Coaxial Circles and Conics.

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The following notes are intended to introduce a simple method of treating elementary geometrical conics, and at the same time to supply a missing link in the chain of continuity between Euclidean geometry and the modern methods of treating the conics, which at present are treated more as different subjects than as a continuous whole.

Figure 12.

§ 1. Let Xx be the line of centres of the coaxial system, whose common points are S and S_1, and the radical axis of the system orthogonal to the first; then S and S_1 are common inverse points to the second system. Let F be the centre of any circle of the second system, and x the centre of any circle of the first. The tangents to x, or any circle of the x-system, from F are then of constant length, being always equal to the radius, R, of F. Let Ff_1 and Ff_2 be the two tangents from F to x, QSP and $Q'S_1P'$ the tangents to x at S and S_1, cutting the tangents to x from F in P, Q, P′, Q′. Then, since Pf_1 and PS are tangents to x from P, $Pf_1 = PS$. Therefore $PF - Pf_1 = PF - PS = R$ and $QF - QS = R$. Hence the locus of P or Q is a curve the difference of any point on which from two fixed points, F and S, is constant, that is an hyperbola.

Again, $P'f_1$ and $P'S_1$, being also tangents to x from P′, are equal. Therefore $P'F + P'S = P'F + P'f_1 = R$, and the locus in this case is an ellipse.

This gives a very simple method of tracing either curve. For x may be any point on Xx and, x being the centre of the circle to which SP or S_1P' is a tangent, xSP, or xS_1P', is always a right angle. Take therefore any point x_a; join x_aS; draw at S a line at right angles to x_aS and, with x_aS as radius, mark f_a on the circumference of F; draw Ff_a cutting SP_a in P_a, and so on.

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FIGURE 12.

§ 2. Let PM be the perpendicular from P on the radical axis, Xx, of the F-system, and PK a tangent to F from P; then, by a well-known property of coaxial circles, we have

$$2SF \,.\, PM = PK^2 - PS^2 \quad - \quad - \quad - \quad - \quad (1)$$

S being a point circle of the F-system.

Now $\qquad PK^2 = (Pf_1 + R)^2 - R^2$

and $\qquad Pf_1 = PS.$

Putting these values in (1) we get

$$2SF \,.\, PM = R^2 + 2PS \,.\, R + PS^2 - R^2 - PS^2;$$

therefore $\qquad SF \,.\, PM = PS \,.\, R$ or $PM : PS = R : SF$.

Hence $CA : CS = PM : PS$ and, R and SF being fixed quantities, $PM : PS$ is a constant ratio. Thus we see that the radical axis is also the directrix.

For the ellipse, the proof is slightly different. P′ being inside the circle F, take P′K′ (not shown in figure) at right angles to $P'f_1$;

then $\qquad 2S_1F \,.\, P'M' = P'K'^2 + P'S_1^2. \quad - \quad - \quad - \quad (2)$

Now $P'K'^2 = R^2 - (R - P'f_1)^2$ and $P'f_1 = P'S_1$.

Substituting these values in (2), we have

$$2S_1F \,.\, P'M' = R^2 - R + 2P'S_1 \,.\, R - P'S_1^2 + P'S_1^2$$

or, as before, $S_1F \,.\, P'M' = P'S_1 \,.\, R$, and therefore $P'M' : P'S_1 = R : S_1F$ a constant ratio.*

When the two tangents of the F-system are taken parallel to SS_1, the points of contact f_1 and f_2 will be on xX. But PM will then be equal to $Pf_1 = PS$ and therefore $PM : PS = 1$; so that in all cases the radical axis is the directrix. In the last case, that of the parabola, the orthogonal, or director circle, coincides with the radical axis, F being at infinity.

FIGURE 12.

