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ON JAMES' QUASI-REFLEXIVE BANACH SPACE AS A BANACH ALGEBRA

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1. Introduction. In [**4**] and [**5**], R. C. James introduced a non-reflexive Banach space J which is isometric to its second dual. Developing new techniques in the theory of Schauder bases, James identified J^{**}, showed that the canonical image of J in J^{**} is of codimension one, and proved that J^{**} is isometric to J.

In Section 2 of this paper we show that J, equipped with an equivalent norm, is a semi-simple (commutative) Banach algebra under pointwise multiplication, and we determine its closed ideals. We use the Arens multiplication and the Gelfand transform to identify J^{**}, which is in fact just the algebra obtained from J by adjoining an identity.

In Section 3, we show that the multiplier algebras of J and of J^{**} can be identified isometrically and isomorphically with the Banach algebra J^{**}. Throughout the paper, we have tried to use a minimum of basis theory, exploiting instead the multiplication on J. From this point of view, the choice of the operator norm, where J^{**} is regarded as the multiplier algebra of J, is the most natural one. This approach also gives a basis-free characterization of the multipliers on J. Indeed, the definition of "multiplier" (after [**6**]) makes no assumption of continuity or linearity and no assumption that the multiplier coincides with multiplication by a sequence, although all these properties follow immediately from this characterization.

Section 4 is devoted to the characterization of the automorphism group $\mathrm{Aut}(J)$ of J. We show that every automorphism of J is bounded, that each automorphism corresponds to a permutation of the natural numbers $\mathbf{N}$, and that the only automorphism of norm less than $\sqrt{2}$ is the identity. Moreover, a permutation σ of $\mathbf{N}$ induces an automorphism of J if and only if σ and σ^{-1} satisfy a certain non-overlapping, or non-mixing, condition with regard to finite subsets of $\mathbf{N}$. This section also contains a discussion of the shift operator and of James' map [**5**] of J^{**} onto J. This latter map is not an isometry when J and J^{**} have the operator norm.

In Section 5 we discuss topological properties of the group $\mathrm{Aut}(J)$. We show that every automorphism of J is the strong operator limit of a sequence of automorphisms induced by permutations moving only finitely many integers. We provide the permutation group $\mathscr{S}(\mathbf{N})$ with a metric

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which makes it a complete separable topological group. The topology induced by this metric coincides on bounded subsets of $\mathrm{Aut}(J)$ with the strong operator topology, and the group

$$\mathscr{F}(\mathbf{N}) = \{\alpha \in \mathscr{S}(\mathbf{N}): \quad \alpha(i) \neq i \text{ for only finitely many } i\}$$

is dense in $\mathrm{Aut}(J)$ and in $\mathscr{S}(\mathbf{N})$. Since the norm on J may be regarded as being induced by a certain generating set for $\mathscr{F}(\mathbf{N})$, the results of Sections 4 and 5 are a reflection of the close relationship between $\mathrm{Aut}(J)$ and the norm on J. Finally, we obtain a characterization of the strong operator relatively compact subgroups of $\mathrm{Aut}(J)$.

Our notation will generally follow that of [**7**]. We let $l_\infty = l_\infty(\mathbf{N})$ denote the Banach *-algebra of all bounded sequences of complex numbers, and c_0 the closed ideal in l_∞ consisting of all null sequences. We write l for the identity of l_∞, δ_m for the characteristic function of the singleton $\{m\}$, χ_A for the characteristic function of a set $A \subset \mathbf{N}$, and χ_m for $\sum_{i=1}^m \delta_i$. We shall also find it convenient to have $\chi_0 = 0$ and $\chi_\infty = 1$. If X is a Banach space, we use X^* for the dual of X, span S for the linear span of a subset S of X, $\mathscr{B}(X)$ for the Banach algebra of all bounded linear maps of X into X, and (when X is complex) $X_{\mathbf{R}}$ for the real part of X. In a vector lattice, we write x^+ and x^- for the positive and negative parts of x. Throughout this paper, we use the term "projection" to mean an idempotent element of an algebra.

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2. J and J^{**} as Banach algebras. If $\mathscr{F} = \{p_1 < p_2 < \ldots < p_k\}$ is a finite subset of $\mathbf{N}$, and if $a \in l_\infty$, put

$$N(a,\mathscr{F}) = 2^{-1/2}\left[\sum_{i=1}^{k-1} |a(p_{i+1}) - a(p_i)|^2 + |a(p_k) - a(p_1)|^2\right]^{1/2},$$

and let $N(a)$ be the (possibly infinite) supremum of $\{N(a,\mathscr{F})\}$, where $\mathscr{F}$ ranges over all such finite subsets of $\mathbf{N}$. It is well known (and easy to verify) that if $a \in l_\infty$ and $N(a) < \infty$, then a converges. In particular, $N(a) = 0$ if and only if a is constant. Simple arguments show that if a has finite support, then $N(a) < \infty$, and that $N(\delta_m) = N(\chi_m) = 1$ for all $m \in \mathbf{N}$. The proof of the following proposition is routine and well known.

PROPOSITION 2.1. *The function N is a norm on the subspace $J = \{a \in c_0: N(a) < \infty\}$ of c_0 and a seminorm on the subspace $A = J + \mathbf{C}1$ of l_∞, and J is complete in N. If $a \in J$ and $\lambda \in \mathbf{C}$, then $N(a + \lambda 1) = N(a)$. In particular, $A = \{a \in l_\infty : N(a) < \infty\}$.*

Remark. If $a \in l_\infty$, then

$$N(a) \geqq \sup_{j,k} 2^{-1/2}[2|a(j) - a(k)|^2]^{1/2}.$$

Since $a(j) \to 0$ whenever $a \in J$, it follows that $N(a) \geqq \|a\|_\infty$ for all $a \in J$.

The space J was introduced by R. C. James in [**4**] and [**5**], and we shall refer to it as James' space. The sequence $\{\delta_m\}$ is a monotone Schauder basis for J [**5**]. That is, $N(a - \chi_m a) \to 0$ for all $a \in J$, and if $m \geqq k$, then $N(\chi_m a) \geqq N(\chi_k a)$. In fact, this inequality holds for all complex sequences a. Note that if $a \in J$, then $N(\chi_m a)$ converges monotonically up to $N(a)$.

LEMMA 2.2. *Let a be a complex sequence. Then $a \in A$ if and only if the sequence $\{N(\chi_m a)\}_{m \in \mathbf{N}}$ is bounded. In particular, if $a \in A$, then $\lim_m N(\chi_m a)$ exists.*

Proof. If $\mathscr{F} = \{p_1 < p_2 < \ldots < p_k\} \subseteq \mathbf{N}$, then for any $m \geqq p_k$, we have

$$N(a, \mathscr{F}) = N(\chi_m a, \mathscr{F}) \leqq N(\chi_m a).$$

Thus if $\{N(\chi_m a)\}_{m \in \mathbf{N}}$ is bounded, then $N(a) < \infty$, i.e., $a \in A$. If $a = a_0 + \lambda 1$ with $a_0 \in J$ and $\lambda \in \mathbf{C}$, then for any $m \in \mathbf{N}$, we have

$$N(a\chi_m) \leqq N(a_0\chi_m) + N(\lambda\chi_m) \leqq N(a_0) + |\lambda|.$$

Let $A_{\mathbf{R}}$ and $J_{\mathbf{R}}$ have the partial orderings (pointwise) which they inherit from l_∞.

PROPOSITION 2.3. *The (real) vector spaces $A_{\mathbf{R}}$ and $J_{\mathbf{R}}$ are vector lattices. Moreover, $N(a^+) \leqq N(a)$, $N(a^-) \leqq N(a)$, and $N(|a|) \leqq N(a)$.*

Proof. Let $a \in A_{\mathbf{R}}$, and let $a = a^+ - a^-$ be its decomposition in l_∞ into positive and negative parts. Since

$$|a^+(j) - a^+(k)| \leqq |a(j) - a(k)|$$

for all j and k in $\mathbf{N}$, it follows that $N(a^+) \leqq N(a)$. Similar arguments establish the other inequalities. In particular, a^+ and a^- lie in $A_{\mathbf{R}}$. Since $a \in c_0$ if and only if a^+ and a^- lie in c_0, we have a^+ and a^- in J whenever $a \in J$. It follows that $A_{\mathbf{R}}$ and $J_{\mathbf{R}}$ are vector lattices.

