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ON HERMITE-FEJÉR TYPE INTERPOLATION

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For the Hermite-Fejér interpolation operator of higher order $K_m^{(\alpha,\beta)}$ constructed on the roots $x_{km}^{(\alpha,\beta)}$, $1 \le k \le m$, of the Jacobi-polynomial $P_m^{(\alpha,\beta)}$ it is shown that $K_m^{(\alpha,\beta)}$ is positive for all $m \in \mathbb{N}$, if $(\alpha, \beta) \in [-\frac{3}{4}, -\frac{1}{4}]^2$. Further there is given an error bound, which implies $\lim_{m\to\infty} \left\|f-K_m^{(\alpha,\beta)}f\right\| = 0$ for arbitrary $f \in C(I)$ and $(\alpha, \beta) \in [-\frac{3}{4}, -\frac{1}{4}[^2$.

1. Formulation of the problem and main results

In this paper we investigate the question of convergence for Hermite-Fejér interpolation of higher order, introduced by Kryloff and Stayermann [8]. To state the problem, let $m \in \mathbb{N}$, $(\alpha, \beta) \in]-1, \infty[^2$ and let

$$(1.1) \qquad -1 < x_{mm}^{(\alpha,\beta)} < x_{m-1,m}^{(\alpha,\beta)} < \ldots < x_{1m}^{(\alpha,\beta)} < 1$$

be the roots of the Jacobi-polynomial $P_m^{(\alpha,\beta)}$ of degree m (with regard to the weight function $x \mapsto (1-x)^{\alpha} \cdot (1+x)^{\beta}$). We denote by $C(I)$ the Banach-space of all continuous real-valued functions on $I = [-1, 1]$ with the sup-norm $\|\cdot\|$. For any $f \in C(I)$ there is an uniquely determined polynomial $K_m^{(\alpha,\beta)}f$ of degree at most $4m - 1$ satisfying the conditions

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$$K_m^{(\alpha,\beta)} f\left(x_{km}^{(\alpha,\beta)}\right) = f\left(x_{km}^{(\alpha,\beta)}\right) , \quad 1 \le k \le m ,$$

$$\left(K_m^{(\alpha,\beta)} f\right)^{(i)} \left(x_{km}^{(\alpha,\beta)}\right) = 0 , \qquad 1 \le k \le m , \quad i = 1, 2, 3 .$$

This polynomial can be represented in the following form:

$$K_m^{(\alpha,\beta)} f(x) = \sum_{k=1}^{m} u_{km}^{(\alpha,\beta)}(x) \cdot \left(l_{km}^{(\alpha,\beta)}(x)\right)^4 \cdot f\left(x_{km}^{(\alpha,\beta)}\right) ,$$

where $l_{km} = l_{km}^{(\alpha,\beta)}$ is the kth Lagrange polynomial of degree $m - 1$ determined by the nodes (1.1) and where $u_{km}^{(\alpha,\beta)}$ is given by

$$(1.2) \qquad u_{km}^{(\alpha,\beta)}(x) = \sum_{i=0}^{3} a_{km}^{(i)} \cdot \left(x - x_{km}^{(\alpha,\beta)}\right)^i$$

with

$$a_{km}^{(0)} = 1 ,$$

$$a_{km}^{(1)} = -4 l'_{km}\left(x_{km}^{(\alpha,\beta)}\right) ,$$

$$a_{km}^{(2)} = 10 \cdot \left(l'_{km}\left(x_{km}^{(\alpha,\beta)}\right)\right)^2 - 2 l''_{km}\left(x_{km}^{(\alpha,\beta)}\right) ,$$

and

$$a_{km}^{(3)} = 10 l'_{km}\left(x_{km}^{(\alpha,\beta)}\right) \cdot l''_{km}\left(x_{km}^{(\alpha,\beta)}\right) - 20\left(l'_{km}\left(x_{km}^{(\alpha,\beta)}\right)\right)^3 - \tfrac{2}{3} l_{km}^{(3)}\left(x_{km}^{(\alpha,\beta)}\right) .$$

As in the case of Hermite-Fejer interpolation the question arises for which $(\alpha, \beta) \in]-1, \infty[^2$ we have

$$(1.3) \qquad \lim_{m\to\infty} \left\| f - K_m^{(\alpha,\beta)} f \right\| = 0 \quad \text{for all} \quad f \in C(I) .$$

