ON THE DISJOINT PRODUCT OF IRREDUCIBLE REPRESENTATIONS OF THE SYMMETRIC GROUP

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1. Introduction. The results of the present paper can be interpreted (a) in terms of the theory of the representations of the *symmetric group*, or (b) in terms of the corresponding theory of the *full linear group*. In the latter connection they give a solution to the problem of the expression of an invariant matrix of an invariant matrix as a sum of invariant matrices, in the sense of Schur's Dissertation. D. E. Littlewood[1] has pointed out the significance of this problem for invariant theory and has attacked it via Schur functions, i.e. characters of the irreducible representations of the full linear group. We shall confine our attention here to the interpretation (a). Our results are explicit and yield the interpretation (b) merely by changing[2] brackets [ ] and multiplication $(\cdot)$ into brackets $\{\ \}$ and multiplication $(\times)$; moreover, they are self-contained from a group-theoretic point of view.

Consider the symmetric group S_{mn} on mn symbols and the subgroup

1.1 $$H = S_m \times S_m \times \ldots \times S_m, \qquad n \text{ factors},$$

which is the direct product of n factors S_m on n different sets of the mn symbols. An irreducible representation of H is necessarily the Kronecker product of irreducible representations of the n factors; such an irreducible representation of H *induces* a representation of S_{mn} which is reducible and of degree[3]

1.2 $$\frac{(mn)!}{(m!)^n}\, x_\alpha x_\beta \ldots x_\gamma\,,$$

where $x_\alpha, x_\beta, \ldots, x_\gamma$ are the degrees of the n irreducible representations $[\alpha]$, $[\beta], \ldots, [\gamma]$ of the S_m in 1.1. We shall assume in the sequel that such irreducible representations are always written in Young's orthogonal form.

Murnaghan[4] studied these representations of S_{mn} and gave a method for finding their irreducible components. He called such an induced representation the *direct product* of the n irreducible representations $[\alpha]$, $[\beta], \ldots, [\gamma]$. In a previous paper the author has shown[3] that this induced representation is associated with a skew Young diagram which has the n right diagrams $[\alpha]$, $[\beta], \ldots, [\gamma]$ as its disjoint constituents; the skew diagram may be denoted

1.3 $$[\alpha]\cdot[\beta]\cdot\ldots\cdot[\gamma], \qquad n \text{ factors},$$

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[1][2] and [3]. Cf. also [13] p. 110.

[2][8] section 3.

[3][7] and [8].

[4][5].

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where the order of writing the factors is immaterial. This suggests the term *disjoint product*, which designation will be used throughout this paper.

Having associated the induced representation under consideration with a skew Young diagram we have available a second method for finding the irreducible components, namely that based on the Littlewood-Richardson rule.[5] We may assume, then, that these irreducible components are known in any given case.

The problem to be considered here is a certain reduction of the disjoint product in the case where *all the disjoint constituents are the same*; 1.3 becomes

1.4 $$\Pi_n^{[\alpha]} = [\alpha] \,.\, [\alpha] \,.\, \dots \,.\, [\alpha]\,, \qquad n \text{ factors,}$$

and the reduction in question arises from the interchangeability of the n factors. As one might expect, there is a very close analogy here with Schur's celebrated derivation of the irreducible representations of order n of the full linear group.[6]

The notation $\{\alpha\}\otimes\{\beta\}$ was introduced by Littlewood to designate his "new multiplication" of Schur functions.[7] Restricting our attention to the symmetric group, we obtain the desired reduction of $\Pi_n^{[\alpha]}$ in the following form:

1.5 $$\Pi_n^{[\alpha]} = \sum_\beta x_\beta [\alpha]\odot[\beta]\,,$$

where β runs over all partitions of n, and the reducible component $[\alpha]\odot[\beta]$, which corresponds to Littlewood's $\{\alpha\}\otimes\{\beta\}$, appears with a frequency x_β. The degree of $[\alpha]\odot[\beta]$ is

1.6 $$\frac{(mn)!}{(m!)^n n!}\,(x_\alpha)^n x_\beta.$$

The frequency **f** with which a given irreducible representation $[\lambda]$ of S_{mn} appears as a component of $[\alpha]\odot[\beta]$ is obtained in **5** below and this leads to an immediate proof of Littlewood's Theorem of Conjugates.[8] Littlewood has given a number of methods for obtaining the reduction of $\{\alpha\}\otimes\{\beta\}$ or analogously of $[\alpha]\odot[\beta]$, but none lead to explicit results in the general case. The application of the method developed here is illustrated in **7** below.

