(M, \boldsymbol{Z}) \,,$$

we shall consider the matrices $L(\chi, s) = L_{\exp(f + s g_v), \chi}$, $s \in \boldsymbol{R}$. Let $\lambda_1(\chi, s)$, $\cdots, \lambda_d(\chi, s)$ $(|\lambda_1(\chi, s)| \ge \cdots \ge |\lambda_d(\chi, s)|)$ denote the eigenvalues of $L(\chi, s)$ and put $S_m(\chi, s) = \mathrm{Trace}\,(L(\chi, s)^m)$. By an easy calculation of matrix elements we have

$$(1) \qquad S_m(\chi, s) = \sum_c \chi(\llbracket c \rrbracket) \exp\{f(c) + sg_v(c)\},$$

where c runs over all $m\nu$-step closed paths in (V^r, E^r) with $o(c) \in V_1^r$. The presence of the associated involution κ guarantees that $S_m(\chi, 0)$ is a real number for any m and $\chi \in \hat{H}_1$. In fact, we have

$$
\begin{aligned}
S_m(\chi, 0) &= \sum_c \chi([p \circ \iota \circ \omega(c)]) \exp f(c) \\
&= \sum_c \chi([p(\Phi(\langle \omega(c) \rangle))]) \exp f(c) \\
&= \sum_c \chi([- p(\Phi(\langle \kappa\omega(c) \rangle))]) \exp f(\kappa c) \\
&= \sum_c \overline{\chi([p(\Phi(\langle \kappa\omega(c) \rangle))])} \exp f(\kappa c) = \overline{S_m(\chi, 0)}.
\end{aligned}
$$

Hence $\lambda_1(\chi, 0) \in \boldsymbol{R}$ if $|\lambda_1(\chi, 0)| > |\lambda_2(\chi, 0)|$.

LEMMA 7. $\lambda(s) = \lambda_1(1, s)$ *is a smooth function on $\boldsymbol{R}$ with $\lambda'(s) > 0$.*

Proof. Since $\lambda(s)$ is an isolated eigenvalue of $L(s) = L(1, s)$, by perturbation theory of matrices, we know that $\lambda(s)$ is smooth and that there is a smooth map $u(s): \boldsymbol{R} \to \boldsymbol{R}^d$ with $L(s)u(s) = \lambda(s)u(s)$ and $u(s) > 0$. It is an easy consequence of 3) of Proposition 4 that $\lambda'(s) \geq 0$. Now suppose $\lambda'(s_0) = 0$ for some s_0. Put $C = \min_{1 \leq j \leq d}\{-u'_j(s_0)/u_j(s_0)\} - 1$ and define $h(s) = \exp\{C(s - s_0)\}u(s)$. Differentiating the equality $L(s)h(s) = \lambda(s)h(s)$, we have

$$L'(s_0)h(s_0) + L(s_0)h'(s_0) = \lambda(s_0)h'(s_0).$$

Since $h(s_0) > 0$ and $L'(s_0) = L_{g_v \exp(f + s_0 g_v), 1}$ is a non-zero and non-negative matrix, we get

$$L'(s_0)h(s_0) \geq 0, \quad L'(s_0)h(s_0) \neq 0, \quad L(s_0)h'(s_0) \leq \lambda(s_0)h'(s_0).$$

Again 3) of Proposition 4 implies

$$L(s_0)h'(s_0) = \lambda(s_0)h'(s_0),$$

which leads us to a contradiction $L'(s_0)h(s_0) = 0$.