§ 3. The line xP, or xP′, from the centre x to the point of contact, is the tangent to the curve at P, or P′. Since it bisects S_1f_1 at

* See the appended proof that $2S_1F \,.\, P'M' = P'K'^2 + P'S_1^2$.

right angles, it is the locus of points from which equal lines can be drawn to S_1 and f_1. Taking, therefore, any point P_1 on it, we have $P_1S_1 = P_1f_1$. But $P_1f_1 + P_1F$ is greater than R, and therefore greater than $P'S_1 + P'F$. Hence no other point on the line xP', except P', is on the curve.

A similar proof applies to the hyperbola and the parabola.

FIGURE 12.

§ 4. The proofs of the following standard propositions become extremely simple by this method.

(i) The tangent to a conic bisects the angle between the focal distances of the point of contact. For FP and SP are tangents to x and therefore xP bisects the angle between them.

(ii) Tangents at the ends of a focal chord meet on the directrix. For they both pass through the centre of x which is on the directrix.

(iii) The intercept on the tangent between P and the directrix subtends a right angle at the focus. For SP is a tangent and xS a radius at S to x; hence xSP is a right angle.

(iv) Perpendiculars on the tangents to a conic from the focus cut the tangents on a fixed circle. For Sf_1 is always bisected at right angles by the tangent. The cut a is therefore always the mid-point of the line from S to the circumference of F, and hence the locus of a is a circle whose radius is $\frac{1}{2}$R and whose centre C is the mid-point of SF. (See Mackay's *Euclid*, Appendix I., Prop. 6.)

(v) The normal at P cuts the axis at G so that

$$SG : SP = SP : PM.$$

For, since PG is parallel to Sf_1,

$$SG : Pf = FS : R = SP : PM \text{ and } Pf_1 = SP.$$

Hence

$$SG : SP = SP : PM.$$

The proofs of many other standard theorems can be much simplified in the same way.

FIGURE 12.

§ 5. Since P is clearly the pole of Sf_1 to x, the conic is the locus of the poles of the varying chord Sf of the varying circle x.

FIGURE 13.

§6. In figure 12 it is evident that, S_1 being inside F, S_1P' will always cut Ff internally to F; hence P′ will always be internal to F, and the curve is therefore closed and finite, lying wholly inside the circle F. But when S is external to F, the intersection of Ff, and SP will move away from f_1 as the angle FPS becomes less, *i.e.*, as x increases in radius; and when Sf_1 becomes the tangent to F from S, it is evident that FP and SP, being at opposite ends of the diameter Sf_1 of the circle x, will be parallel, and therefore xP will also be parallel to them. The point P will then have no existence or be at infinity, and, however far xP be produced, it can never meet the curve; for FP and SP can never meet. It is evident from fig. 12, however, that P continually approaches the asymptote; for the point a continually approaches x until it coincides with it, when Sf_1 becomes the tangent to F. When Sf_1 becomes the tangent to F, it is bisected by Xx, and xP_∞, being parallel to Ff_1, passes through C, the mid-point SF; and $Cx = \frac{1}{2}R$. Hence the asymptote, directrix, and auxiliary circle all pass through x. Since a second tangent can be drawn to F from S, another asymptote exists, equally inclined to SS_1 but in an opposite sense of direction. Thus each hyperbola has two asymptotes symmetrically situated with regard to the axis and passing through the centre C.

FIGURE 14.

§7. We might infer from the symmetrical position of the asymptotes and focal radii in the last note that the curve would have a corresponding branch on the opposite side of C. But we have only to continue the process of construction of figure 12 to arrive at the locus of the second branch. For when x is taken at a greater distance from X than the cut of the tangent to F from S, then P_1 will fall on the tangent from S on the opposite side of S from P so that $$SP_1 = P_1f_1;$$
whence again $$P_1S - P_1F = Ff_1 = R$$
and $$SA' - A'F = Ff = R.$$
From this we derive $SA' = SA$ and $CA' = CA$;

showing that F has the same relation to the second branch that S has to the first, and that the asymptotes to the first branch are also asymptotes to the second.