It was shown first that (1.3) is valid in the case $\alpha = \beta = -0.5$ (*cf.* Kryloff and Stayermann [8], Laden [9] as well as Sharma and Tzimbalario [12]). Then it was shown that estimations of

$$\left\| f - K_m^{(-0.5,-0.5)} f \right\| \quad \text{and of} \quad \left| f(x) - K_m^{(-0.5,-0.5)} f(x) \right|$$

(for $x \in I$) by the modulus of continuity and by the modulus of smoothness of f can be given (*cf*. Stancu [*13*], Florica [*2*], Haussmann and Knoop [*6*], Mills [*10*], Prasad [*11*] and Gonska [3, *4*]). Moreover, there exist error bounds for subspaces of $C(I)$ (*cf*. Gonska [*3*, *4*] as well as Goodenough and Mills [*5*]). In these investigations the positivity of the operators $K_m^{(-0.5,-0.5)}$ plays a fundamental role. Now the positivity of these operators is equivalent to

$$u_{km}^{(-0.5,-0.5)}(x) \geq 0 \quad \text{for all} \quad x \in I \ , \quad 1 \leq k \leq m \ .$$

In [*7*] it was shown that for all $m \in \mathbb{N}$ we have

$$u_{km}^{(-0.5,-0.5)}(x) \geq \tfrac{1}{8} \quad \text{for all} \quad x \in I \ , \quad 1 \leq k \leq m \ .$$

Other pairs (α, β) were considered by Laden. He has shown in [*9*] that (1.3) is valid for all pairs (α, β) with

$$(\alpha, \beta) \in [-\tfrac{3}{4}, -\tfrac{1}{2}]^2 \cup [-\tfrac{1}{2}, -\tfrac{1}{4}[^2 \ ,$$

and that there is a function $f \in C(I)$ such that $K_m^{(\alpha,\alpha)}f(1)$ does not converge to $f(1)$ in the case $\alpha = -0.25$.

In this paper we show that

$$K_m^{(\alpha,\beta)} : C(I) \ni f \mapsto K_m^{(\alpha,\beta)}f \in C(I)$$

is a positive operator for a wider field of pairs (α, β) , namely for all $(\alpha, \beta) \in [-\tfrac{3}{4}, -\tfrac{1}{4}]^2$, and that (1.3) holds even for all $(\alpha, \beta) \in [-\tfrac{3}{4}, -\tfrac{1}{4}[^2$. To formulate more precisely

THEOREM 1. *Let* $(\alpha, \beta) \in [-\tfrac{3}{4}, -\tfrac{1}{4}]^2$ *, then for any* $m \in \mathbb{N}$ *, any* $k \in \{1, \ldots, m\}$ *and any* $x \in I$ *we have*

$$u_{km}^{(\alpha,\beta)}(x) \geq -\tfrac{4}{3}\eta^3 + \tfrac{23}{12}\eta^2 - 2\eta + 1 \geq \tfrac{1}{64}$$

with $\eta = \max(\alpha, \beta) + 1$.

THEOREM 2. *Let* $(\alpha, \beta) \in [-\tfrac{3}{4}, -\tfrac{1}{4}[^2$ *, then we have for any* $f \in C(I)$,

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$$\left\|f-K_m^{(\alpha,\beta)}f\right\| \le 2 \cdot D \cdot \begin{cases} \omega\left(f, \sqrt{m^{4\Theta+1}}\right), & \text{if } \min(\alpha,\beta) > -\tfrac{3}{4}, \\ \omega\left(f, \sqrt{m^{4\Theta+1}\cdot \log m}\right), & \text{if } \min(\alpha,\beta) = -\tfrac{3}{4}. \end{cases}$$

Here $\Theta = \max(\alpha, \beta, -0.5)$, ω *denotes the usual modulus of continuity and* D *is a positive constant independent of* f *and* m .

2. Proof of Theorem 1

Let $m \in \mathbb{N}$ and $(\alpha, \beta) \in [-\tfrac{3}{4}, -\tfrac{1}{4}]^2$ be given. Then for the sake of simplicity we put

$$\gamma := \alpha - \beta ,$$

$$\delta := \alpha + \beta + 2 ,$$

and

$$M := m(m+\alpha+\beta+1) .$$

We further use the notation

$$x_k := x_{km}^{(\alpha,\beta)} , \quad l_k := l_{km}^{(\alpha,\beta)} \text{ and } u_k := u_{km}^{(\alpha,\beta)} .$$