For each $\alpha \in H_1(M, Z)$, we denote by $\mathscr{C}_n(V^r; \alpha)$ the set of n-step closed paths c in (V^r, E^r) with $[p(\Phi\langle \omega c \rangle)] = \alpha$, and by $\mathscr{P}_n(V^r; \alpha)$ the set of all prime $c \in \mathscr{C}_n(V^r; \alpha)$ with prime $\Phi(\langle \omega c \rangle)$. Put

$$C_m(\alpha) = \sum_{c \in \mathscr{C}_{m\nu}(V^r; \alpha), o(c) \in V_1^r} g_v(c) \exp f(c),$$

$$G_m(\alpha) = \sum_{c \in \mathscr{P}_{m\nu}(V^r; \alpha), o(c) \in V_1^r} g_v(c) \exp f(c).$$

By the definition of g_v, to prove Proposition 5, it is enough to show $G_m(\alpha) > 0$ for infinitely many m.

Let $d\chi$ denote the Haar measure on $\hat{H}_1$ with $\int_{\hat{H}_1} d\chi = 1$. Using the orthogonal relation of characters and (1), we have

$$C_m(\alpha) = \int_{\hat{H}_1} \chi(-\alpha) S_m'(\chi, 0) d\chi,$$

where $S_m'(\chi, 0) = (d/ds)|_{s=0} S_m(\chi, s)$. Since f and g_v are real-valued, one can rewrite this equality as follows;

$$C_m(\alpha) = \int_{\hat{H}_1} \mathscr{R}e \left\{ \chi(-\alpha) S_m'(\chi, 0) \right\} d\chi.$$

In order to estimate $C_m(\alpha)$ we shall deal with eigenvalues of the matrix $L(\chi, s)$. By Proposition 4, $L(\chi, s)$ has an eigenvalue of the form $\lambda_1(\chi, s) = \exp(\sqrt{-1}\psi)\lambda(s)$, $0 \leq \psi < 2\pi$, if and only if

$$\chi([\![c]\!]) = \exp(\sqrt{-1}\psi |c|/\nu)$$

for any closed path c in (V^r, E^r). As we have

$$\chi([\![c]\!]) = \exp(\sqrt{-1}\psi |\kappa c|/\nu) = \chi([\![\kappa c]\!]) = \overline{\chi([\![c]\!])},$$

such case occurs if and only if $\chi = \mathbf{1}$ or $\chi = \chi_0$, where $\chi_0 \in \hat{H}_1$ (if exists) uniquely determined by the formula

$$\chi_0([\![c]\!]) = \exp(\sqrt{-1} |c|/\nu)$$

for any closed path c in (V^r, E^r).

Given a positive B with $B < \lambda$, we set

$$U_B = \{\chi \in \hat{H}_1 \,|\, |\lambda_1(\chi, 0)| > B\},$$
$$U_B^+ = \{\chi \in \hat{H}_1 \,|\, \lambda_1(\chi, 0) > B\},$$

and

$$U_B^- = \{\chi \in \hat{H}_1 \,|\, \lambda_1(\chi, 0) < -B\}.$$

By petturbation theory of eigenvalues, if B is sufficiently close to λ, we have $|\lambda_1(\chi, 0)| > |\lambda_2(\chi, 0)|$. Hence $U_B = U_B^+ \cup U_B^-$, and U_B^+ and U_B^- are open neighborhoods of $\mathbf{1}$ and χ_0 respectively. It should be also noted that U_B^+ and U_B^- shrink to $\{\mathbf{1}\}$ and $\{\chi_0\}$ as B goes to λ.

We now estimate $C_m(\alpha)$ from below.

LEMMA 8. *There is a constant C_1 such that if $\chi \in \hat{H}_1 \setminus U_B$ then $|S_m'(\chi, 0)| \leq C_1 m B^{m-1}$ for any m.*

Proof. For each $\chi_* \in \hat{H}_1$ there exist open neighborhoods $U(\chi_*)$, $W(\chi_*)$ of χ_* and continuous maps $h_1, \cdots, h_d : U(\chi_*) \to C^d$ such that

1) the closure of $W(\chi_*)$ is contained in $U(\chi_*)$.