§ 8. There being an infinite series of circles orthogonal to the x-system on each side of the radical axis, there will therefore be a double infinite series of conics, related in pairs, corresponding to the circles similar to the two discussed; an ellipse in every circle and a double-branched hyperbola to every two equal circles which have their centres equidistant from X.

FIGURE 12.

§ 9. The radius of the circle F is cut harmonically by the ellipse and hyperbola, that is, $FP'F_1P$ and $FQ'f_2Q$ are harmonic ranges. For the two tangents at S and S_1 meet on the directrix at Z, and F lies on the polar of Z to x; therefore Z lies on the polar of F to x; and hence FS_1OS is harmonic, ZO, the polar of F, passes through f_1 and f_2, and, the pencil $Z(FS_1OS)$ being harmonic, the ranges $FP'f_1P$ and $FQ'f_2Q$ are harmonic. Similarly $F(f_1Of_2Z)$ is harmonic, and therefore $ZQ'S_1P'$ and ZQSP are harmonic. Hence any line through F cutting the ellipse, hyperbola, and circle, is divided harmonically, and the tangents at S and S_1 are divided harmonically by the curve, the focus, and the directrix.

FIGURE 13.

§ 10. The range $FP'f_1P_\infty$ being harmonic and P_∞ being at infinity, $FP' = f_1P'$ and similarly $ZQ = SQ$. As commonly put in the theory of harmonics, this results from the equation

$$FP : f_1P' = FP_\infty : f_1P_\infty.$$

This is, however, not an equality; for the difference of FP_∞ and f_1P_∞ is clearly $Ff_1 = R$. But in view of the fact that the harmonic conjugate to a point in a segment changes its direction when the point passes from one side of the centre of the segment to the other, if we take $P_{1\infty}$ in an opposite sense from P_∞, we have an absolutely true equation $FP' : fP' = FP_\infty : f_1P_{1\infty}$ and $FP_\infty = f_1P_{1\infty}$. The two equalities $FP' = f_1P'$ and $ZQ = SQ$ can, however, be easily proved otherwise. For $FP' = f_1P'$, we have, when Sf_1 is the tangent from the inverse of the focus to the circle F, S_1f_1 is at right angles to the axis. Therefore xP', which is at right angles to S_1f_1 through its mid-point a, bisects Ff_1. For $ZQ = SQ$, we have that x is the mid-point of Sf_1 and f_1Z is parallel to xQ, both being at right angles to Sf_2, and f_1Z passes through f_2; therefore $ZQ = QS$.

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FIGURE 13.

§ 12. The following simple relations may be noted.

S_1f_1 is equal to the minor axis of the ellipse, and Ff_1, or R, is equal to the major axis.

S_1F, the distance between the foci, forms with S_1f_1 and Ff_1 a right-angled triangle; whence $R^2 - BB_1^2 = S_1F^2$. In the hyperbola $R^2 + BB^2 = SF^2$. Halving these lines, we have the usual forms

$$CA^2 - CB^2 = CS_1^2, \quad CA^2 + CB^2 = CS^2.$$

Again $FS_1 . FS = R^2$. Taking the mid-points of FS_1 or FS and SS_1, we have $CS . CX = CA^2$. Therefore Xx is the polar of S or S_1 to C the auxiliary circle.

§ 13. It is easy to extend these notes to show how the second focus and directrix can be found and a second coaxial system of circles along with them. The only other point, however, which need be noted just now is that there is one, and only one, circle which belongs to both directrices. Its centre is C, and the radius evidently is $\sqrt{CA^2 \pm CB^2}$ or $CR^2 = CA^2 \pm CB^2$. It is this circle which is sometimes called the director circle or orthocycle, and which is the locus of intersection of tangents at right angles to each other. It is the doubly orthogonal circle which has the directrices for radical axis and the foci and their inverses in common with the director circles F and S_1 or S and the systems to which they belong.

FIGURE 16.