PROPOSITION 2.4. *The Banach space J is a subalgebra of c_0, and if $a, b \in J$, then $N(ab) \leqq 2N(a)N(b)$.*

Proof. If $x, y \in \mathbf{C}$, then

$$|x + y|^2 \leqq |x|^2 + 2 \operatorname{Re} x\bar{y} + |y|^2 \leqq 2(|x|^2 + |y|^2),$$

so for any i and j in $\mathbf{N}$, we have

$$\begin{aligned} |a(i)b(i) - a(j)b(j)|^2 &= |a(i)b(i) - a(i)b(j) + a(i)b(j) - a(j)b(j)|^2 \\ &\leqq 2[|a(i)b(i) - a(i)b(j)|^2 + |a(i)b(j) - a(j)b(j)|^2] \\ &\leqq 2[\|a\|_\infty{}^2|b(i) - b(j)|^2 + |a(i) - a(j)|^2 \, \|b\|_\infty{}^2]. \end{aligned}$$

Now let $\mathscr{F} = \{p_1 < p_2 < \ldots < p_k\} \subseteq \mathbf{N}$ be arbitrary. Taking i and j to be the appropriate elements of $\mathscr{F}$ and summing gives us

$$\begin{aligned} 2N(ab, \mathscr{F})^2 &\leqq 2[\|a\|_\infty{}^2(2N(b, \mathscr{F})^2) + (2N(a, \mathscr{F})^2)\|b\|_\infty{}^2] \\ &\leqq 8N(a)^2N(b)^2, \end{aligned}$$

since N dominates $\| \ \|_\infty$ on J. The proposition follows if we take suprema over $\mathscr{F}$.

COROLLARY. *The space A is a subalgebra of l_∞, and J is an ideal in A. If $a \in A$ and $ab = 0$ for all $b \in J$, then $a = 0$.*

Proof. Let $a, b \in A$, and put $a = a_0 + \lambda 1$ and $b = b_0 + \mu 1$, with a_0 and b_0 in J and λ and μ in $\mathbf{C}$. Then

$$N(ab) = N(a_0b_0 + \lambda b_0 + \mu a_0 + \lambda\mu 1) = N(a_0b_0 + \lambda b_0 + \mu a_0) < \infty,$$

since $a_0b_0 + \lambda b_0 + \mu a_0 \in J$. Thus $ab \in A$, so A is an algebra. If $a \in A$ and $b \in J$, then $ab \in c_0$ and $N(ab) < \infty$, so $ab \in J$. Thus J is an ideal in A. If $a\delta_m = 0$ for all $m \in \mathbf{N}$, then $a = 0$.

By the last proposition and its corollary, the function $\| \ \|$ defined by

$$\|a\| = \sup\{N(ab) \colon b \in J \text{ and } N(b) = 1\}$$

is a norm on A. Since $N(a) = \lim_m N(a\chi_m) \leqq \|a\| \leqq 2N(a)$ for all $a \in J$, $\| \ \|$ and N are equivalent on J. Note however that $\| \ \|$ is submultiplicative (i.e. $\|ab\| \leqq \|a\| \, \|b\|$) on J (or on A), while N is not. Indeed, if $1/2 < \theta < 1$ and $a = (1, \theta, 1, 0, 0, \ldots)$, then $N(a^2) > N(a)^2$.

Remarks. Note that $\|a\|$ may be computed by taking the supremum over all those b in J such that b has finite support and $N(b) \leqq 1$. If $a \in J_{\mathbf{R}}$ then

$$\|a^+\| \leqq 2N(a^+) \leqq 2N(a) \leqq 2\|a\|.$$

Similarly $\|a^-\| \leqq 2\|a\|$, and if $b = |a|$, then $\|b\| \leqq 2\|a\|$.

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LEMMA 2.5. *If $a \in l_\infty$, then $\|\chi_m a\|$ is monotone increasing in m. If $a \in A$, then $\|\chi_m a\|$ converges to $\|a\|$.*

Proof. If $a \in l_\infty$, if $b \in l_\infty$ has finite support, and if $k \leqq m$ in $\mathbf{N}$, then $N(\chi_k ab) \leqq N(\chi_m ab) \leqq N(ab)$. Taking suprema over those b with $N(b) \leqq 1$ gives $\|\chi_k a\| \leqq \|\chi_m a\|$ for any $a \in l_\infty$ and $\|\chi_m a\| \leqq \|a\|$ for any $a \in A$. Now fix $a \in A$, let $\epsilon > 0$, and choose b with finite support such that $N(b) \leqq 1$ and $\|a\| - \epsilon \leqq N(ab)$. For all sufficiently large m, we have $N(a\chi_m b) = N(ab)$, and it follows that $\|a\chi_m\|$ converges to $\|a\|$.

THEOREM 2.6. *With the norm $\| \ \|$, J and A are commutative Banach algebras with isometric involutions (given by complex conjugation). We have $\|1\| = 1$ and $\|\chi_m\| = 1$ for all $m \in \mathbf{N}$, and the sequence $\{\chi_m\}$ is an approximate identity in J.*

Proof. For all m, we have $1 = N(\chi_m) \leqq \|\chi_m\|$. Since $N(\chi_m a) \leqq N(a)$ for all $a \in J$, we have $\|\chi_m\| \leqq 1$ for all m. Since N and $\| \ \|$ are equivalent on J and $N(a\chi_m - a) \to 0$ for all $a \in J$, $\{\chi_m\}$ is an approximate identity for J with respect to either of these norms. The rest is immediate.

If $m \in \mathbf{N}$, let $\epsilon_m : l_\infty \to \mathbf{C}$ be evaluation at m. Clearly ϵ_m is a character on l_∞, hence on A and on J, and distinct integers give rise to distinct characters on each of these algebras. The (maximal) ideal J of A is the kernel of the character $a \to \lim_m a(m)$, which we denote by ϵ_∞. Let $\mathbf{N}$ have the discrete topology, and let $\mathbf{N}^* = \mathbf{N} \cup \{\infty\}$ be the one point compactification of $\mathbf{N}$.

PROPOSITION 2.7. *The algebras J and A are semisimple. We have* $\operatorname{spec}(J) = \{\epsilon_m : m \in \mathbf{N}\}$ *and* $\operatorname{spec}(A) = \{\epsilon_m : m \in \mathbf{N}^*\}$. *The map $m \to \epsilon_m$ is a homeomorphism of $\mathbf{N}$ onto* $\operatorname{spec}(J)$ *and of $\mathbf{N}^*$ onto* $\operatorname{spec}(A)$.

Proof. Suppose $\phi : A \to \mathbf{C}$ is multiplicative. Then on any idempotent of A, ϕ is either zero or one. In particular $\phi(\delta_m) = 0$ or 1 for any $m \in \mathbf{N}$, and $\phi(\delta_k + \delta_m) = 0$ or 1 if $k \neq m$. If ϕ is also linear (and hence bounded) on A, and if $\phi(\delta_m) = 1$, then $\phi(\delta_k)$ must be zero for all $k \neq m$. Thus if $\phi \neq 0$ on J, then

$$\phi(a) = \phi\left(\sum_k a(k)\delta_k\right) = a(m)$$

for some $m \in \mathbf{N}$, i.e., $\phi = \epsilon_m$ for some $m \in \mathbf{N}$. In particular,

$$\operatorname{spec}(J) = \{\epsilon_m : m \in \mathbf{N}\}.$$

If $\phi = 0$ on J but $\phi \neq 0$ on A, then

$$J \subseteq \operatorname{kernel}(\phi) \subsetneqq A,$$

so $J = \operatorname{kernel}(\phi)$, since the dimension of A/J is 1. Thus

$$\operatorname{spec}(A) = \{\epsilon_m : m \in \mathbf{N}^*\}.$$

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Since A has an identity, spec(A) is compact. Since $\phi \to \phi(\delta_m)$ is weak* continuous on J^* or on A^*, and since $\{0, 1\}$ is discrete, it follows that spec(J) is discrete and is discretely embedded in spec(A). By minimality of the one point compactification, spec(A) is homeomorphic to $\mathbf{N}^*$.

THEOREM 2.8. *Let I be a closed ideal in J. Then I contains a monotone increasing approximate identity $\{\phi_m\}_{m\in\mathbf{N}}$ with the following properties: each ϕ_m is a projection such that $\phi_m \leqq \chi_m$, and*

$$I = \{a \in J : \phi_m a = \chi_m a \text{ for all } m \in \mathbf{N}\}.$$

If $K = \{i \in \mathbf{N} : \phi_m(i) = 0$ for all $m \in \mathbf{N}\}$, then

$$I = \bigcap_{i\in K} \text{kernel } \epsilon_i = \{a \in J : a(i) = 0 \text{ for all } i \in K\}.$$

Proof. Let $m \in \mathbf{N}$. Since $(a(m)^{-1}\delta_m)(a) = \delta_m$ whenever $a(m) \neq 0$, we have $\delta_m \in I$ if and only if there exists $a \in I$ with $a(m) \neq 0$. Put

$$\phi_m = \sum_{\delta_k \in I, k \leqq m} \delta_k,$$

where we take $\phi_m = 0$ if no such δ_k lies in I. It follows readily that $\phi_m a = \chi_m a$ for all $a \in I$ and all $m \in \mathbf{N}$. Since $\chi_m a \to a$ in norm for all $a \in J$, this implies that $\{\phi_m\}$ is an approximate identity for I. Moreover, if $a \in J$ with $\phi_m a = \chi_m a$, then $a = \lim \phi_m a \in I$, since I is a closed ideal. Clearly each $a \in I$ satisfies $\phi_m a = \chi_m a \to a$ pointwise on $\mathbf{N}$, so each $a \in I$ vanishes on K. Suppose conversely that $a \in J$ and a vanishes on K. To complete the proof, it suffices to show that $\phi_m a = \chi_m a$ for all m. Let $i \in \mathbf{N}$. Since $\{\phi_m\}$ is increasing, $\phi_m(i)$ is eventually zero if and only if $\phi_m(i)$ is zero for all m, if and only if $i \in K$. If $\phi_k a \neq \chi_k a$ for some $k \in \mathbf{N}$, then there exists $i \leqq k$ such that $a(i) \neq 0$ and $\phi_k(i) = 0$. If any $\phi_{j+k}(i) \neq 0$, then $\delta_i \in I$, which contradicts $\phi_k(i) = 0$. Thus $\phi_m(i)$ is eventually zero, so $i \in K$. But then a fails to vanish on K. Thus we must have $\phi_m a = \chi_m a$ for all m, i.e., $a \in I$.