From the differential equation for the Jacobi-polynomials (see Szegö [15]) we conclude

$$\begin{aligned} (2.1)\quad u_k(x) = 1 - 2 \cdot \frac{\gamma+\delta x_k}{1-x_k^2} \cdot (x-x_k) + \tfrac{11}{6}\left(\frac{\gamma+\delta x_k}{1-x_k^2}\right)^2 \cdot (x-x_k)^2 \\ + \tfrac{1}{6}\left(\frac{x-x_k}{1-x_k^2}\right)^2 \cdot s_k(x) \cdot \left[4\cdot(M-\delta)\left(1-x_k^2\right) - 8\cdot x_k\cdot\left(\gamma+\delta x_k\right)\right] \\ - \left[\left(\frac{\gamma+\delta x_k}{1-x_k^2}\right)^3 + \tfrac{1}{3}\cdot x_k \cdot \frac{\left(\gamma+\delta x_k\right)^2}{\left(1-x_k^2\right)^3} + \tfrac{1}{6}(\delta+2)\cdot \frac{\gamma+\delta x_k}{\left(1-x_k^2\right)^2}\right] \cdot (x-x_k)^3 \end{aligned}$$

with

$$s_k(x) = s_{km}^{(\alpha,\beta)}(x) = 1 - \frac{2\gamma+(2\delta-1)\cdot x_k}{1-x_k^2} \cdot (x-x_k) .$$

Now the assertion of Theorem 1 is an immediate consequence of the two following lemmas.

LEMMA 1. *For any* $(\alpha, \beta) \in [-\frac{3}{4}, -\frac{1}{4}]^2$ *, any* $k \in \{1, \ldots, m\}$ *and any* $x \in I$ *we have*

$$s_{km}^{(\alpha,\beta)}(x) \geq \min\left(-(2\alpha+\tfrac{1}{2}), -(2\beta+\tfrac{1}{2})\right) \geq 0 .$$

Proof. s_k being a linear function it is sufficient to consider $s_k(1)$ and $s_k(-1)$. We compute

$$s_k(1) = -(2\alpha+\tfrac{1}{2}) + \frac{4\beta+3}{2} \cdot \frac{1-x_k}{1+x_k}$$

and

$$s_k(-1) = -(2\beta+\tfrac{1}{2}) + \frac{4\alpha+3}{2} \cdot \frac{1+x_k}{1-x_k} . \qquad \square$$

Now taking into consideration the estimation

$$4(M-\delta) \cdot \left(1-x_k^2\right) - 8 \cdot x_k \cdot \left(\gamma+\delta x_k\right) \geq \left(\gamma+\delta x_k\right)^2$$

(*cf.* Laden [9, Lemma 4]) and putting for fixed $k \in \{1, \ldots, m\}$ and, for fixed $x \in I$,

$$t := \frac{\gamma+\delta x_k}{1-x_k^2} , \quad y := x - x_k ,$$

we get for any $(\alpha, \beta) \in [-\frac{3}{4}, -\frac{1}{4}]^2$,

$$u_k(x) \geq (1-ty)^2 + \tfrac{5}{6}(ty)^2 + \tfrac{2}{3}(ty)^2 \cdot s_k(x) - \tfrac{1}{3}(ty)^2 \cdot \left[1 - 2ty + \frac{y \cdot x_k}{1-x_k^2}\right]$$
$$- (ty)^2 \cdot \left[ty + \tfrac{1}{3}\frac{y \cdot x_k}{1-x_k^2}\right] - \tfrac{1}{6}(\delta+2)\frac{t \cdot y^3}{1-x_k^2} .$$

Thus we have for any $x \in I$ and all u_k the estimation

$$u_k(x) \geq h\left(x, x_k, \gamma, \delta\right) \tag{2.2}$$

with

$$h(x, x_k, \gamma, \delta) = -\frac{t}{6}\left(8t^2 + t\cdot\frac{x_k}{1-x_k^2} + \frac{\delta+2}{1-x_k^2}\right)y^3 + 2t^2y^2 - 2ty + 1 .$$

Now h can be estimated from below as shown in the following lemma.

LEMMA 2. *For any* $(\alpha, \beta) \in [-\frac{3}{4}, -\frac{1}{4}]^2$ *we have*

$$\inf_{x_k\in]-1,1[}\ \min_{x\in[-1,1]} h(x, x_k, \gamma, \delta) = -\frac{4}{3}\eta^3 + \frac{23}{12}\eta^2 - 2\eta + 1$$

with $\eta = \max(\alpha, \beta) + 1$.