§ 14. By varying the position of F, we obtain a clear view of how the curves of the three classes—ellipse, parabola, and hyperbola—are related to each other, taking two circles of the x-system and drawing the tangents to them at S, namely, SZ and SZ_1. (In the figure SZ_1 is taken at right angles to the axis, and therefore determines the latus rectum in each case.)

The tangents to X and x from F determine points P, P,′ P_1, P_1' on an ellipse greater or less as F is taken further from or nearer to S. At S we have a point circle as the limit. The ellipse increases as F is taken further and further from S, until we reach the parallel position at infinity, when the curve developes into a parabola and

F changes sides and appears as $-F$. The tangents from $-F$ then determine points Q, Q', Q_1, Q_1' on a hyperbola which, when $-F$ is taken at S_1, has the directrix Xx for its limit; for then the tangents at S intersect the tangents at S_1 on the directrix.

§ 15. The property of the parabola that $SA = AX$ has the corresponding property in the other conics that $SA = Af$, as Xx is in the case of the parabola the limit circle of the orthogonal circles of the F-system.

FIGURE 17.

§ 16. Being given the foci F and S, and R, either as the sum of the focal distances, or from $R : SF = PM : PS$, we can find the orthogonal system with its radical axis related to the conic as in § 1.

For let F be a circle of given radius and S_1 any point on Ff. Through the ends of any diameter of F, such as f_3f_4, draw chords through S_1; these cut F again in f_1 and f_2. Produce f_4f_1 to meet f_3f_2 in S_2. Then, because of the right angles at f_1 and f_2, $S_1f_1S_2f_2$ are cyclic and Fx bisects the angles f_1Ff_2 and $f_1S_2f_2$. But half these angles equals a right angle; for they are equal to $f_2f_4S_2 + f_2Sf_4$, and therefore Ff_1x and Ff_2x are right angles. Hence x is orthogonal to F, and xX, which is at right angles to FS_1, is the directrix. We might have taken S as any external point, such as the inverse of S_1, in which case the diameter would be that indicated by the line f_5f_6, making equal angles with the directrix but in an opposite sense to f_3f_4.

FIGURE 17.

§ 17. The locus of S_2, the intersection of f_4f_1 and f_3f_2, is the polar of S_1. For S_1SS_2 is a right angle and S is the inverse of S_1.

FIGURE 17.

§ 18. $P'S_1Q'$ is parallel to the diameter through the ends of which S_1f_1 and S_1f_2 pass.

For $$\angle P'S_1f_1 = P'f_1f_3 = f_1f_3F$$

and therefore the focal chord $P'S_1Q'$ and the diameter f_3f_4 are parallel.

FIGURE 17.

§ 19. If C be the auxiliary circle and xP′, xQ′ the tangents to a focal chord P′S$_1$Q′, the tangents cut C again at the ends of a parallel diameter. For S$a = af_1$, and aP′ is parallel to f_1f_3; therefore Sf_4 is bisected at c by the tangent and, since cac_1 and $ca_1'c_1$ are at right angles in the same semicircle, the tangent xa_1Q$_1$C$_1$ passes through the end of the diameter cCc_1 and, since aP′ is parallel to f_1f_4, cCc_1 is parallel to f_3f_4. This gives a very ready method of determining the point of contact of a given tangent. For let ac be the tangent; then SP parallel to Cc gives the required point, and a and c can by this always be found if R is given.

FIGURE 17.

§ 20. (a) xF bisects ∠ P′FQ′; for it bisects f_1Ff_2 and we know P′S$_1x$ and Q′S$_1x$ to be both right angles, hence xS bisects P′S$_1$Q′.

(b) Cx bisects P′Q′; for C is the mid-point of the base cc_1 of xcc_1 and P′Q′ is parallel to cc_1.

From (a) and (b), by aid of a generalisation of some elementary geometry of the circle we can get the general case of tangents from a point bisecting the focal angle of points of contact, and the line xC (or xy if y be the point on second direction) bisecting all chords of a conic parallel to PSQ or cc_1.