Two additional remarks should be made: (*i*) The reason for not applying Schur's original procedure to the Kronecker product $\{\alpha\}^n$ is that the interpretation of the results must ultimately be based, as in Schur's case, upon the theory of the symmetric group. Basing the whole argument on this theory is of some intrinsic interest. (*ii*) While the results of this paper do lead to a systematic study of the structure of the representation $[\alpha]\odot[\beta]$, they do not provide a means of identifying the particular $[\alpha]\odot[\beta]$ to which a given irreducible component $[\lambda]$ of $\Pi_n^{[\alpha]}$ belongs. This identification is important in

[5] Applied to skew diagrams in [7].

[6] [11].

[7] In particular [3] Part II. Note that we use the symbol $\{\alpha\}$ to designate an irreducible representation of the full linear group and not merely its character or corresponding Schur function.

[8] [3] p. 359.

the application to invariant theory.[1] However, the definition of $[\alpha]\odot[\beta]$ as an induced representation in **6** points the way in which this refinement of the theory may be sought.

In concluding this introduction I should like to express my thanks to R. Steinberg and W. T. Sharp for reading the manuscript and suggesting, in particular, the insertion of section **6**.

2. The normalizer of *H*. The subgroup H, as defined in 1.1, has a normalizer in S_{mn} which can be written

2.1 $$\mathfrak{N}(H) = H + Hs^*_2 + \ldots + Hs^*_n.$$

If each of the n sets of m symbols be considered as an *ordered* set, then the element s^*_i of S_{mn} permutes the sets amongst themselves, preserving this order. The factor group $\mathfrak{N}(H)/H$ satisfies the following relation

2.2 $$\mathfrak{N}(H)/H \sim S^*_n \sim S_n,$$

and the s^*_i generate the subgroup S^*_n of $\mathfrak{N}(H)$. The normalizer $\mathfrak{N}(H)$ is not a direct product but its structure is sufficiently simple for us to obtain the necessary information concerning it.

With regard to the isomorphism between S_n and S^*_n, let us assume that an element s_i of S_n has ω_λ cycles of length λ so that

2.3 $$n = \omega_1 + 2\omega_2 + 3\omega_3 + \ldots + n\omega_n;$$

then the corresponding element s^*_i of S^*_n has $m\omega_\lambda$ cycles of length λ.

Given an irreducible representation

2.4 $$[\alpha]^n = [\alpha] \times [\alpha] \times \ldots \times [\alpha], \qquad n \text{ factors},$$

of H one naturally asks whether it can be extended to yield a representation of $\mathfrak{N}(H)$, of the same degree. Taking the x_α standard diagrams as variables of $[\alpha]$ and forming all combinations of n of these diagrams, one diagram being associated with each of the n sets of m symbols, we obtain the variables of the Kronecker product $[\alpha]^n$. *The matrices s_i which must be adjoined to those of $[\alpha]^n$ to yield a representation of $\mathfrak{N}(H)$ are just those permutation matrices which arise through transformation by s^*_i of the $(x_\alpha)^n$ variables of $[\alpha]^n$.* We shall say that this representation of $\mathfrak{N}(H)$, which is necessarily irreducible, is *generated*[9] by the representation $[\alpha]^n$ of H.

It will be important in the sequel to have specific information concerning these permutation matrices. To this end let us consider those elements of H which commute with a given s^*_i; clearly they will form a subgroup of H which we call the *centralizer* of s^*_i in H, denoting it $\mathfrak{C}(s^*_i)$. In particular, if $s^*_i = I$ then $\mathfrak{C}(I) = H$. On the other hand, if s^*_i contains m cycles of length n then $\mathfrak{C}(s^*_i) \sim S_m$; each element of $\mathfrak{C}(s^*_i)$ contains n cycles of equal length, in which case we shall say that the n sets of symbols are *linked*. In general, to each of the ω_λ cycles of length λ of s_i there corresponds a factor of $\mathfrak{C}(s^*_i)$ which is

[9]For a further discussion of the irreducible representations of $\mathfrak{N}(H)$ see **6** below.

isomorphic to S_m and in which λ sets of symbols are linked. Thus

2.5 $$\mathfrak{C}(s_i) \sim S_m \times S_m \times \ldots \times S_m, \qquad \omega \text{ factors},$$

where $0 < \omega = \omega_1 + \omega_2 + \ldots + \omega_n \leqslant n$.