Proof. For any $(\alpha, \beta) \in [-\frac{3}{4}, -\frac{1}{4}]^2$ we have $\gamma \in [-\frac{1}{2}, \frac{1}{2}]$ and $\delta \in [\frac{1}{2}, \frac{3}{2}]$. Since $h(x, x_k, \gamma, \delta) = h(-x, -x_k, -\gamma, \delta)$ we assume $\gamma \ge 0$ in the following. At first we show that h is a monotone function of x . Let $t \ne 0$. If there is a zero of $\partial h/\partial x$ we then have

$$\Delta := -4t^2 - t\cdot\frac{x_k}{1-x_k^2} - \frac{\delta+2}{1-x_k^2} \ge 0 .$$

This implies $t\cdot x_k < 0$, hence $t > 0$ and $x_k < 0$. From this we conclude $x_k \ge -0.5$. Since $\Delta \ge 0$ implies that Δ has a zero (as a polynomial in t) we have

$$\frac{x_k^2}{\left(1-x_k^2\right)^2} - 16\,\frac{\delta+2}{1-x_k^2} \ge 0 .$$

But this is impossible since $x_k \in [-0.5, 0[$. Therefore we have for $x_k \in]-1, 1[$, $\gamma \in [0, 0.5]$ and $\delta \in [0.5, 1.5]$,

$$\min_{x\in[-1,1]} h(x, x_k, \gamma, \delta) = \min\left(h(-1, x_k, \gamma, \delta), h(1, x_k, \gamma, \delta)\right) .$$

Next we compare $h(-1, x_k, \gamma, \delta)$ with $h(1, x_k, \gamma, \delta)$. We put $\tau := (\gamma+\delta x_k)/(1-x_k)$ and compute

$$h(-1, x_k, \gamma, \delta) = \tfrac{4}{3}\tau^3 + \left(2 + \tfrac{1}{6}\,\frac{x_k}{1-x_k}\right)\cdot \tau^2 + \left(2 + \tfrac{1}{6}\,\frac{(\delta+2)(1+x_k)}{1-x_k}\right)\cdot \tau + 1 =: u(\tau) .$$

If $x_k \le 0$ it is easily seen that $(\partial/\partial\tau)u(\tau)$ has no zero and hence $(\partial/\partial\tau)u(\tau) > 0$. In this case we get

$$h(1, -x_k, \gamma, \delta) = u\left(\tau - \frac{2\gamma}{1-x_k}\right) \le u(\tau) = h(-1, x_k, \gamma, \delta) .$$

If $x_k \ge 0$ we have $\tau \ge 0$ and therefore $h(-1, x_k, \gamma, \delta) \ge 1$. Since we will see later on that

$$\inf_{x_k \in]-1,1[} h(1, x_k, \gamma, \delta) < 1 ,$$

there follows for any $\gamma \in [0, 0.5]$, $\delta \in [0.5, 1.5]$,

$$\inf_{x_k \in]-1,1[}\ \min_{x \in [-1,1]} h(x, x_k, \gamma, \delta) = \inf_{x_k \in]-1,1[} h(1, x_k, \gamma, \delta) .$$

We now consider the function $h_1 :]-1, 1[\ni z \mapsto h(1, z, \gamma, \delta)$ for fixed γ and δ ,

$$h_1(x_k) = 1 - \frac{\gamma+\delta x_k}{1+x_k}\left(2 + \tfrac{1}{6}\,\frac{(\delta+2)(1-x_k)}{1+x_k}\right) + 2\left(\frac{\gamma+\delta x_k}{1+x_k}\right)^2 - \tfrac{1}{6}\left(\frac{\gamma+\delta x_k}{1+x_k}\right)^2\cdot\frac{x_k}{1+x_k} - \tfrac{4}{3}\left(\frac{\gamma+\delta x_k}{1+x_k}\right)^3 .$$

Obviously h_1 has a continuous extension to a function on $]-1, 1]$. Evidently, for $x_k \le -(\gamma/\delta)$ we have $h_1(x_k) \ge 1$. We now show that

$$\inf_{x_k \in]-1,1]} h_1(x_k) = h_1(1) .$$

Since we will see that $h_1(1) < 1$, it is sufficient to show that there is no x_k with $-(\gamma/\delta) < x_k < 1$ and $h_1(x_k) < h_1(1)$. It is evident that for the derivative h_1' we have $h_1'(-(\gamma/\delta)) < 0$. On the other hand it

