COMMUTATIVE RINGS WHOSE FACTORS HAVE
ARTINIAN RINGS OF QUOTIENTS

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Let R be a commutative ring with unity. Then every factor ring
of R has an Artinian total quotient ring if and only if R is
a direct sum of one-dimensional Noetherian domains and local
Artinian rings.

The problem of characterizing those commutative rings for which every
factor ring has some particular finiteness property has been studied for
some time. Cohen [2] has shown that those commutative rings enjoying the
property that every proper factor ring is an Artinian ring are either one-
dimensional Noetherian domains or are themselves Artinian. Camillo [1]
proved that a commutative ring is Noetherian if and only if every factor
ring is a Goldie ring. The purpose of this paper is to consider those
commutative rings whose factor rings have an Artinian ring of fractions.

All rings considered here will be commutative rings with unity.
Recall that a ring is said to be an order in an Artinian ring if the ring
of fractions we obtain when we localize at the set of all non zero-divisors
is an Artinian ring. A ring is said to be a Goldie ring if it has no
infinite direct sums of ideals and has the ascending chain condition on
annihilator ideals.

LEMMA 1. *If every factor ring of R is an order in an Artinian ring
then R is a Noetherian ring.*

Proof. Let $\overline{R}$ be a factor ring of R and let $\overline{R}$ be an order in

Q . Since Q is an Artinian ring it satisfies the ascending chain condition on annihilators and this property passes to the subring $\overline{R}$. If $I_1 \oplus \ldots \oplus I_n$ is a direct sum of ideals of $\overline{R}$ then it is easy to see that $I_1 Q \oplus \ldots \oplus I_n Q$ is a direct sum of ideals of Q . Since Q has no infinite direct sum of ideals, $\overline{R}$ has no infinite direct sum of ideals. Thus $\overline{R}$ is a Goldie ring. Since every factor ring of R is a Goldie ring we have that R is a Noetherian ring by Camillo's theorem.

The next lemma essentially appears in Small [3].

LEMMA 2. *If every factor ring of R is an order in an Artinian ring then each ideal of R has no embedded prime ideals.*

Proof. R is a Noetherian ring by Lemma 1. Let I be an ideal of R and let $P_1, \ldots, P_t$ denote the associated prime ideals of I . Let $\overline{R} = R/I$ and $\overline{P}_j = P_j/I$ for $j = 1, \ldots, t$. Note that $\overline{P}_1, \ldots, \overline{P}_t$ are the associated prime ideals of zero in $\overline{R}$ and, as such, no $\overline{P}_j$ contains a regular element of $\overline{R}$. If $P_i \subsetneq P_k$ for some $i \neq k$ then $\overline{P}_i \subsetneq \overline{P}_k$ and if Q denotes the Artinian ring of quotients of $\overline{R}$, then $\overline{P}_i Q \subsetneq \overline{P}_k Q \subsetneq Q$ since no $\overline{P}_j$ contains a regular element. Since $\overline{P}_i Q$ and $\overline{P}_k Q$ are prime ideals of Q , and since all prime ideals of an Artinian ring are maximal, we have a contradiction. Thus I has no embedded primes.

LEMMA 3. *Let R be a local Noetherian ring such that each ideal has no embedded primes. If R is not an Artinian ring then R is a one-dimensional domain.*

Proof. Let M be the maximal ideal of R and suppose that P is a prime ideal such that $P \subsetneq M$. Let $k \in \mathbb{N}$ and consider $I = M^k \cap P$. Since I has no embedded primes, we have $I = P$ for otherwise the primary ideal M^k would be contained in P and so M would be equal to P . Thus $P \subseteq M^k$ for all $k \in \mathbb{N}$ and $P \subseteq \bigcap_{k=1}^{\infty} M^k$. But $\bigcap_{k=1}^{\infty} M^k = (0)$ by the Krull intersection theorem. Hence R is a one-dimensional domain.

LEMMA 4. *Let $(0) = Q_1 \cap \ldots \cap Q_s$ be the irredundant decomposition of (0) in a Noetherian ring R in which each ideal has no embedded*

primes. Let Q_i *be* P_i-*primary and suppose that* P_i *is a non-maximal prime ideal for* $i = 1, \ldots, t$ *and that* P_i *is a maximal ideal for* $i = t+1, \ldots, s$. *Then for* $1 \leq i < j \leq s$, Q_i *and* Q_j *are comaximal ideals and for* $1 \leq i \leq t$, $Q_i = P_i$. *Furthermore, for* $1 \leq i \leq t$, R/P_i *is a one-dimensional domain.*

Proof. For $1 \leq i < j \leq s$, assume that the maximal ideal M contains both Q_i and Q_j . The localization R_M inherits the property that each ideal has no embedded primes and so by Lemma 3, $Q_i = P_i = P_j = Q_j$ or one of P_i or P_j is M , in which case we have a containment relation between P_i and P_j . In either case we have a contradiction and so Q_i and Q_j are comaximal. If $i \leq t$ then $P_i \subseteq M$ for some maximal ideal M and again Lemma 3 implies that $P_i R_M$ is the zero ideal in R_M and so $P_i = Q_i$, and R/P_i is a one-dimensional domain.

We now have the necessary ingredients for our main result.

THEOREM. *Let* R *be a commutative ring. Then every factor ring of* R *is an order in an Artinian ring if and only if* R *is a finite direct sum of one-dimensional Noetherian domains and Artinian local rings.*

Proof. Since a proper factor ring of a one-dimensional Noetherian domain is Artinian, and since the ring of quotients of a direct sum is the direct sum of the rings of quotients of each factor, it follows that every factor ring of a direct sum of one-dimensional Noetherian domains and Artinian rings is a ring of this form, and thus is an order in an Artinian ring.

Conversely, suppose that every factor ring of R is an order in an Artinian ring. By Lemma 1, R is a Noetherian ring and each ideal of R has no embedded primes by Lemma 2. We apply Lemma 4 to obtain a primary decomposition of the zero ideal: $(0) = P_1 \cap \ldots \cap P_s \cap Q_1 \cap \ldots \cap Q_t$, where each P_i is a non-maximal prime ideal and each Q_j is a primary ideal whose associated prime ideal is maximal. (Either s or t may be 0 .) In this decomposition each pair of ideals is comaximal and R/P_i is

a one-dimensional domain for $i = 1, \ldots, s$. Thus

$$R \simeq R/P_1 \oplus \ldots \oplus R/P_s \oplus R/Q_1 \oplus \ldots \oplus R/Q_t$$

where each R/P_i is a one-dimensional domain and each R/Q_j is a local Artinian ring.

We observe that if R is a ring for which every proper factor ring is an order in an Artinian ring then R may not itself be an order in an Artinian ring. An example of such a ring, due to Kaplansky, appears in Small [3].

EXAMPLE. Let F be a field and consider the power series ring $F[[x, y]]$ in two indeterminants. Set $R = F[[x, y]]/(x^2, xy)$. Then R is its own ring of quotients but R is not Artinian. Every proper factor ring of R is itself Artinian except R/I where $I = (x)/(x^2, xy)$. But $R/I \simeq F[[x, y]]$ which is a domain and so R/I does have an Artinian ring of quotients.

References

[1]   Victor P. Camillo, "Commutative rings whose quotients are Goldie", *Glasgow Math. J.* 16 (1975), 32-33.

[2]   I.S. Cohen, "Commutative rings with restricted minimum condition", *Duke Math. J.* 17 (1950), 27-42.

[3]   Lance W. Small, "Orders in Artinian rings", *J. Algebra* 4 (1966), 13-41.

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