We can now prove the following theorem:

2.6 *The characteristic of the permutation matrix $\bar{s}_i$ representing s^*_i in the representation of $\mathfrak{N}(H)$ generated by the representation $[\alpha]^n$ of H is $(x_\alpha)^\omega$ where ω is the number of cycles in s_i.*

To see this it is only necessary to look for the number of variables of $[\alpha]^n$ which remain unaltered after transformation by s^*_i. But this is precisely x_α for each factor S_m in 2.5. Since we are dealing with a Kronecker product the total number of such variables is $(x_\alpha)^\omega$.

Of course if $[\alpha] = [m]$ this whole analysis falls to the ground for then $x_\alpha = 1$ and the representation of $\mathfrak{N}(H)$ under consideration is merely the identity representation.

To illustrate these ideas consider the case $m = 3$, $n = 2$ with $[\alpha] = [2, 1]$. We take the two sets of symbols to be 1, 2, 3; 4, 5, 6, so that

2.7 $$\begin{aligned} H = &\{1 + (12) + (13) + (23) + (123) + (132)\} \\ &\times \{1 + (45) + (46) + (56) + (456) + (465)\}, \end{aligned}$$

2.8 $$\mathfrak{N}(H) = H + H(14)(25)(36),$$

2.9 $$S^*_2 = 1 + (14)(25)(36) \sim S_2.$$

If $s^* = (14)(25)(36)$, then clearly

2.10 $$\begin{aligned} \mathfrak{C}(s^*) = 1 &+ (12)(45) + (13)(46) + (23)(56) \\ &+ (123)(456) + (132)(465). \end{aligned}$$

In order to construct the matrix $\bar{s}$ which represents s^* in the representation of $\mathfrak{N}(H)$ generated by $[2, 1]^2$, let us denote the variables of $[2, 1]^2$ by the products of the corresponding standard Young diagrams:

2.11 $$\begin{pmatrix}12\\3\end{pmatrix}\begin{pmatrix}45\\6\end{pmatrix}, \begin{pmatrix}13\\2\end{pmatrix}\begin{pmatrix}45\\6\end{pmatrix}, \begin{pmatrix}12\\3\end{pmatrix}\begin{pmatrix}46\\5\end{pmatrix}, \begin{pmatrix}13\\2\end{pmatrix}\begin{pmatrix}46\\5\end{pmatrix},$$

in this order. Transforming by s^* we have

2.12 $$\bar{s} : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. The reduction of $\Pi_n^{[\alpha]}$. We are interested in the representation of S_{mn} induced[10] by the representation $[\alpha]^n$ of H. If

3.1 $$S_{mn} = \mathfrak{N}(H) + \mathfrak{N}(H)R_2 + \ldots + \mathfrak{N}(H)R_d,$$

where $d = (mn)!/(m!)^n n!$, we can set $s^*_i = s_i$ with no ambiguity for the time

[10] Cf. [12] pp. 198-202.

being and write the group matrix of the induced representation in question in the form

3.2 $$\Pi_I(H) = \Pi_n^{[\alpha]}(H) = [r_p^{-1}s_i^{-1}\mathbf{H}s_j r_q]$$

where $i, j = 1, 2, \ldots, n!$; $p, q = 1, 2, \ldots, d$ and $\mathbf{H}$ is the group matrix of the representation $[\alpha]^n$ of H. The elements s, r to the right and left of $\mathbf{H}$ operate on the *labels* attached to the independent variables of $\mathbf{H}$ to yield the independent variables of the group matrix of S_{mn}.