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follows that

$$h_1'(x_k) = \frac{1}{(1+x_k)^4}\left(A\cdot x_k^2+B\cdot x_k+C\right)$$

with

$$A = -\tfrac{47}{6}\alpha^2 - 23\alpha - 8\beta^3 - \tfrac{98}{3}\beta^2 - 43\beta - 8\alpha^2\beta - 16\alpha\beta^2 - 40\alpha\beta - 18 ,$$

$$B = -\tfrac{47}{3}\alpha^2 - 14\alpha + 16\beta^3 + 49\beta^2 + 42\beta - 16\alpha^2\beta - 16\alpha\beta + \tfrac{32}{3} ,$$

$$C = -\tfrac{47}{6}\alpha^2 + 9\alpha - 8\beta^3 - \tfrac{101}{6}\beta^2 - 15\beta - 8\alpha^2\beta + 16\alpha\beta^2 + 24\alpha\beta - \tfrac{16}{3} .$$

Now sign $h_1'(1) = \mathrm{sign}(A+B+C)$ and

$$A + B + C \le -\tfrac{22}{3}\alpha^2 - 4\alpha - \tfrac{2}{3} < 0 .$$

If we now assume that there is an $x_k \in\,]-(\gamma/\delta), 1[$ with $h_1(x_k) < h_1(1)$, it follows that h_1' has two different zeros in the interval $]-(\gamma/\delta), 1[$ and therefore $B^2 > 4AC$ and $A < 0$. Now

$$C = \alpha^2(-\tfrac{47}{6}-8\beta) + 16\alpha(\beta+\tfrac{3}{4})^2 - 8\beta^3 - \tfrac{101}{6}\beta^2 - 15\beta - \tfrac{16}{3} < 0 .$$

We consider two cases.

CASE 1. $B > 0$. We show that in this case the middle-point $-(B/2A)$ of the zeros of h_1' satisfies $-(B/2A) > 1$, that is, a contradiction. Indeed, from $-(B/2A) \le 1$ there follows $4AC \ge -2BC$ and $B \ge -2C$. Now

$$B + 2C = \alpha^2(-\tfrac{94}{3}-32\beta) + \alpha\left(4+32\beta+32\beta^2\right) + \tfrac{46}{3}\beta^2 + 12\beta .$$

It is $B + 2C \ge 0$ only if

$$D(\beta) := \left(4+32\beta+32\beta^2\right)^2 + 4(\tfrac{94}{3}+32\beta)\left(\tfrac{46}{3}\beta^2+12\beta\right) \ge 0 .$$

But using Descartes' rule we see that D has exactly one zero β_0 with $\beta_0 \ge -\tfrac{3}{4}$ and from $D(-\tfrac{1}{4}) < 0$ we conclude that $D(\beta) < 0$ if $\beta \in [-0.75, -0.25]$. Thus $B + 2C < 0$ and therefore $-(B/2A) > 1$.

CASE 2. $B < 0$. We will show that in this case we have $-(B/2A) \le -1$. From the assumption $-1 < -(B/2A)$ we conclude similar as

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above $B < 2C$. Now we have

$$2C - B = 32\alpha(\beta+1)^2 - r(\beta)$$

where r with

$$r(\beta) = 32\beta^3 + \tfrac{248}{3}\beta^2 + 72\beta + \tfrac{64}{3}$$

is an increasing function of β . From this we see that $2C - B < 0$; hence we get $-(B/2A) \le -1$.

From Case 1 and Case 2 we conclude that h_1' cannot have two distinct zeros in $]-(\gamma/\delta), 1[$. Therefore

$$\inf_{x_k \in]-1,1]} h_1(x_k) = h_1(1) .$$

The result is

$$\inf_{x_k \in]-1,1[} \ \min_{x \in [-1,1]} h(x, x_k, \gamma, \delta) = -\tfrac{4}{3}\eta^3 + \tfrac{23}{12}\eta^2 - 2\eta + 1 .$$

Now the right side of this equation is a strictly decreasing function of η . Therefore

$$\tfrac{1}{64} \le h_1(1) \le \tfrac{115}{192} ,$$

which completes the proof. □

From Lemma 2 we get in view of formula (2.2) the assertion of Theorem 1.

In the case of $\alpha = \beta = -\frac{1}{2}$ we have - as an improvement of the result in [7] - the estimation

$$u_{km}^{(-0.5,-0.5)}(x) \ge \tfrac{5}{16}$$

for any $m \in \mathbb{N}$, any $k = 1, \ldots, m$ and any $x \in I$.