2) for any $\chi \in U(\chi_*)$, $h_1(\chi), \cdots, h_d(\chi)$ are linearly independent eigenvectors for $L(\chi, 0)$. By means of the Schmidt's orthogonalization we get a continuous map P_{χ_*} of $U(\chi_*)$ into the vector space $\mathrm{Mat}(d, C)$ of d-dimensional complex matrices such that $P_{\chi_*}(\chi)^{-1} L(\chi, 0) P_{\chi_*}(\chi)$ is an upper triangular matrix for each $\chi \in U(\chi_*)$.

Since $\hat{H}_1$ is compact, one can choose finite $\chi_1, \cdots, \chi_J \in \hat{H}_1$ so that $\hat{H}_1 = \cup_{j=1}^J W(\chi_j)$. Put $P_j = P\chi_j$ and define $Q_j(\chi, s) = P_j(\chi)^{-1} L(\chi, s) P_j(\chi)$ for $\chi \in U(\chi_j)$ and $s \in R$. We have

$$S'_m(\chi, 0) = m \, \mathrm{Trace}\, \{L'(\chi, 0) L(\chi, 0)^{m-1}\}$$
$$= m \, \mathrm{Trace}\, \{Q'_j(\chi, 0) Q_j(\chi, 0)^{m-1}\}\,.$$

Given $P \in \mathrm{Mat}(d, C)$ we shall denote by $P_{;kl}$ the (k, l)-entry of P. Put

$$C = \max_{1 \le j \le J}\, \max_{\chi \in W(\chi_j)}\, \max_{k < l} |Q_j(\chi, 0)_{;kl}|,$$
$$C' = \max_{1 \le j \le J}\, \max_{\chi \in W(\chi_j)}\, \max_{k < l} |Q'_j(\chi, 0)_{;kl}|,$$

and

$$N = \begin{bmatrix} 0 & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \end{bmatrix} \in \mathrm{Mat}(d, C).$$

Since N^{d+1} is a zero matrix and $|Q_j(\chi, 0)_{;kk}| \le B$ if $\chi \in \hat{H}_1 \backslash U_B$, there is a constant C'' such that

$$|Q_j(\chi, 0)^{m-1}_{;kl}| \le (BI + CN)^{m-1}_{;kl} \le C'' B^{m-1}$$

for each $\chi \in \hat{H}_1 \backslash U_B$ and $k \le l$. As $Q_j(\chi, 0)^{m-1}$ is an upper triangular matrix, we have

$$|\mathrm{Trace}\, \{Q'_j(\chi, 0) Q_j(\chi, 0)^{m-1}\}| \le \tfrac{1}{2} d(d + 1) C' C'' B^{m-1}\,,$$

and get the conclusion.

LEMMA 9. *There exist positive constants C_2, C_3, A and B with $\sqrt{\lambda} < B < A < \lambda$ such that if $\chi_0(-\alpha) = 1$ (or if χ_0 does not exist)*

$$C_{2m}(\alpha) \geq 2m\{C_2 A^{2m-1} - C_3 B^{2m-1}\},$$

and if $\chi_0(-\alpha) = -1$

$$C_{2m+1}(\alpha) \geq (2m+1)\{C_2 A^{2m} - C_3 B^{2m}\}$$

for any m.

Proof. Since $\lambda_j(\chi, s)$ varies continuously, one can find positive δ, B_1 and B_2 so that

1) $\sqrt{\lambda} < B_1 < B_2 < \lambda$,
2) $|\lambda_2(\chi, s)| < B_1$ if $|s| < \delta$ and $\chi \in U_{B_2}$.

In particular, $\lambda_1(\chi, s)$ is a simple eigenvalue of $L(\chi, s)$, hence the map $\lambda_1(\chi, s)$: $U_{B_2} \times (-\delta, \delta) \to C$ is smooth with respect to s. Here we note that $\lambda_1'(\chi_0, 0) = -\lambda_1'(1, 0)$ because $\lambda_1(\chi_0, s) = -\lambda_1(1, s)$ for any s.