Following Schur[6] we seek matrices which will commute with $\Pi_I(H)$. Now it is well known[11] that every matrix of the *left* regular representation ($g_i \to g^{-1}g_i$) commutes with every matrix of the *right* regular representation ($g_i \to g_i\, g$) of a given group G. The form 3.2 of the group matrix of S_{mn} is derived from the *right* regular representation of $S_n \sim \mathfrak{N}(H)/H$. Let us construct the matrix

3.3 $$\Pi_s(I) = (s_j s_k^{-1}) \times I_{(x\alpha)^n} \times I_d,$$

where $(s_j s_k^{-1})$ is the matrix of the *left* regular representation of S_n representing $s = s_j s_k^{-1}$ and the I's are unit matrices of indicated degrees. By definition,

3.4 $$\Pi_s(H) = \Pi_s(I)\,.\,\Pi_I(H) = \Pi_I(H)\,.\,\Pi_s(I)$$

where the matrix $I_{(x\alpha)^n}$ in $\Pi_s(I)$ is placed in the position of $\mathbf{H}$ so that $\Pi_s(I)$ commutes with $\Pi_I(H)$. Multiplying the matrices in 3.4, we obtain a contribution only when $s_j = ss_k$ so that

3.5 $$\Pi_s(H) = [r_p^{-1}s_i^{-1}s\,\mathbf{H}\,s_k r_q] = [r_p^{-1}s_i^{-1}\,\mathbf{H}\,ss_k r_q]\,.$$

These two apparently different designations of the independent variables of the group matrix are actually the same, as can be seen by setting $ss_k \equiv s_k$ so that $s_i^{-1} \equiv s_i^{-1}s$.

Varying s, it is clear that the matrices $\Pi_s(I)$ generate a representation of S_n. By a suitable choice of M the matrices $M^{-1}\Pi_s(I)M$ may be taken in completely reduced form, and

$$\begin{aligned} M^{-1}\Pi_s(H)M &= M^{-1}\Pi_I(H)M.M^{-1}\Pi_s(I)M \\ &= M^{-1}\Pi_s(I)M.M^{-1}\Pi_I(H)M. \end{aligned}$$

It follows from Schur's Lemma[12] that, for any element A of S_{mn},

3.6 $$\operatorname{tr}\Pi_s(A) = \sum_\beta \operatorname{tr}(A) \text{ in } [\alpha]\odot[\beta]\,.\,\chi_\beta(s),$$

where $\chi_\beta(s)$ is the character of s in the irreducible representation $[\beta]$ of S_n. The concluding step in this part of the argument is to solve 3.6 by multiplying by $\chi_\beta(s)$ and summing over s to yield

3.7 $$\chi(A) \text{ in } [\alpha]\odot[\beta] = \frac{1}{n!}\sum_s \operatorname{tr}\Pi_s(A)\,.\,\chi_\beta(s),$$

which can serve to define the reducible representation $[\alpha]\odot[\beta]$ of S_{mn} for the

[11] [1] pp. 22-24 and 231-242.

[12] [10]. This paper was translated and published by Dickson as chapter XI of his *Modern Algebraic Theories* (Chicago, 1926).

time being. Setting $A = I$ in 3.7 we obtain a contribution from the right-hand side only for $s = I$ so, from 3.3, it follows that the

3.8 $$\text{degree of } [\alpha]\odot[\beta] = \frac{(mn)!}{(m!)^n n!}(x_\alpha)^n x_\beta,$$

and the frequency of $[\alpha]\odot[\beta]$ in $\prod_I(H)$ is x_β. Thus

3.9 $$\prod_n{}^{[\alpha]} = \sum_\beta x_\beta[\alpha]\odot[\beta].$$

An explicit definition of $[\alpha]\odot[\beta]$ will be given in **6** below.

4. The character of $[\alpha]\odot[\beta]$. It is necessary to return briefly to the abstract theory and consider further the relation of the subgroup H to $\mathfrak{C}(s^*)$ for a given s^* of S^*_n. We can write

4.1 $$H = \mathfrak{C} + \mathfrak{C}g_2 + \ldots + \mathfrak{C}g_r,$$

where $r = (m!)^{n-\omega}$. Now the g_i can be chosen to be the elements of certain of the original S_m's so that

4.2 $$G = \{g_i\} = S_m \times S_m \times \ldots \times S_m, \qquad n - \omega \text{ factors.}$$

In particular, any $\lambda - 1$ of the λ factors S_m linked in a given factor of $\mathfrak{C}$ will appear in G; it is immaterial which factor is omitted. Collecting together these omitted factors, however, we can construct

4.3 $$K = S_m \times S_m \times \ldots \times S_m, \qquad \omega \text{ factors,}$$

and $H = K \times G$.