Thus each closed ideal in J is the intersection of the maximal ideals which contain it, and the closed ideals in c_0 are in one-to-one correspondence, via intersection with J, with those in J. By adjoining identities, one can show that the same assertions hold with J replaced by A and c_0 replaced by c, the algebra of all convergent sequences. Since we shall show in Theorem 2.11 that $A = J^{**}$, we have the corresponding assertions for J^{**} and c.

Theorem 2.8 also asserts that each closed ideal in J is the span of a subsequence of $\{\delta_i\}_{i=1}^{\infty}$. Hence, by a result of [**2**], each closed ideal in J is a complemented subspace of J.

Now consider the dual space J^*, where we compute the norm in J^* as a supremum over the unit ball of $(J, \| \ \|)$. If $a \in A$ and $\phi \in J^*$, define

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$\phi a \in J^*$ by $(\phi a)(b) = \phi(ab)$. It is easy to check that

$$\|\phi a\| \leqq \|\phi\| \, \|a\|$$

and that this action of A on J^* makes J^* into a Banach A-module. If $\phi \in J^*$ and $m \in \mathbf{N}$, then

$$\phi\delta_m = \phi(\delta_m)\epsilon_m,$$

so $\phi\chi_m = \sum_{i=1}^m \phi(\delta_i)\epsilon_i$ lies in the span of $\{\epsilon_m : m \in \mathbf{N}\}$. If $\phi \in J^*$, we write $\hat{\phi}$ for the function defined by $\hat{\phi}(m) = \phi(\delta_m)$, $m \in \mathbf{N}$.

Remark. The mapping $\phi \to \hat{\phi}$ is an injective norm decreasing linear map of J^* into l_∞. Since $\{\delta_m\}$ is a shrinking basis for J, i.e., the biorthogonal sequence $\{\epsilon_m : m \in \mathbf{N}\}$ is a basis for J^* [**4**], each $\hat{\phi}$ lies in c_0. If $a \in J$ and $\phi \in J^*$, then

$$(\phi a)\hat{}\,(m) = \phi(a\delta_m) = a(m)\hat{\phi}(m),$$

so $(\phi a)\hat{}$ is the product of ϕ and a in c_0.

PROPOSITION 2.9. *The annihilator in J^{**} of* spec(J) *is zero.*

Proof. By the Hahn–Banach Theorem, this assertion is equivalent to norm density of the span of spec(J) in J^*. But this follows from Proposition 2.7 and the fact that $\{\delta_m : m \in \mathbf{N}\}$ is shrinking.

Recall that the double dual J^{**} of J is a Banach algebra with the Arens multiplication [**1**, pp. 50–51]: if $F, G \in J^{**}$ and $\phi \in J^*$, put

$$(F\phi)(a) = F(\phi a), \; a \in J$$

and

$$(FG)(\phi) = F(G\phi).$$

In particular, we have

$$|(FG)(\phi)| \leqq \|FG\| \, \|\phi\| \leqq \|F\| \, \|G\| \, \|\phi\|$$

for all F and G in J^{**} and all $\phi \in J^*$. If $F \in J^{**}$, define $\hat{F} : \mathbf{N} \to \mathbf{C}$ by $\hat{F}(m) = F(\epsilon_m)$. Then $\hat{F} \in l_\infty$, and it is easy to check that $(FG)\hat{} = \hat{F}\hat{G}$ for all F and G in J^{**}, i.e., that $F \to \hat{F}$ is an algebra homomorphism. If $\phi \in J^*$ and $F \in J^{**}$, we also have $(F\phi)\hat{} = \hat{F}\hat{\phi}$. If we identify spec$(J)$ with $\mathbf{N}$ and the Gelfand transform on J with the identity map, then we have $\hat{a} = a$ for all $a \in J$.

The following proposition allows us to identify J^{**} as a sequence space.

PROPOSITION 2.10. *The mapping $F \to \hat{F}$ is a norm decreasing injective algebra isomorphism of J^{**} into l_∞. Let K be a bounded subset of J^{**}. Then the weak* topology coincides on K with the weak topology from* spec(J). *Moreover, $F \to \hat{F}$ is a weak* homeomorphism of K onto its image in l_∞.*

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Proof. The injectivity follows immediately from Proposition 2.9, and the map $F \to \hat{F}$ is clearly norm decreasing. We may thus identify J^{**} as an algebra with its image in l_∞, and $\text{spec}(J)$ with $\mathbf{N}$. Since the image of K is bounded, and since $F \to \hat{F}$ is clearly a homeomorphism when K and its image have the topology of pointwise convergence on $\mathbf{N}$, it suffices for us to check that in J^{**} and in l_∞ this topology coincides on bounded subsets with the weak* topology. Let $\bar{K}$ be the weak* closure of K, and let τ be the topology on $\bar{K}$ of pointwise convergence on $\mathbf{N}$. Since $\text{spec}(J)$ separates points of J^{**}, τ is Hausdorff. Since the identity map is weak* $-\tau$ continuous and $\bar{K}$ is weak compact, τ coincides with the weak* topology, as desired. A similar argument establishes the corresponding result for bounded subsets of l_∞.

COROLLARY. *The Banach algebra J^{**} is commutative and semi-simple and contains an identity, whose image in l_∞ is* 1. *If $F \in J^{**}$, then $\chi_m F \to F$ weak*, and $\|\chi_m F\| \to \|F\|$. In particular, we may identify A with a closed *-subalgebra of J^{**}.*

Proof. Commutativity and semisimplicity follow from the last proposition, or from [**3**, Theorem 3.7] and Proposition 2.9. Since J^* is separable, the weak* topology is first countable on bounded subsets of J^{**}. Since $\{\chi_m : m \in \mathbf{N}\}$ is uniformly bounded in J^{**}, it has a weak* convergent subsequence. Any limit point e of such a sequence is clearly an identity for J^{**}, since $\chi_m(\epsilon_k) = \epsilon_k(\chi_m) = 1$ whenever $m \geqq k$. But $\{\chi_m\}$ is pointwise monotone on $\mathbf{N}$, hence is weak* convergent in J^{**}. Thus $\chi_m \to e$ weak* in J^{**}. Consequently $\hat{e} = 1$, since $\chi_m \to 1$ weak* in l_∞. Since

$$\|\chi_m F\| \leqq \|\chi_m\| \, \|F\| = \|F\|,$$

$\overline{\lim}_m \|\chi_m F\|$ exists and is at most $\|F\|$. Since the Arens multiplication is weak* continuous in its first variable, $\chi_m F \to eF = F$ weak*, and hence

$$\underline{\lim}_m \|\chi_m F\| \geqq \|F\|.$$

Thus $\|\chi_m F\|$ converges to $\|F\|$ for all $F \in J^{**}$.

Now the norm in A of any $a \in A$ is also given by $\lim \|\chi_m a\|$, by Lemma 2.5. If $F \in J^{**}$, then $\chi_m F \in J$, so $\|\chi_m F\| = \|(\chi_m F)\hat{}\|$, and it follows that $\|F\| = \|\hat{F}\|$ whenever $\hat{F} \in A$. Thus we may identify A and $J + \mathbf{C}e$ (as Banach subalgebras of J^{**}).

THEOREM 2.11. *If $F \in J^{**}$, then $N(F) < \infty$. That is, $J^{**} = A$.*

Proof. For any $m \in \mathbf{N}$, we have

$$N(\chi_m F) \leqq \|\chi_m F\| \leqq \|F\|,$$

since $\|\chi_m F\|$ converges up to $\|F\|$. The result now follows from Lemma 2.2.

COROLLARY. *The Banach algebra J^{**} has an isometric involution given by $a^*(m) = \overline{a(m)}$, where $a \in J^{**}$ and $m \in \mathbf{N}$.*

Proof. From the definition of N, we have easily $N(b^*) = N(b)$ for all $b \in J$. It then follows directly from the definition of $\|\ \|$ that $a \to a^*$ is $\|\ \|$-isometric, since $(a^*b)^* = ab^*$.

Remarks. In [**5**], James gave an explicit description of a linear N-isometry between J and J^{**}. Since J has no identity, these spaces cannot be isomorphic as algebras. In Section 4, we shall show that James' map is no longer an isometry if J and J^{**} have the operator norm $\|\ \|$.