By Lemma 8, for any $B_2 < B < \lambda$, we have $|S_m'(\chi, 0)| \leq C_1 m B^{m-1}$ for $\chi \in \hat{H}_1 \setminus U_B$. Applying the same argument as Lemma 8, there is a constant C such that

$$|S_m'(\chi, 0) - m\lambda_1'(\chi, s)\lambda_1(\chi, s)^{m-1}| \leq CmB_1^{m-1}$$

for any m and $\chi \in \overline{U_B}$. In fact, as the eigenspace associated with $\lambda_1(\chi, s)$ varies smoothly with respect to s and continuously with respect to χ, we have a map $\tilde{L}(\chi, s)$: $U_B \times (\delta, \delta) \to \mathrm{Mat}(d-1, C)$ which is s-smooth and χ-continuous, and which satisfies

$$\mathrm{Trace}\,(\tilde{L}(\chi, s)^m) = S_m(\chi, s) - \lambda_1(\chi, s)^m.$$

Now in the first place, we suppose $\chi_0(-\alpha) = 1$. Since $\lambda_1'(1, 0) > 0 > \lambda_1'(\chi_0, 0)$, there is $B_2 < B < \lambda$ such that

$$\mathscr{R}e\,\{\chi(-\alpha)\lambda_1'(\chi, 0)\} > 0 \qquad \text{for } \chi \in U_B^+$$

and

$$\mathscr{R}e\,\{\chi(-\alpha)\lambda_1'(\chi, 0)\} < 0 \qquad \text{for } \chi \in U_B^-.$$

Choose positive A so that $B < A < \lambda$ and put $\Lambda = \min\{\mathscr{R}e\,\{\chi(-\alpha)\lambda_1'(\chi, 0)\}|\chi \in \overline{U_A^+}\}$. Since $\mathscr{R}e\,\{\chi(-\alpha)\lambda_1'(\chi, 0)\}\lambda_1(\chi, 0)^{2m-1} > 0$ for any $\chi \in U_B$, we have

$$C_{2m}(\alpha) = \int_{U_B} \mathscr{R}e\,\{\chi(-\alpha)S_{2m}'(\chi, 0)\}d\chi + \int_{\hat{H}_1 \setminus U_B} \mathscr{R}e\,\{\chi(-\alpha)S_{2m}'(\chi, 0)\}d\chi$$

$$\geq 2m\int_{U_B} \mathscr{R}e\,\{\chi(-\alpha)\lambda_1'(\chi, 0)\lambda_1(\chi, 0)^{2m-1}\}d\chi$$

$$- \int_{U_B} | S'_{2m}(\chi, 0) - 2m\lambda'_1(\chi, 0)\lambda_1(\chi, 0)^{2m-1} | \, d\chi$$

$$- \int_{\hat{H}_1 \setminus U_B} | S'_{2m}(\chi, 0) | \, d\chi$$

$$\geq 2m \int_{U_A^+} \mathscr{R}e \, \{\chi(-\alpha)\lambda'_1(\chi, 0)\lambda_1(\chi, 0)^{2m-1}\} d\chi$$

$$- 2CmB_1^{2m-1} - 2C_1 m B^{2m-1}$$

$$\geq 2mA^{2m-1}\Lambda \operatorname{vol}(U_A^+) - 2m(C + C_1)B^{2m-1} .$$

(In case χ_0 does not exist this argument can also be applied.)

In the second we suppose $\chi_0(-\alpha) = -1$. There is $B_2 < B < \lambda$ with $\mathscr{R}e \, \{\chi(-\alpha)\lambda'_1(\chi, 0)\} > 0$ for any $\chi \in U_B$. Repeating the same computation we get the conclusion.