3. Proof of Theorem 2

For $(\alpha, \beta) \in [-\frac{3}{4}, -\frac{1}{4}]^2$ the mappings $K_m^{(\alpha,\beta)}$ form a sequence of positive linear operators from $C(I)$ into $C(I)$. If e is a constant function we have $K_m^{(\alpha,\beta)}e = e$; therefore the theorem of Bohman-Korovkin

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(*cf.* DeVore [1]) yields for any $x \in I$,

$$\left|f(x)-K_m^{(\alpha,\beta)}f(x)\right| \le 2\omega\left(f, \left|\varepsilon_m(x)\right|\right) \tag{3.1}$$

with

$$\varepsilon_m^2(x) = \left(K_m^{(\alpha,\beta)}g_x\right)(x)$$

and

$$g_x(t) = (x-t)^2 .$$

The function ε_m^2 can be estimated from above. We have

LEMMA 3. *For Jacobi abscissas, with* $(\alpha, \beta) \in [-\frac{3}{4}, -\frac{1}{4}]^2$ *, there exists a positive constant* $D = D_{\alpha,\beta}$ *independent of* m *such that we have for any* $x \in I$ *,*

$$\varepsilon_m^2(x) \le D \cdot \begin{cases} m^{4\Theta+1} & , \text{ if } \min(\alpha, \beta) > -\frac{3}{4} , \\ m^{4\Theta+1} \log m , & \text{if } \min(\alpha, \beta) = -\frac{3}{4} \end{cases}$$

with $\Theta = \max(\alpha, \beta, -\frac{1}{2})$.

Proof. We get from (2.1) with $P_m = P_m^{(\alpha,\beta)}$,

$$\varepsilon_m^2(x) = \sum_{k=1}^{m} u_k(x) \cdot \left(x-x_k\right)^2 \cdot l_k^4(x)$$

$$\le D_1 \cdot \sum_{k=1}^{m} \frac{P_m^2(x)}{\left(1-x_k^2\right)\left(P_m'(x_k)\right)^2} l_k^2(x)$$

$$+ D_2 \cdot \sum_{k=1}^{m} \frac{m^2P_m^4(x)}{\left(1-x_k^2\right)^2 \cdot \left(P_m'(x_k)\right)^4} + D_3 \cdot \sum_{k=1}^{m} \frac{P_m^4(x)}{\left(1-x_k^2\right)^3 \left(P_m'(x_k)\right)^4} ,$$

where D_1, D_2, D_3 are positive constants depending only on (α, β) . Because of the uniform boundedness of

$$\sum_{k=1}^{m} \frac{1}{\left(1-x_k^2\right) \cdot \left(P_m'(x_k)\right)^2} \quad \text{and} \quad \sum_{k=1}^{m} l_k^2(x)$$

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(*cf.* for example, Szász [14]) we get in view of Szegö [15, Chapter 7.32, Chapter 8.9],

$$\varepsilon_m^2(x) \le D_4 \cdot m^{2\cdot\max(\alpha,\beta,-0.5)}$$
$$+ D_5 \cdot m^{4\cdot\max(\alpha,\beta,-0.5)} \cdot \sum_{k=1}^{m} \frac{m^6}{k^4}\left(\frac{k^{4\alpha+6}}{m^{4\alpha+8}} + \frac{k^{4\beta+6}}{m^{4\beta+8}}\right)$$

(*cf.* also Laden [9, proof of Lemma 2]).

Since

$$m^{-4\varphi}m^{-2} \cdot \sum_{k=1}^{m} k^{4\varphi+2} \le D_6 \cdot \begin{cases} m , & \text{if } \varphi > -\tfrac{3}{4} , \\ m \log m , & \text{if } \varphi = -\tfrac{3}{4} , \end{cases}$$

and

$$2\varphi \le 4\varphi+1 \le 0 \quad \text{if} \quad -\tfrac{1}{2} \le \varphi \le -\tfrac{1}{4} ,$$

the proof is complete. □

Combining the estimation of Lemma 3 and formula (3.1) yields the proof of Theorem 2. In the special case $\alpha = \beta = -0.5$ we derive from Theorem 2 the relation

$$\left\|f-K_m^{(-0.5,-0.5)}f\right\| \le 2D\omega\left(f, \sqrt{m^{-1}}\right)$$

(*cf.* Stancu [13]).

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