We are seeking the distribution of the conjugate sets of S_{mn} in Hs, where again we suppose that $s^* = s$. If g is an element of G,

$$g^{-1}Ksg = Kg^{-1}sg.$$

From the definitions of G and K it follows that $sgs^{-1} = gkg'$, where $g' \neq I$ in G and k is an element of K, so that

$$g^{-1}Ksg = Kkg's = Kg's.$$

If we assume that $g_1^{-1}Ksg_1 = g_2^{-1}Ksg_2$ then $s(g_1g_2^{-1})s^{-1} = (g_1g_2^{-1})k$, for which $g' = I$ and this is impossible unless $g_1 = g_2$; this implies that all such conjugate cosets of K are distinct and make up $K \times G = H$. Thus

4.4 *The $(m!)^{n-\omega}$ cosets of K in Hs are conjugate to Ks under transformation by elements of G.*

We are now in a position to proceed to the calculation[10] of tr $\prod_s(A)$. From 3.5 we have

4.5 $$\text{tr} \prod_s(H) = \sum r_p^{-1}s_i^{-1}s\mathbf{H}s_ir_p = \sum r_p^{-1}s_i^{-1}\mathbf{H}ss_ir_p,$$

and we obtain a contribution only for elements A of S_{mn} contained in $sH = Hs$. Each of the terms in 4.5 yields the same contribution, and there are $(mn)!/(m!)^n$ terms. Moreover, changing the labels attached to the independent variables of the group matrix in 4.5 can be accomplished by multiplying $\mathbf{H}$ by the operator $\bar{s}$ either on the right or on the left. Thus *the con-*

tribution of each element conjugate to A in Hs is $\chi(A)$, where $\chi(A)$ is the character of A in the representation of $\mathfrak{N}(H)$ generated by $[\alpha]^n$; there are $n_A{}^{Hs}$ such elements. But there are n_A elements conjugate to A in S_{mn}, so to obtain tr $\Pi_s(A)$ for a given A we must divide by n_A, thus

4.6 $$\text{tr}\,\Pi_s(A) = \frac{(mn)!n_A{}^{Hs}}{(m!)^n n_A} \cdot \chi(A).$$

In view of 4.4 we need only consider elements $A = ks$ in calculating $\chi(A)$.

As we saw in 2.6, the structure of $\bar{s}$ depends on $\mathfrak{C}(s)$. The subgroup K was chosen so that every set of symbols which are linked in $\mathfrak{C}(s)$ is represented by just one factor S_m in 4.3. Now the matrix representing a substitution k of K in $[\alpha]^n$ consists of the matrix representing k in $[\alpha]^\omega$ repeated $x_\alpha{}^{n-\omega}$ times down the diagonal. Multiplying by $\bar{s}$ yields a matrix whose diagonal elements are all zeros except for those of k in $[\alpha]^\omega$, each of which appears once corresponding to the 1's in the diagonal of $\bar{s}$. If $\chi(A)$ is the character of $A = ks$ in $[\alpha]^n$ and $\theta(k)$ is the character of k in $[\alpha]^\omega$, this proves that

4.7 $$\chi(A) = \theta(k),$$

and 4.6 becomes

4.8 $$\text{tr}\,\Pi_s(A) = \frac{(mn)!n_A{}^{Hs}}{(m!)^n n_A} \cdot \theta(k).$$

In the particular case $[\alpha] = [m]$, the identity representation of S_m, $\theta(k) = 1$ and 4.8 becomes

4.9 $$\text{tr}\,\Pi_s(A) = \frac{(mn)!n_A{}^{Hs}}{(m!)^n n_A},$$

which should be compared with Frobenius' formula

4.10 $$\text{tr}\,\Pi_I(A) = \frac{(mn)!n_A{}^{H}}{(m!)^n n_A}.$$

Inserting the expression 4.6 in 3.7 we have

4.11 $$\chi(A) \text{ in } [\alpha]\odot[\beta] = \frac{(mn)!}{(m!)^n n! n_A} \sum_s \chi_\beta(s) n_A{}^{Hs}\chi(A);$$

we shall see the significance of this formula in **6** below. Setting $A = I$, it follows that $s = k = I$ and we verify 3.8.