For $v \in V$ and $\alpha \in H_1(M, \mathbf{Z})$, let $\mathscr{C}_n(\alpha; v)$ denote the set of $c \in \mathscr{C}_n(V; \alpha)$ with $o(c) = v$. If $c \in \mathscr{C}_n(V; \alpha)$ satisfies $g_v(c) > 0$, then there exists $c' \in \mathscr{C}_n(\alpha; v)$ which is equivalent to c. Note also that

$$g_v(c) \exp f(c) \leq n \cdot \exp \{(\varepsilon - h_{\mathrm{top}})n \cdot \min(\ell)\} .$$

Therefore we have

COROLLARY 10. *If* $\chi_0(-\alpha) = 1$, *then*

$$\#\mathscr{C}_{2m\nu}(\alpha; v) \geq \frac{1}{2m\nu} (C_2 A^{2m-1} - C_3 B^{2m-1}) \exp \{2(h_{\mathrm{top}} - \varepsilon)m\nu \cdot \min(\ell)\}$$

and if $\chi_0(-\alpha) = -1$, *then*

$$\#\mathscr{C}_{(2m+1)\nu}(\alpha; v) \geq \frac{1}{(2m+1)\nu} (C_2 A^{2m} - C_3 B^{2m}) \exp \{(h_{\mathrm{top}} - \varepsilon)(2m+1)\nu \cdot \min(\ell)\} .$$

In particular, $\mathscr{C}_n(\alpha; v)$ *is not empty for infinitely many* n.

In order to get an estimate of $G_m(\alpha)$, we need auxiliary graphs (see [3] and [6]). Using informations on them, we can conclude the following by the same way as Lemmas 2–3 and 2–4 in [3].

LEMMA 11. *There exist positive constants* C_4 *and* C_5 *such that if* $\chi_0(-\alpha) = 1$

$$(2) \qquad G_{2m}(\alpha) \geq 2m \{C_2 A^{2m-1} - C_3 B^{2m-1}\} - 8C_4 m^3 \lambda^m - 4C_5 m^2 ,$$

and if $\chi_0(-\alpha) = -1$

$$(3) \quad G_{2m+1}(\alpha) \geq (2m+1)\{C_2 A^{2m} - C_3 B^{2m}\} - 8C_4(m+1)^3 \lambda^{m+1} - 4C_5(m+1)^2$$

for any $m \geq 1$.

This completes the proof of Proposition 5. In fact, since the right sides of (2) and (3) are dominated by A^{2m-1} and A^{2m} respectively, these estimates imply that $\#\{c \in \mathscr{P}_{2m,\nu}(V; \alpha) \mid o(c) = v\}$ or $\#\{c \in \mathscr{P}_{(2m+1)\nu}(V; \alpha) \mid o(c) = v\}$ grows exponentially with respect to m.

Proof of Theorem 1. Given an open set D in UM, we choose a Markov family (V, E) for φ_t with associated involution and of small size so that there is $v \in V$ with $v \subset D$. The previous argument implies that there is a prime closed path $c \in \cup_k \mathscr{C}_k(\alpha; v)$ with prime $\mathfrak{p} = \varPhi(\langle c \rangle)$. Since $\mathfrak{p} \cap v \neq \varnothing$, this leads us to the conclusion.

§5. Equidistribution of prime closed geodesics

We now mention the equidistribution of closed geodesics with a preassigned homology class. It has been noted that there is a resemblance between prime closed geodesics and prime ideals in number fields ([9], [10], [17]). Our work in this section is an analogue of Chebotarev's density theorem and refines Bowen's result (cf. [8]). Throughout this section, let H be a finite index subgroup of $H_1(M, \mathbf{Z})$ and $\mathscr{A} \in G = H_1(M, \mathbf{Z})/H$. We denote by $\hat{G}$ the character group of G. Given $\chi \in \hat{G}$ and a smooth function g on UM, we define $\eta_g(s; \chi)$ by

$$\sum_{n=1}^{\infty} \sum_{\mathfrak{p}; \varphi_t \text{ closed orbit}} \chi([\![\mathfrak{p}]\!])^n \tau_g(\mathfrak{p}) h_{\text{top}} \exp\{-snh_{\text{top}} \tau(\mathfrak{p})\},$$

where

$$\tau_g(\mathfrak{p}) = \int_0^{\tau(\mathfrak{p})} g(\varphi_t(x))dt, \qquad x \in \mathfrak{p},$$