If $\phi \in J^*$, then $\hat{\phi}$ is positive in c_0 if and only if $\phi(a^*a) \geqq 0$ for all $a \in J$. Although each $a \in J_{\mathbf{R}}$ is the difference of two positive elements of $J_{\mathbf{R}}$, the corresponding decomposition in J^* does not hold [**8**, Remarks after Corollary 9, p. 198].

3. Multipliers of J and of J^{**}.

In this section we show that the multiplier algebras of J and of J^{**} are J^{**}.

A *multiplier* T on a Banach algebra A is a mapping $T:A \to A$ such that $a(Tb) = (Ta)b$ for all a and b in A, and a multiplier is necessarily bounded and linear [**6**, p. 13]. The multiplier algebra $M(A)$ is the subalgebra (with the inherited norm) of $\mathscr{B}(A)$ consisting of all multipliers on A. Since J^{**} is commutative and has an identity, the regular representation of J^{**} on itself is an isometric isomorphism of J^{**} onto $M(J^{**})$ [**6**, pp. 15–16]. By the corollary to Proposition 2.4 this representation also embeds J^{**} injectively into $M(J)$. Let us write (temporarily) $\|\ \|_{M(J)}$ for the norm in $M(J)$, and (as usual) $\|\ \|$ for the norm in J^{**}. Let $a \in J^{**}$. Since J^{**} and $M(J^{**})$ are isometric,

$$\|a\| \geqq \|a\|_{M(J)} \geqq \sup\{\|a\chi_m\| : m \in \mathbf{N}\} = \|a\|.$$

Thus the norms from J^{**}, $M(J^{**})$, and $M(J)$ all coincide on J^{**} if we identify J^{**} with its images in the other algebras.

Since we may identify $\operatorname{spec}(J)$ with $\mathbf{N}$, there is a natural one-to-one algebra homomorphism, say $T \to m_T$, of $M(J)$ into l_∞ such that for all $a \in J$, $Ta = m_T a$ [**6**, Theorem 1.2.2, p. 19].

THEOREM 3.1. *The representation*

$$a \to \textit{multiplication by } a$$

of J^{**} on J is an isometric algebra isomorphism of J^{**} onto $M(J)$ with inverse $T \to m_T$.

Proof. Since we have

$$N(m_T\chi_k) \leqq \|m_T\chi_k\| \leqq \|T\chi_k\| \leqq \|T\|_{M(J)}\|\chi_k\| = \|T\|_{M(J)},$$

the sequence $N(m_T\chi_k)$ is bounded, and m_T lies in J^{**}, by Theorem 2.11

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and Lemma 2.2. It is easy to check that the map $T \to m_T$ is an algebra homomorphism, and its image is clearly $J^{**} = J + \mathbf{C}1$. The rest of the theorem now follows from the fact that the composition $T \to m_T \to$ multiplication by m_T is the identity map on $M(J)$.

Remark. If $\chi \in J^{**}$ is idempotent, then the corresponding multiplier T is also idempotent. Moreover, T has norm one precisely when χ has the form $\chi_k - \chi_n$, where k and n lie in $\mathbf{N} \cup \{0, \infty\}$ with $n < k$.

THEOREM 3.2. *Let T be a multiplier operator with corresponding sequence $m_T = \{m_T(i)\} \in J^{**}$. Then*

(a) *T is compact if and only if $m_T \in J$;*

(b) *$T = \lambda I + S$, where I is the identity, $\lambda \in \mathbf{C}$, and S is a compact multiplier.*

Proof. If $m_T \in J$, then T is the norm limit of the finite rank multipliers $\chi_n m_T$, and is hence compact. On the other hand, suppose $m_T \notin J$. Since $m_T(i)$ converges, there exist $\epsilon > 0$ and $n \in \mathbf{N}$ such that $k > n$ implies $|m_T(k)| > \epsilon$. But then $\{T\delta_i\}_{i \geqq n}$ has no norm convergent subsequence. This proves (a). Part (b) follows immediately from part (a) and Theorem 3.1.

Remarks. If $\{x_n\}$ is a Schauder basis for a Banach space, then a scalar sequence $\{b_n\}$ is said to be a *multiplier sequence* for $\{x_n\}$ if the convergence of $\sum a_n x_n$ implies that of $\sum b_n a_n x_n$. A basis is said to be *unconditional* if its space of multiplier sequences is isomorphic to l_∞ under the obvious map. If a basis is conditional it is generally difficult to identify its multiplier sequences, and it is well known that the unit vector basis $\{\delta_n\}$ for J is conditional. It follows, however, from Theorem 3.1 that the multipliers for the algebra J correspond to the multiplier sequences for the basis $\{\delta_n\}$, and these sequences are precisely the elements of J^{**}. We note that the definition of a multiplier on the algebra J is basis-free in the sense that it depends only on the multiplication in J.

4. Endomorphisms and automorphisms of J.

In this section we will be interested chiefly in maps which are algebra endomorphisms of J and of J^{**}. Any such map must take indempotents (i.e., characteristic functions) to idempotents. If, moreover, ψ is an idempotent in J, then $\|\psi\| \geqq N(\psi) \geqq \sqrt{2}$ unless ψ has the form $\chi_m - \chi_k$ for some non-negative integers m and k. These facts will enable us to show that no non-trivial automorphism of J or of J^{**} can have norm less than $\sqrt{2}$.

PROPOSITION 4.1. *Let $T: J \to J$ be a bounded linear map which takes $\{\delta_m : m \in \mathbf{N}\}$ into itself, let J have either the norm N or the norm $\| \ \|$, and let $\|T\|$ denote the operator norm of T acting on J. If $\|T\| < \sqrt{2}$, then for each $m \in \mathbf{N}$, there exists $k \in \mathbf{N}$ such that $\{T(\delta_m), T(\delta_{m+1})\} = \{\delta_k, \delta_{k+1}\}$.*

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Proof. Let J be equipped with the norm N. If $T(\delta_m)$ and $T(\delta_{m+1})$ are not adjacent, then

$$\sqrt{2} \leqq N(T(\delta_m) + T(\delta_{m+1})) \leqq \|T\|N(\delta_m + \delta_{m+1}) = \|T\|.$$

The proof in the case of $\| \ \|$ is identical.

COROLLARY. *Let T be a bounded linear map of J onto J which permutes $\{\delta_m : m \in \mathbf{N}\}$. If $\|T\| < \sqrt{2}$, then T is the identity map.*

Proof. Let σ be the permutation of $\mathbf{N}$ induced by T. If σ is not the identity, then there exists a least integer j such that $\sigma(j + 1) < \sigma(j)$. By the last proposition, $\sigma(j + 1) = \sigma(j) - 1$. But then $\sigma(j - 1)$, which must be adjacent to $\sigma(j)$, is $\sigma(j) + 1$. Since then $\sigma(j) < \sigma(j - 1)$, we have contradicted the minimality of j.

LEMMA 4.2. *Let α be an algebra endomorphism of J. Then α is monotone on the idempotents in J. If α is an automorphism, then α permutes $\{\delta_m : m \in \mathbf{N}\}$.*

Proof. The first assertion follows from the fact that the ordering on the idempotents in J is determined by the ring structure of J. If α is invertible, then α must take minimal non-zero idempotents to minimal non-zero idempotents, i.e., α must map $\{\delta_m : m \in \mathbf{N}\}$ into itself.

PROPOSITION 4.3. *If α is an automorphism of J, then α has a unique extension to an automorphism of J^{**}, and conversely every automorphism of J^{**} carries J into J. Every automorphism of J (or of J^{**}) is bounded.*

Proof. It is easy to check that if α is an automorphism of J, then $\alpha(a + \lambda 1) = \alpha(a) + \lambda 1$, where $a \in J$ and $\lambda \in \mathbf{C}$, defines an automorphism of J^{**}. Clearly this is the only linear extension of α to J^{**} which fixes the identity of J^{**}. If α is an automorphism of J^{**}, then for each $\phi \in \operatorname{spec}(J^{**})$, $\phi \circ \alpha \in \operatorname{spec}(J^{**})$. Thus α induces a map of $\operatorname{spec}(J^{**})$ into itself which is easily seen to be weak* continuous. Since α^{-1} is also an automorphism, this induced map is a homeomorphism. It therefore carries ϵ_∞ to ϵ_∞, since every other point of $\operatorname{spec}(J^{**})$ is isolated. But J = kernel of ϵ_∞, so α carries J into J.

To show α is continuous, it suffices to show α has a closed graph. Let $a_n \to a$ and $\alpha(a_n) \to x$ (in norm) in J. Then $a_n \to a$ and $\alpha(a_n) \to x$ pointwise on $\mathbf{N}$. For each $\phi \in \operatorname{spec}(J)$, $\phi \circ \alpha(a_n) \to \phi \circ \alpha(a)$, since $\phi \circ \alpha \in \operatorname{spec}(J)$. Thus $\alpha(a_n) \to \alpha(a)$ pointwise on $\mathbf{N}$, so $\alpha(a) = x$, and the graph of α is closed.

THEOREM 4.4. *If α is an automorphism of J or of J^{**}, and if $\|\alpha\| < \sqrt{2}$, then α is the identity map.*

Proof. Apply Lemma 4.2 and the corollary to Proposition 4.1.