FIGURE 17.

§ 21. Since the tangent P′H is parallel to f_1f_4, and the normal P′J is parallel to f_1f_3, we have FH and FJ each equal FP′. Hence the tangent and normal at any point P′ can be determined very easily. For draw JH through F parallel to SP′; then a circle, with FP′ as radius, cuts this line in J and H which, on being joined to P′, give the required lines. This holds for conics generally, as nearly the whole of these notes do.

FIGURE 17.

§ 22. The following points are too obvious to require proof.

(1) The triangle xcc_1 has its sides half the parallel sides of S$_2f_4f_3$.

(2) The focus is the orthocentre or polar centre of both triangles.

(3) The cut of the polar of the inverse of the focus to F and C determines the polar radius Sk and Sk_1, that is, a perpendicular to FS_1 at S_1 cuts F and C in k and k_1 and Sk and Sk_1; this, in the case of $S_1f_4f_3$, is equal to the minor axis BB_1 and, in the case of xcc_1, half the minor axis or CB.

(4) xcc_1 is a self-conjugate triangle. For cc_1 is the polar of x, c_1x is the polar of c and c is the polar of c_1. So that any triangle is self conjugate.

FIGURE 17.

§ 23. Since $CS = \frac{1}{2}FS$, $SX = \frac{1}{2}SS_1$, $CA = \frac{1}{2}R$, $FS \, . \, FS_1 = R^2$, we have $CS \, . \, CX = CA^2$ and therefore xX is the polar of S to C.

If the tangent at P′ were produced to cut the axis in T, and a perpendicular from P′ cuts the axis in N, then, since P′, N, S_1, a are cyclic, cCS_1a are also cyclic and, C being the centre and Ca a chord cutting the diameter CN in T, we have

$$CN \, . \, CT = CA^2 = CSCX.$$

Hence $TaPc$ is harmonic and therefore, if NP cut C in t, tT is the tangent to C from T; or, in the case of the hyperbola, if the perpendicular from T cut C in t, tN is the tangent from N to C.

FIGURE 17.

§ 24. If through c and c_1 lines cY and c_1Y be drawn parallel to xc_1 and xc, then yY, perpendicular to SF, is the second directrix. For $YF = xS$ and so on, the position of y being identical with that of x to every line in the figure, if xcc_1 were revolved round C till c coincided with c_1.

FIGURE 17.

§ 25. Cc and Cx are conjugate diameters and are conjugate lines to the polar of circle xcc_1. For, by § 19 (a), Cx bisects P′Q′ which is parallel to cc_1.

Again, since x is the pole of cc_1 and C the pole of xX, the pole of Cx is the point R in which Cc_1 meets xX. But $Sc \, . \, Sx = CB^2$, or Sx is the polar circle of cc_1x. Hence cc_1 and Cx are conjugate lines to this circle.

FIGURE 15.

Addenda. Proof that, when P is a point inside a circle, S and S_1 two inverse points, and XM the radical axis, then

$$2PM \, . \, FS = PK^2 + PS^2,$$

PM being the perpendicular on the radical axis and PK the semi-chord at right angles to FP.

Let F be the centre of the circle and S and S_1 the two inverse points which determine XM. Draw PM, PE and PK perpendicular to XM, FX, and Pf respectively;

then since $$FX^2 - SX^2 = R^2$$

and $$FX = PM + FE, \quad SX = PM - ES,$$

we have $$(PM + FE)^2 - (PM - ES)^2 = R^2$$

$$2PM(FE + ES) + FE^2 - ES^2 = R^2$$

$$2PM \, . \, FS = R^2 - FE^2 + ES^2.$$

Now $$FE^2 - ES^2 = FP^2 - PS^2.$$

Substituting we get $2PM \, . \, FS = R^2 - FP^2 + PS^2$

and therefore $$2PM \, . \, FS = PK^2 + PS^2$$

which reduces to $PM \, . \, FS = R \, . \, PS$ when Pf = PS.