If one defines a "weighted" L-function for φ_t by

$$L_g(s, z; \chi) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\mathfrak{p}; \varphi_t-\text{closed orbit}} \chi([\![\mathfrak{p}]\!])^n \exp\{-nh_{\text{top}}(s\tau(\mathfrak{p}) + z\tau_g(\mathfrak{p}))\},$$

then

$$\eta_g(s; \chi) = -\frac{d}{dz} \log L_g(s, z; \chi)_{z=0}.$$

Analyzing these functions (see [2] and [8]), we have

PROPOSITION 12. 1) $\eta_g(s; \chi)$ *is analytic when* $\mathscr{R}e\, s > 1$.
2) $\eta_g(s; \mathbf{1})$ *has an analytic extension to a neighborhood of* $\mathscr{R}e\, s > 1$

except for a simple pole at $s = 1$ of residue $\int g d\mu$.

3) *If $\chi \neq \mathbf{1}$, $\eta_g(s; \chi)$ is analytic in some neighborhood of $\mathscr{R}_e s \geq 1$.*

For each $\mathscr{A} \in G$, we set

$$F_g(\mathscr{A}, T) = \frac{1}{\sharp G} \sum_{\mathfrak{p}} \tau_g(\mathfrak{p}) h_{\text{top}},$$

$$\pi_g(\mathscr{A}, T) = \frac{1}{\sharp G} \sum_{\mathfrak{p}} \tau_g(\mathfrak{p})/\tau(\mathfrak{p}),$$

where in the summations $\mathfrak{p}$ runs over all φ_t-orbit cycles with $[p(\mathfrak{p})] \in \mathscr{A}$ and $\tau(\mathfrak{p}) \leq (\log T)/h_{\text{top}}$. Since

$$\int_1^\infty T^{-s} d F_g(\mathscr{A}, T) = \frac{1}{\sharp G} \sum_{\chi \in \hat{G}} \chi(-\mathscr{A}) \eta_g(s; \chi),$$

Tauberian theorem implies that

$$F_g(\mathscr{A}, T)/T \longrightarrow \frac{1}{\sharp G} \int g d\mu.$$

We can also conclude $\pi_g(\mathscr{A}, T) \log T \sim F_g(\mathscr{A}, T)$. Therefore

$$\hat{\mu}_{T, \varepsilon, \mathscr{A}}(g) = \pi_g(\mathscr{A}, \exp(T + \varepsilon) h_{\text{top}}) - \pi_g(\mathscr{A}, \exp(T - \varepsilon) h_{\text{top}})$$

$$\sim \{h_{\text{top}} T \sharp G\}^{-1} \exp(h_{\text{top}} T) \int_{UM} g d\mu \{\exp(\varepsilon h_{\text{top}}) - \exp(-\varepsilon h_{\text{top}})\},$$

hence

$$\mu_{T, \varepsilon, \mathscr{A}}(g) \longrightarrow \int_{UM} g d\mu \quad \text{and} \quad \mu_{T, \varepsilon, \mathscr{A}}(g)/\mu_{T, \varepsilon, \mathscr{B}}(g) \longrightarrow 1$$

for any $\mathscr{A}, \mathscr{B} \in G$.

We now examine the measure $\nu_{T, \varepsilon, \mathscr{A}}$. Since

$$\sum \chi(\llbracket \mathfrak{p} \rrbracket) \tau_g(\mathfrak{p}) h_{\text{top}} \exp\{- nsh_{\text{top}} \tau(\mathfrak{p})\}$$

contains the essential part of $\eta_g(s; \chi)$, one obtains

$$h_{\text{top}} \sum_{\mathfrak{p}; \tau(\mathfrak{p}) \leq T} \tau_g(\mathfrak{p}) \sim \exp(h_{\text{top}} T) \int g d\mu,$$

which leads us to Theorem 2.

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