5. The irreducible components of $[\alpha]\odot[\beta]$. In order to obtain the frequency $\mathbf{f}$ with which a given irreducible representation $[\lambda]$ of S_{mn} appears as a component of $[\alpha]\odot[\beta]$ we multiply 4.11 by $\chi_\lambda(A)$ and sum over A to yield

5.1 $$\begin{aligned}\mathbf{f}_\lambda([\alpha]\odot[\beta]) &= \frac{1}{(m!)^n n!} \sum_s \chi_\beta(s)\,[\chi(hs)\cdot\phi_\lambda(hs)],\\ &= \frac{1}{(m!)^n n!}\, \chi_\beta(s)\cdot[\chi(hs)\cdot\phi_\lambda(hs)].\end{aligned}$$

Here we are denoting by $\phi_\lambda(hs)$ a vector whose $(m!)^n$ components are the

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characteristics of the elements hs in the representation $[\lambda]$ of S_{mn}; similarly, $\chi(hs)$ denotes a vector whose components are the characteristics of the elements hs in the representation of $\mathfrak{N}(H)$ generated by $[\alpha]^n$ and calculated as described in the preceding section. The dot denotes scalar multiplication.

In the case $[\beta] = [n]$, which is important for the theory of invariants, $\chi_\beta(s) = 1$ and 5.1 simplifies somewhat to give

5.2 $$\mathrm{f}_\lambda([\alpha]\odot[n]) = \frac{1}{(m!)^n n!} \sum_s \chi(hs) \, . \, \phi_\lambda(hs).$$

If we also set $[\alpha] = [m]$, 5.2 reduces to the familiar formula[13] giving the frequency with which $[\lambda]$ appears as an irreducible component of the representation of S_{mn} induced by the identity representation of $\mathfrak{N}(H)$; this representation is in fact the permutation representation to which the subgroup $\mathfrak{N}(H)$ gives rise under right multiplication by the elements of S_{mn}.

It is interesting to note the effect of changing the representations $[\alpha]$ and $[\beta]$ into their conjugates,[14] that is, interchanging the rows and columns of the corresponding Young diagrams. It is clear that we must distinguish two cases according as m is even or odd. If m is even every element of S^*_n, considered as a permutation on mn symbols, is even. On the other hand, if m is odd $\frac{1}{2}(n!)$ of the permutations of S^*_n are even and $\frac{1}{2}(n!)$ are odd. Changing $[\alpha]$ into $[\alpha']$ will change the sign of the component of $\chi = \theta$ associated with an odd permutation $h = k$, but this will be compensated for in 5.1 by changing $[\lambda]$ into $[\lambda']$, provided m is *even*; if m is *odd* ks^* may be odd or even depending on s^*, and to achieve compensation it will be necessary also to change $[\beta]$ into $[\beta']$. This yields D. E. Littlewood's[8]

5.3 *Theorem of Conjugates*:

$$\begin{aligned} \mathrm{f}_\lambda([\alpha]\odot[\beta]) &= \mathrm{f}_{\lambda'}([\alpha']\odot[\beta]) \ , \ m \ even, \\ &= \mathrm{f}_{\lambda'}([\alpha']\odot[\beta']) \ , \ m \ odd. \end{aligned}$$

6. Explicit definition of $[\alpha]\odot[\beta]$. Let us return to the representation of $\mathfrak{N}(H)$ generated by $[\alpha]^n$ as defined in **2**. Clearly, the representation $[\beta]$ of S_n is also an irreducible representation of $\mathfrak{N}(H)$ in which every element of H is represented by the identity matrix. These are the two extreme members of a family of representations of $\mathfrak{N}(H)$ which we can easily construct. The remaining irreducible representations of $\mathfrak{N}(H)$ are similar in character, but we do not need to refer to them here.