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Remarks. If $a \in J$ and $m \in \mathbf{N}$, then for each automorphism α of J, we have

$$\alpha(\chi_m a^*) = \sum_{k=1}^{m} \overline{a(k)}\alpha(\delta_k) = \alpha(a)^*\chi_m.$$

Since $\chi_m a^* \rightarrow a^*$ and α is bounded, $\alpha(a^*) = \alpha(a)^*$. That is, every automorphism of J is a *-automorphism.

We remarked in Lemma 4.2 that every automorphism of J corresponds to a permutation of $\mathbf{N}$. Let σ be a permutation of $\mathbf{N}$, and let T_σ be the map of l^∞ which it induces. We now derive necessary and sufficient conditions for T_σ to be bounded on J.

We shall find it convenient to use an equivalent norm on J, defined as follows. Let $G = \{p_1 < p_2 < \ldots < p_{2n}\} \subset \mathbf{N}$ and define

$$M(a, G) = \left[\sum_{i=1}^{n} |a(p_{2i-1}) - a(p_{2i})|^2\right]^{1/2}.$$

Let

$$M(a) = \sup\{M(a, G): n, G = \{p_1 < \ldots < p_{2n}\}\}.$$

For integers m, n, we denote the interval $(\min(m, n), \max(m, n))$ by (m, n).

Definition 4.5. The set $G = \{p_1 < \ldots < p_{2n}\}$ is *non-overlapping* for σ if $i \neq j$ implies

$$(\sigma p_{2i-1}, \sigma p_{2i}) \cap (\sigma p_{2j-1}, \sigma p_{2j}) = \emptyset.$$

If G is non-overlapping for σ^{-1}, then both G and $\sigma^{-1}G$ may be used in the computation of $M(a)$. Specifically

PROPOSITION 4.6. *If G is non-overlapping for σ^{-1}, then*

$$M(a, \sigma^{-1}G) = M(T_\sigma a, G) \text{ for all } a \in J.$$

Proof. Since G is non-overlapping for σ^{-1}, the sum

$$\left[\sum_{i=1}^{n} |a(\sigma^{-1}p_{2i-1}) - a(\sigma^{-1}p_{2i})|^2\right]^{1/2}$$

is permissible in the computation of $M(a)$. We have, moreover,

$$M(T_\sigma a, G)^2 = \sum_{i=1}^{n} |T_\sigma a(p_{2i-1}) - T_\sigma a(p_{2i})|^2$$

$$= \sum_{i=1}^{n} |a(\sigma^{-1}p_{2i-1}) - a(\sigma^{-1}p_{2i})|^2 = M(a, \sigma^{-1}G)^2.$$

The main result concerning operators induced by permutations is

THEOREM 4.7. *T_σ is a bounded operator on J if and only if there exists*

$K > 0$ such that each $G = \{p_1 < \ldots < p_{2n}\}$ is the disjoint union of sets $G_1, \ldots, G_l$ with $l \leqq K$ such that

(1) *$p_{2i-1} \in G_k$ if and only if $p_{2i} \in G_k$,*

(2) *each G_k is non-overlapping for σ^{-1}.*

The proof of this theorem will be accomplished in several lemmas. The point of the non-overlapping condition is that it limits the shuffling of disjoint blocks of unit vectors δ_k. This will be made explicit in Lemmas 4.10 and 4.11. Lemma 4.8 provides the "if part" of Theorem 4.7.

LEMMA 4.8. *Let σ be a permutation of* **N**, *and suppose there exists K such that each $G = \{p_1 < \ldots < p_{2n}\}$ may be written as the disjoint union of sets $G_1, \ldots, G_l$, $l \leqq K$, satisfying* (1) *and* (2). *Then T_σ is a bounded algebra endomorphism of J.*

Proof. For $a \in J$, and $G = \{p_1 < \ldots < p_{2n}\}$,

$$\begin{aligned} M(T_\sigma a, G)^2 &= \sum_{i=1}^{n} |T_\sigma a(p_{2i-1}) - T_\sigma a(p_{2i})|^2 \\ &= \sum_{i=1}^{l} \left[\sum_{\{j: p_{2j} \in G_i\}} |T_\sigma a(p_{2j-1}) - T_\sigma a(p_{2j})|^2 \right] \\ &= \sum_{i=1}^{l} M(T_\sigma a, G_i)^2 \\ &= \sum_{i=1}^{l} M(a, \sigma^{-1} G_i)^2, \quad \text{by Proposition 4.6,} \\ &\leqq \sum_{i=1}^{l} M(a)^2 \leqq K M(a)^2. \end{aligned}$$

Taking the supremum over G yields $\|T_\sigma\| \leqq K^{1/2}$.

Definition 4.9. Let $\{m, n\}$, $\{p, q\}$ be pairs of integers. We say a *type* 1 *overlap* occurs if either (a) $(m, n) \subset (p, q)$ or (b) $(p, q) \subset (m, n)$. We say a *type* 2 *overlap* occurs if (a) and (b) fail, but $(m, n) \cap (p, q) \neq \emptyset$.

Lemma 4.10 estimates $\|T_\sigma\|$ in the case that σ^{-1} produces a chain of type 1 overlaps, and Lemma 4.11 estimates $\|T_\sigma\|$ if σ^{-1} produces many type 2 overlaps.

LEMMA 4.10. *Let σ be a permutation of* **N**, *let $G = \{p_1 < \ldots < p_{2n}\}$, and let $q_i = \sigma^{-1} p_i$. Suppose that $q_1 < q_3 < q_5 < \ldots < q_6 < q_4 < q_2$. Then $\|T_\sigma\| \geqq \sqrt{n}$.*

Proof. Define $a \in J$ by

$$a(i) = \begin{cases} 1 & i \leqq q_{2n-1} \\ 0 & i > q_{2n-1} \end{cases}.$$

Then $M(a) = 1$, yet

$$M(T_\sigma a, G)^2 = \sum_{i=1}^{n} |(T_\sigma a)(p_{2i-1}) - (T_\sigma a)(p_{2i})|^2$$

$$= \sum_{i=1}^{n} |a(q_{2i-1}) - a(q_{2i})|^2 = n.$$

Hence $\|T_\sigma\| \geqq \sqrt{n}$.

LEMMA 4.11. *Let σ be a permutation of* $\mathbf{N}$, *let*

$$G = \{p_1 < p_2 < \ldots < p_{2n}\},$$

and let $q_i = \sigma^{-1}p_i$. Suppose that each $(q_{2j-1}, q_{2j}), j \neq 1$, has a type 2 overlap with (q_1, q_2). Then $\|T_\sigma\| \geqq \sqrt{n/2}$.

Proof. We suppose without loss of generality that $q_1 < q_2$, and let $a \in J$ be defined by

$$a(i) = \begin{cases} 1 & q_1 \leqq i < q_2 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$M(a) = \begin{cases} 1 & q_1 = 1 \\ \sqrt{2} & q_1 > 1, \end{cases}$$

but $M(T_\sigma a, G) = \sqrt{n}$. Hence $\|T_\sigma\| \geqq \sqrt{n/2}$.

The next lemma provides the "only if part" of Theorem 4.7.

LEMMA 4.12. *Let σ be a permutation of* $\mathbf{N}$, *and suppose there exists a constant $k > 0$ and a set of integers $G = \{p_1 < p_2 < \ldots < p_{2n}\}$ such that whenever G is written as the disjoint union of sets G_i, $i \leqq i \leqq l$, satisfying (1) and (2) of Theorem 4.7 we have $l \geqq 2^{4k}$. Then*

$$\|T_\sigma\| \geqq \min(\sqrt{3k}, 2^{(k-1)/2}).$$

Proof. The proof depends on producing either a chain of $3k$ type 1 overlaps, in which case the first estimate holds, or a situation to which Lemma 4.11 applies, in which case the second estimate of $\|T_\sigma\|$ will be shown to hold. As the construction will show, these estimates may be significantly improved in some cases.

Let $\pi_i = \{p_{2i-1}, p_{2i}\}$, and let $q_i = \sigma^{-1}p_i$. Let $l(\pi_i) = |q_{2i-1} - q_{2i}|$, and let $>$ be any linear ordering of $\{\pi_i\}_{i=1}^{n}$ such that $\pi_i > \pi_j$ implies $l(\pi_i) \geqq l(\pi_j)$. We choose sets G_i, $1 \leqq i \leqq l$, inductively. Let π_1' be maximal, and put $\pi_1' \subset G_1$. Assume $\pi_i' \subset G_1$, $1 \leqq i \leqq j$, have been chosen so that $\pi_1' > \pi_2' > \ldots > \pi_j'$ and $\bigcup_{i=1}^{j} \pi_i'$ is non-overlapping for σ^{-1}. Let π_{j+1}' be the largest pair such that $\bigcup_{i=1}^{j+1} \pi_i'$ is non-overlapping for σ^{-1} and let $\pi_{j+1}' \subset G_1$. If no such π_{j+1}' exists, the construction of G_1 is completed. Assuming $G_1, \ldots, G_j$ have been chosen, satisfying (1) and (2),

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and that $\{\pi_i\}_{i=1}^n$ has not been exhausted, let π_1^{j+1} be the largest pair such that $\pi_1^{j+1} \not\subset G_i$, $i \leqq j$. Put $\pi_1^{j+1} \subset G_{j+1}$, and construct G_{j+1} in the same manner as G_1.

Then G is the disjoint union of sets $G_1, \ldots, G_l$ satisfying (1) and (2), so by hypothesis, $l \geqq 2^{4k}$. The important aspect of this construction follows from the use of the linear ordering on $\{\pi_i\}_{i=1}^n$, and is that if $i < j$ and $\pi' \subset G_j$, then there exists $\pi'' \subset G_i$ such that $\sigma^{-1}\pi'$ and $\sigma^{-1}\pi''$ overlap. The ordering also allows the construction of chains of type 1 overlaps. We use these observations repeatedly to construct a sequence of pairs $\{\pi_i'\}$ to which we may apply either Lemma 4.10 or Lemma 4.11.

To this end, let $\pi_1' \in G_{2^{4k}}$. If there exist $\pi_2', \ldots, \pi'_{2^{4k-1}}$ such that $\bigcup_{i=1}^{2^{4k-1}} \pi_i'$ satisfies the hypotheses of Lemma 4.11, we have

$$\|T_\sigma\| \geqq [2^{4k-2}]^{1/2},$$

and we are through. Otherwise, there exist $j > 2^{4k-1}$, and $\pi_2' \in G_j$ such that $\sigma^{-1}\pi_2'$ and $\sigma^{-1}\pi_1'$ have a type 1 overlap, and $l(\pi_2') > l(\pi_1')$.

Continuing, if there exist $\{\pi_i'\}_{i=3}^{2^{4k-2}+1}$ such that $\bigcup_{i=2}^{2^{4k-2}+1} \pi_i'$ satisfies the hypotheses of Lemma 4.11, we have

$$\|T_\sigma\| \geqq [2^{4k-3}]^{1/2},$$

and we are through. Otherwise, there exist $j > 2^{4k-2}$ and $\pi_3' \in G_1$ such that $\sigma^{-1}\pi_1'$, $\sigma^{-1}\pi_2'$, $\sigma^{-1}\pi_3'$ form a chain of type 1 overlaps.

Continuing as above for at most $3k$ steps, we produce either (i) a set $F = \{r_{2i-1}, r_{2i}\}_{i=2}^{2k}$ satisfying the hypotheses of Lemma 4.11 or (ii) a set $F = \{r_{2i-1}, r_{2i}\}_{i=1}^{3k}$ to which the argument of Lemma 4.10 applies. In the first case we have $\|T_\sigma\| \geqq \sqrt{2^{k-1}}$ and in the second $\|T_\sigma\| \geqq \sqrt{3k}$.

Notice now that Theorem 4.7 follows from Lemma 4.8 and the contrapositive of Lemma 4.12.

COROLLARY 4.13. *Let σ be a permutation of* $\mathbf{N}$. *The following are equivalent:*

(a) *T_σ is a bounded operator on J,*

(b) $\sup_m \|T_\sigma(\chi_m)\| < \infty$,

(c) $\sup_{m,n} \|T_\sigma(\chi_m - \chi_n)\| < \infty$.

Moreover, $T_\sigma \in$ Aut(J) *if and only if both T_σ and $T_{\sigma^{-1}}$ satisfy one of conditions* (a), (b), *and* (c).

Proof. (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are obvious. That (c) $\Rightarrow$ (a) follows from Lemma 4.12 and the proofs of Lemmas 4.10 and 4.11.

Remark. In particular, $T_\sigma \in$ Aut(J) if and only if there exists $K > 0$ such that for any projection $\chi \in J$ with $\|\chi\| = 1$ (or $N(\chi) = 1$), we have

$$\|T_\sigma(\chi)\| \leqq K \text{ and } \|T_{\sigma^{-1}}(\chi)\| \leqq K.$$

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Theorem 4.7 may also be rephrased to characterize the automorphism group of J. We summarize this in

COROLLARY 4.14. *A permutation σ of* **N** *corresponds to an automorphism of J if and only if there exists a constant $K > 0$ such that any*

$$F = \{p_1 < \ldots < p_{2n}\}$$

may be written as disjoint unions

$$F = \bigcup_{i=1}^{l} F_i = \bigcup_{i=1}^{k} G_i,$$

where

(a) $l, k \leqq K$,

(b) $p_{2i-1} \in F_j(G_j) \Leftrightarrow p_{2i} \in F_j(G_j)$, *and*

(c) *each $F_j(G_j)$ is nonoverlapping for $\sigma^{-1}(\sigma)$.*

Remarks. 1. There do exist permutations σ for which T_σ is bounded yet $T_{\sigma^{-1}}$ is unbounded as an operator on J. An example is provided by taking, for each n, and $1 \leqq k \leqq 2^n$

$$\sigma(2^n + k) = \begin{cases} 2^n + j & k = 2j - 1 \\ 3(2^{n-1}) + j & k = 2j. \end{cases}$$

Then, considering T_σ as an operator on (J, M), $\|T_\sigma\| \leqq 2$ by Theorem 4.7, yet $T_{\sigma^{-1}}$ is not bounded by Corollary 4.13. The range of T_σ is a dense subspace of J.

2. Theorem 4.7 may also be interpreted in terms of Schauder basis theory. A basis $\{x_n\}$ for a Banach space X is said to be *symmetric* if for each permutation σ of **N**, $\{x_n\}$ is equivalent to $\{x_{\sigma(n)}\}$, in the sense that a series $\sum a_n x_n$ converges if and only if $\sum a_n x_{\sigma(n)}$ converges. A basis is symmetric if and only if the operator T_σ defined by $T_\sigma x_n = x_{\sigma(n)}$ is bounded for each permutation σ of **N**. The sequence $\{\delta_n\}$ is known to be a Schauder basis for J, and is not symmetric. Theorem 4.7, however, describes the symmetry properties of $\{\delta_n\}$ by characterizing those permutations which correspond to bounded operators.

There exist isometric *-endomorphisms of J and of J^{**} which map these algebras properly into themselves. One such endomorphism is a shift operator S, which is defined as follows: if a is a sequence, put

$$(Sa)(1) = 0 \text{ and } (Sa)(i) = a(i - 1) \text{ if } i > 1.$$

Then S is an isometric *-isomorphism of l_∞ onto $l_\infty(2, 3, 4, \ldots)$, and it is easy to check that $N(Sa) = N(a)$ for all $a \in J$. Moreover, the inverse map $S^*: l_\infty(2, 3, 4, \ldots) \rightarrow l_\infty$ has a unique extension to a *-endomorphism of l_∞ which annihilates δ_1. We write S^* for this extension, and it is easy to check that $N(S^*a) \leqq N(a)$ for all $a \in J$. If follows that S and S^* map J into J, and J^{**} into J^{**}.

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PROPOSITION 4.15. *If* $a \in J^{**}$, *then* $\|Sa\| = \|a\|$ *and* $\|S^*a\| \leqq \|a\|$.

Proof. Let $b \in J$. Since $Sa(1) = 0$, we have $(Sa)(SS^*b) = (Sa)b$. It follows readily that

$$N((Sa)b) = N(S(aS^*b)) \leqq \|a\|N(b),$$

i.e., $\|Sa\| \leqq \|a\|$. On the other hand,

$$N(ab) = N(SaSb) \leqq \|Sa\|N(b),$$

so $\|a\| \leqq \|Sa\|$.

Now consider S^*a. Since $(SS^*a)b = (SS^*)(ab)$, we have

$$N((SS^*a)b) \leqq N(ab) \leqq \|a\|N(b),$$

i.e., $\|SS^*a\| \leqq \|a\|$. But $\|SS^*a\| = \|S^*a\|$, by the first assertion of the proposition.

Suppose now $L: J^{**} \to J$ is the map defined by $L(a) = a - (\lim a)1$. In [**5**], James showed that the composition $L \circ S$ is an isometry of J^{**} onto J, where J has the norm N. It follows immediately from the next proposition that $L \circ S$ cannot be an isometry when J has the norm $\|\ \|$ (and J^{**} has the norm induced by $\|\ \|$).

PROPOSITION 4.16. *Let* $0 < \epsilon < \frac{1}{2}$. *Let* $a = (1, 1 - \epsilon, 1, 1, 1, \ldots)$, *and let* $a' = L \circ S(a) = (-1, 0, -\epsilon, 0, 0, \ldots)$. *For* ϵ *sufficiently small, we have*

$$\|a\|^2 \geqq 1 + 2\epsilon^2 > \|a'\|^2.$$

Proof. Let $c = (1, 1 - \epsilon, 1, 0, 0, \ldots)$. Then

$$\begin{aligned}\|a\|^2 \geqq \|c\|^2 \geqq [N(c^2)/N(c)]^2 &= [1 + (1 - (1 - \epsilon)^2)^2]/[1 + \epsilon^2] \\ &= (1 + 4\epsilon^2 - 4\epsilon^3 + \epsilon^4)/(1 + \epsilon^2) \leqq 1 + 2\epsilon^2\end{aligned}$$

for all sufficiently small ϵ.