If we denote by $s(\beta)$ the matrix representing s^* in the representation $[\beta]$, then the matrices obtained by replacing each 1 of $\bar{s}$ by $s(\beta)$ can be written

6.1 $$\bar{s} \times s(\beta),$$

and these also yield a representation of S^*_n of degree $x_\alpha{}^n \, . \, x_\beta$. Constructing the analogous representation

6.2 $$[\alpha]^n \times I_{x_\beta}$$

[13] [13] p. 287.

[14] [12] p. 202.

of H, and combining 6.1 and 6.2 we obtain a representation of $\mathfrak{R}(H)$ which we may denote by $[\alpha, \beta]$; it follows from Schur's Lemma that $[\alpha, \beta]$ is irreducible. The representation of $\mathfrak{R}(H)$ defined in **2**, for which $[\beta] = [n]$ is here denoted $[\alpha, n]$, and the representation $[\beta]$ of $\mathfrak{R}(H)$ is denoted $[m, \beta]$. The character of any substitution hs^* of $\mathfrak{R}(H)$ in $[\alpha, \beta]$ is clearly $\chi(hs^*) \,.\, \chi_\beta(s^*)$; thus, substituting in Frobenius' formula[10] for the character of an induced representation, we obtain 4.11 and conclude that

6.3 $[\alpha]\odot[\beta]$ *is that representation of* S_{mn} *which is induced by the irreducible representation* $[\alpha, \beta]$ *of* $\mathfrak{R}(H)$.

7. Illustrative examples. Referring to the example of **1**, we calculate the irreducible components[15] of $[2, 1]\odot[2]$. Here K and G are interchangeable; set

7.1 $$K = I + (12) + (13) + (23) + (123) + (132),$$

so that

7.2 $$Hs = Ks + \text{five similar cosets of } K,$$

and

7.3 $$Ks = (14)(25)(36) + (1425)(36) + (1436)(25) + (2536)(14) + (142536) + (143625).$$

With the help of a table of characters S_6 we obtain from 5.1 :

7.4 $$\begin{aligned} [2, 1]\odot[2] &= [4, 2] + [3, 2, 1] + [3, 1^3] + [2^3], \\ [2, 1]\odot[1^2] &= [2^2, 1^2] + [3, 2, 1] + [4, 1^2] + [3^2], \end{aligned}$$

both of degree 40, illustrating 5.3 with m odd.

In order to illustrate the case with m even consider the reduction of the disjoint product $[2]\cdot[2]\cdot[2]$. The subgroups H and S^*_3 are easily seen to be:

7.5 $$H = \{I + (12)\} \times \{I + (34)\} \times \{I + (56)\},$$

7.6 $$S^*_3 = I + (13)(24) + (15)(26) + (35)(46) + (135)(246) + (153)(264).$$

It is only necessary to write down two cosets Hs^* since S^*_3 has only two conjugate sets distinct from the identity:

7.7 $$H(13)(24) = (13)(24) + (1324) + (1423) + (13)(24)(56) + (14)(23) + (1324)(56) + (1423)(56) + (14)(23)(56),$$

7.8 $$H(135)(246) = (135)(246) + (135246) + (146235) + (136245) + (146)(235) + (136)(245) + (145)(236) + (145236).$$

From 5.1 we have:[16]

7.9 $$\begin{aligned} [2]\odot[3] &= [6] + [4, 2] + [2^3], && \text{of degree } 15, \\ [2]\odot[2, 1] &= [5, 1] + [4, 2] + [3, 2, 1], && \text{of degree } 30, \\ [2]\odot[1^3] &= [4, 1^2] + [3^2], && \text{of degree } 15. \end{aligned}$$

[15] [3] p. 336.

[16] [4] p. 289.

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An application of 5.3 yields:

$$
7.10 \qquad \begin{aligned}
&[1^2]\odot[3] &&= [1^6] + [2^2, 1^2] + [3^2], &&\text{of degree } 15,\\
&[1^2]\odot[2, 1] &&= [2, 1^4] + [2^2, 1^2] + [3, 2, 1], &&\text{of degree } 30,\\
&[1^2]\odot[1^3] &&= [3, 1^3] \quad + \quad [2^3], &&\text{of degree } 15.
\end{aligned}
$$

Once the necessary cosets have been written out the irreducible components of any $[\alpha]\odot[\beta]$ can be obtained by purely arithmetic calculations, assuming that the necessary tables of characters are available.

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