Now let $b \in J$ with $N(b) \leqq 1$. We shall show that $N(a'b)^2 \leqq 1 + \epsilon^2$, from which the proposition will follow immediately. Let $r = b_1$ and $\theta = b_3$. We may assume r is real and non-negative, and we write θ_1 and θ_2 for the real and imaginary parts of θ respectively. Since

$$2N(b)^2 \geqq r^2 + |\theta|^2 + |r - \theta|^2 = 2(r^2 + |\theta|^2 - r\theta_1),$$

we have

$$\text{(1)} \quad r^2 + |\theta|^2 \leqq 1 + r\theta_1.$$

Since $a'b = (-r, 0, -\epsilon\theta, 0, 0, \ldots)$, we have

$$N(a'b)^2 = \max\{r^2 + \epsilon^2|\theta|^2, \tfrac{1}{2}(|r - \epsilon\theta|^2 + \epsilon^2|\theta|^2 + r^2), |r - \epsilon\theta|^2\}.$$

Since the second of these quantities is the average of the other two, we have

$$N(a'b)^2 = r^2 + \epsilon^2|\theta|^2 - 2r\epsilon\theta_1 \text{ if } \theta_1 \leqq 0,$$

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and

$$N(a'b)^2 = r^2 + \epsilon^2|\theta|^2 \text{ if } \theta_1 \geqq 0.$$

Suppose $\theta_1 \leqq 0$. Then

$$N(a'b)^2 = r^2 + \epsilon^2|\theta|^2 - 2r\epsilon\theta_1 \leqq r^2 + |\theta|^2 - 2r\epsilon\theta_1 \leqq 1 + r\theta_1 - 2r\epsilon\theta_1,$$

by the inequality (1). But then

$$N(a'b)^2 \leqq 1 + (1 - 2\epsilon)r\theta_1 \leqq 1.$$

If on the other hand, $\theta_1 \geqq 0$, then

$$N(a'b)^2 = r^2 + \epsilon^2|\theta|^2 \leqq 1 + \epsilon^2,$$

so in any case $N(a'b)^2 \leqq 1 + \epsilon^2$.

5. Topological properties of the group Aut(J). In this section we first prescribe a scheme for associating to each $\alpha \in \mathscr{S}(\mathbf{N})$ a sequence $\{\alpha_n\}$ of elements of $\mathscr{F}(\mathbf{N}) = \{\alpha \in \mathscr{S}(\mathbf{N}):\alpha(i) \neq i$ for only finitely many $i\}$. We choose our scheme so that α_n will always converge pointwise on $\{\delta_m\}_{m\in\mathbf{N}}$ to α, and so that α will lie in Aut(J) precisely when the approximating sequences for α and for α^{-1} are uniformly bounded on J. We then introduce a topology on Aut(J) and use the approximating sequences described above to study this topology.

To avoid higher order subscripts, we shall in this section of the paper identify Aut(l_∞) and $\mathscr{S}(\mathbf{N})$ completely, writing σa in place of $T_\sigma a = a \circ \sigma^{-1}$ whenever $a \in l_\infty$ and $\sigma \in \mathscr{S}(\mathbf{N})$.

Let $\alpha \in \mathscr{S}(\mathbf{N})$. For each n, define $\alpha_n \in \mathscr{F}(\mathbf{N})$ as follows. Let $A_n = \{i \leqq n:\alpha(i) \leqq n\}$, and let $B_n = \{i \leqq n:\alpha(i) > n\}$. Define

$$\alpha_n(i) = \begin{cases} \alpha(i) & i \in A_n \\ i & i > n \\ f_n(i) & i \in B_n, \end{cases}$$

where f_n is the unique order preserving function from B_n onto $\{j \leqq n: j \neq \alpha(i)\forall i \leqq n\}$. We shall call $\{\alpha_n\}$ the sequence of *approximators* for α. Indeed, α_n converges pointwise on $\mathbf{N}$ to α. For suppose $n \in \mathbf{N}$ and $k \geqq \max\{\alpha(1), \ldots, \alpha(n)\}$. Then α maps $\{1, \ldots, n\}$ into $\{1, \ldots, k\}$, and so $i \leqq n$ implies $\alpha_k(i) = \alpha(i)$, and the pointwise convergence of α_k to α follows. Note in particular that the subgroup $\mathscr{F}(\mathbf{N})$ is pointwise dense in $\mathscr{S}(\mathbf{N})$.

We introduce some terminology which we will use in the proof of the following theorem. To each projection $\chi \in J$, there are associated unique sequences of nonnegative integers $\{p_i\}_{i=1}^n$ and $\{q_i\}_{i=1}^n$ with $q_i < p_{i+1} < q_{i+1}$ so that

$$\chi = \sum_{i=1}^n (\chi_{q_i} - \chi_{p_i}),$$

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where $\chi_0 = 0$. We shall refer to each summand $(\chi_{q_i} - \chi_{p_i})$ as a *block*, and denote by $g(\chi)$ the number of blocks in this decomposition. Here $g(\chi) = n$. Notice that for projections χ and ϕ, $N(\chi)^2 = g(\chi)$, and if $\chi \cdot \phi = 0$, we have $g(\chi + \phi) \leqq g(\chi) + g(\phi)$.

THEOREM 5.1. *If $\alpha \in \mathscr{S}(\mathbf{N})$ induces an automorphism of J, then $\{\alpha_n\}$ and $\{(\alpha^{-1})_n\}$ are uniformly bounded in norm, where we compute the norm in $\mathscr{B}(J)$.*

Proof. By symmetry it suffices to show that $\|\alpha_n\|$ is bounded, and by Corollary 4.13 it suffices to show that $\{N(\alpha_n(\chi_m)): n, m \in \mathbf{N}\}$ is bounded. Now from the definition of α_n it is clear that for $m \geqq n$, $\alpha_n(\chi_m) = \chi_m$, so that $N(\alpha_n(\chi_m)) = 1$. Thus we need only show that $\{N(\alpha_n(\chi_m)): m < n\}$ is bounded, and to do this we estimate $g(\alpha_n(\chi_m))$. Notice that

$$\begin{aligned}\alpha_n(\chi_m) &= \alpha_n(\chi_{A_n \cap \{1,\ldots,m\}} + \chi_{B_n \cap \{1,\ldots,m\}}) \\ &= \alpha_n(\chi_{A_n \cap \{1,\ldots,m\}}) + \alpha_n(\chi_{B_n \cap \{1,\ldots,m\}})\end{aligned}$$

and that

$$g(\alpha_n(\chi_m)) \leqq g(\alpha_n(\chi_{A_n \cap \{1,\ldots,m\}})) + g(\alpha_n(\chi_{B_n \cap \{1,\ldots,m\}})).$$

Now

$$\alpha_n(\chi_{A_n \cap \{1,\ldots,m\}}) = \chi_n \cdot \alpha(\chi_m),$$

so that

$$g(\alpha_n(\chi_{A_n \cap \{1,\ldots,m\}})) \leqq g(\alpha(\chi_m)) \leqq \|a\|^2.$$

Also, from the order-preserving nature of α_n on B_n, we see that adjacent blocks of $\alpha_n(\chi_{B_n \cap \{1,\ldots,m\}})$ are separated by a block of $\alpha(\chi_n)$. Hence

$$g(\alpha_n(\chi_{B_n \cap \{1,\ldots,m\}})) \leqq g(\alpha(\chi_n)) + 1 \leqq \|\alpha\|^2 + 1.$$

Thus $g(\alpha_n(\chi_m)) \leqq 2\|\alpha\|^2 + 1$,
which, by earlier remarks, implies that $\{\|\alpha_n\|\}$ is bounded.

THEOREM 5.2. *Let $\alpha \in \mathscr{S}(\mathbf{N})$. Then $\alpha_n \to \alpha$ pointwise on $\{\delta_m\}$. Suppose there is a net $\{\beta_\gamma\}$ in $\mathscr{S}(\mathbf{N}) \cap \mathscr{B}(J)$ such that $\beta_\gamma \to \alpha$ pointwise on $\{\delta_m\}$ and such that $\{\beta_\gamma\}$ is uniformly bounded in $\mathscr{B}(J)$. Then $\alpha \in \mathscr{B}(J)$ and $\beta_\gamma \to \alpha$ strong operator on J. In particular, $\alpha \in \mathscr{B}(J)$ if and only if $\{\alpha_n\}$ is uniformly bounded on J, in which case $\alpha_n \to \alpha$ strong operator on J.*

Proof. It was remarked earlier that α_n converges to α pointwise on $\{\delta_m\}$. If $a \in J$ and $\chi_m a = a$ for some m, there exists Λ such that $\gamma > \Lambda$ implies $\beta_\gamma(a) = \alpha(a)$. It follows that $\|\alpha\| \leqq \underline{\lim} \|\beta_\gamma\|$, so $\alpha \in \mathscr{B}(J)$. By an $\epsilon/3$-argument, $\{\beta_\gamma\}$ converges strong operator to α.

Remark. We may replace $\{\delta_m\}$ in Theorem 5.2 by $\{\chi_m\}$ or by the set of all projections in J, since any projection in J is a finite sum of the δ_m.

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COROLLARY 5.3. *Let* $\alpha \in \mathscr{S}(\mathbf{N})$. *Then* $\alpha \in \mathrm{Aut}(J)$ *if and only if the sequences* $\{\|\alpha_n\|: n \in \mathbf{N}\}$ *and* $\{\|(\alpha^{-1})_n\|: n \in \mathbf{N}\}$ *are bounded.*

Thus $\alpha \in \mathscr{S}(\mathbf{N})$ lies in $\mathrm{Aut}(J)$ if and only if α and α^{-1} are strong operator sequential limit points of automorphisms in $\mathscr{F}(\mathbf{N})$, i.e., of automorphisms in the subgroup of $\mathscr{S}(\mathbf{N})$ generated by cyclic permutations of the form $p_1 \to p_2 \to \dots \to p_n \to p_1$.

We topologize $\mathrm{Aut}(J)$ as follows. For each $m \in \mathbf{N}$, define a pseudometric d_m on $\mathscr{S}(\mathbf{N})$ by

$$d_m(\alpha, \beta) = \|\delta_{\alpha(m)} - \delta_{\beta(m)}\| = \|T_\alpha(\delta_m) - T_\beta(\delta_m)\|,$$

and put

$$d(\alpha, \beta) = \sum_{k=1}^{\infty} 2^{-k} d_k(\alpha, \beta).$$

Then d is a metric on $\mathscr{S}(\mathbf{N})$. Suppose $\alpha_n \to \alpha$ in $\mathscr{S}(\mathbf{N})$ with respect to d. Since $\{\delta_m\}$ is a norm-discrete subset of J, we have for each $m \in \mathbf{N}$ that $\alpha_n(m)$ is eventually equal to $\alpha(m)$. Thus the topology τ on $\mathscr{S}(\mathbf{N})$ induced by the metric d is just the topology of pointwise convergence, where $\mathbf{N}$ has the discrete topology. This topology is easily seen to be compatible with the group structure, so $\mathrm{Aut}(J)$ and $\mathscr{S}(\mathbf{N})$ are topological groups when equipped with d. By continuity of inversion, d is equivalent to the metric ρ, where

$$\rho(\alpha, \beta) = d(\alpha, \beta) + d(\alpha^{-1}, \beta^{-1}),$$

and it is easy to check that $\mathscr{S}(\mathbf{N})$ is ρ-complete. Thus we have the following result.

THEOREM 5.4. *There exists a metric* ρ *on* $\mathscr{S}(\mathbf{N})$, *inducing the topology* τ *of pointwise convergence, where* $\mathbf{N}$ *has the discrete topology, and satisfying the following conditions:*

1) $\mathrm{Aut}(J)$ *and* $\mathscr{S}(\mathbf{N})$ *are separable metric groups with respect to* ρ;
2) τ *coincides on norm bounded subsets of* $\mathrm{Aut}(J)$ *with the strong operator topology;*
3) $\mathscr{F}(\mathbf{N})$ *is* τ*-dense in* $\mathrm{Aut}(J)$ *and in* $\mathscr{S}(\mathbf{N})$.

Proof. Condition (3) is easy to verify (use for example the approximators discussed above) and condition (1) is then established as well. Condition (2) follows from Theorem 5.2.

Remarks. In the definition of the metric d above, we may replace $\{d_m\}$ by $\{\chi_m\}$ or by any enumeration of the set of all projections in J, and $\|\ \|$ by any equivalent norm, and we obtain an equivalent metric.

Although $\mathscr{S}(\mathbf{N})$ is ρ-complete, the following result shows that $\mathrm{Aut}(J)$ cannot be made complete in any metric which is equivalent to ρ.

THEOREM 5.5. *In the topology induced by the metric* d, $\mathrm{Aut}(J)$ *is of the first Baire category in itself.*

Proof. Notice that by Corollary 4.12

$$\mathrm{Aut}(J) = \{T_\sigma\colon \sup_m \|T_\sigma(\chi_m)\| < \infty \text{ and } \sup_m \|T_{\sigma^{-1}}(\chi_m)\| < \infty\}.$$

Furthermore, the values of $\|T_\sigma(\chi_m)\|$ form a discrete set $\{v_1 < v_2 < \ldots\}$, and

$$\mathrm{Aut}(J) = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} [\{T_\sigma\colon \|T_\sigma(\chi_m)\| \leqq v_n\} \cap \{T_\sigma\colon \|T_{\sigma^{-1}}(\chi_m)\| \leqq v_n\}].$$

Thus it suffices to show that the sets

$$A_n = \bigcap_{m=1}^{\infty} [\{T_\sigma\colon \|T_\sigma(\chi_m)\| \leqq v_n\} \cap \{T_\sigma\colon \|T_{\sigma^{-1}}(\chi_m)\| \leqq v_n\}]$$

are closed and nowhere dense. Since for each m the map $\alpha \to \alpha(\chi_m)$ is d-continuous, each A_n is closed.

Let $\alpha \in A_n$, and let α_k be the kth approximator of α. Then for $i > k$, $\alpha_k(\delta_i) = \delta_i$. Let $\beta_k \in \mathrm{Aut}(J)$ be induced by a permutation σ_k such that $\sigma_k(i) = i$ if $i \leqq k$ or $i > 3k$, but

$$\|\beta_k(\chi_{2k} - \chi_k)\| = \sqrt{k},$$

and let $\gamma_k = \beta_k\alpha_k$. Then $d(\gamma_n, \alpha) \to 0$, but

$$\|\gamma_k(\chi_{2k})\| \geqq \sqrt{k}.$$

Thus for $k > v_n^2$, $\gamma_k \notin A_n$, and it follows that A_n is nowhere dense.

Theorem 5.4 also allows us to describe the relatively compact subgroups of $\mathrm{Aut}(J)$. A subset S of $\mathscr{S}(\mathbf{N})$ is pointwise relatively compact in $\mathscr{S}(\mathbf{N})$ if and only if each $S\delta_m = \{\delta_{\alpha(m)}\colon \alpha \in S\}$ is norm-relatively compact, i.e., finite. If in addition $S \subseteq \mathrm{Aut}(J)$ and S is uniformly bounded in norm, then by Theorem 5.4, S is strong operator relatively compact if and only if S is pointwise relatively compact. By the uniform boundedness principle, any strong operator relatively compact subset of $\mathrm{Aut}(J)$ is uniformly bounded in norm. Thus a subgroup G of $\mathrm{Aut}(J)$ is strong operator relatively compact in $\mathrm{Aut}(J)$ if and only if G is uniformly bounded and each orbit $G\delta_m$ is finite.

Suppose for example $\{S_i\}_{i\in\mathbf{N}}$ is a disjoint partition of $\mathbf{N}$ into finite subsets of the form $\{m + 1, m + 2, \ldots, n\}$, and that for each i, G_i is a subgroup of the permutation group on S_i. Let G be the subgroup of $\mathscr{S}(\mathbf{N})$ which is generated by the G_i. Then G is pointwise relatively compact in $\mathscr{S}(\mathbf{N})$. If $\sigma \in G$, then we may write σ as a product $\sigma_1\sigma_2 \ldots \sigma_k$ with $\sigma_j \in G_{i_j}$ and $i_1 < i_2 < \ldots i_k$. (Note that $\sigma \in G_i$ and $\tau \in G_j$ imply

$\sigma\tau = \tau\sigma$ if $i \neq j$.) Thus

$$N(T_\sigma(\chi_m)) = N(T\sigma_j(\chi_m)),$$

where $m \in S_{i_j}$. By Corollary 4.13 G is strong operator relatively compact in Aut(J) if and only if the sequence defined by

$$B_i = \sup\{N(T_\sigma(\chi_m)) : \sigma \in G_i \text{ and } m \in \mathbf{N}\}$$

is bounded. This is the case, in particular, if the cardinalities of the S_i are bounded. Suppose on the other hand that these cardinalities are unbounded. For each $k \in \mathbf{N}$, choose i_k and σ_k in the permutation group on S_{i_k} such that $\|T_{\sigma_k}\| \geqq k$. If each G_{i_k} is the group generated by σ_k, then G is not norm bounded, and hence is not strong operator relatively compact.

Remarks. When each $S_i = \{2i - 1, 2i\}$ and each $G_i \cong Z_2$, G has cardinality $2^{\aleph_0}$, and in particular Aut(J) is uncountable. For each $\alpha \in$ Aut(J), $\| \ \| \circ \alpha$ is a Banach algebra norm on J^{**}, and by Theorem 4.4,

$$\| \ \| \circ \alpha = \| \ \| \circ \beta \Leftrightarrow \| \ \| \circ \alpha\beta^{-1} = \| \ \| \Leftrightarrow \alpha = \beta.$$

Thus J^{**} has uncountably many distinct (but equivalent) Banach algebra norms, each of which takes the value one at the identity.

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