The Journal of Navigation (2024), 1–14
doi:10.1017/S0373463324000043

RESEARCH ARTICLE

Latitude by two altitudes of the Sun – Douwes' and Riddle's methods

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Received: 19 April 2023; **Accepted:** 8 January 2024

Keywords: Cornelis Douwes; Edward Riddle; latitude calculation methods

Abstract
In approximately 1740, Cornelis Douwes presented an algorithmic method to determine the latitude when it is impossible to observe the Sun at the meridian passage. To apply Douwes' method, it is necessary to know two altitudes of the Sun, the time elapsed between observations, the Sun's declination at the time when the greater altitude was observed and the latitude by account. Douwes' method, originally written in Dutch, was translated and published in English by Richard Harrison in 1759. This translation made possible the dissemination of this method throughout Europe. In 1821, James Ivory proposed a new method that was independent of the latitude by account. This method was improved by Edward Riddle in 1822. Riddle's proposal was widely disseminated throughout Europe during the 19th century. In this work, our objective is to study the reliability of these two methods. For that purpose, we will apply the algorithmic methods of Douwes and Riddle to determine the latitude using real observations made during the years 2021 and 2022. The results obtained will then be compared with the GPS (Global Positioning System) latitude to assess the reliability of each method.

1. Introduction

At the end of the 15th century, to determine the latitude, sailors used a method based on the observation of the Sun at the meridian passage, that is, at its maximum altitude. However, due to meteorological conditions, it could be difficult or even impossible to observe the Sun at that precise moment. Whenever the observation was not possible, it would be necessary to wait for the following day to obtain the latitude. In practice, due to weather conditions or overcast sky at the meridian passage, it might be impossible to observe the Sun's altitude at the meridian passage for several days. Consequently, the determination of the ship's latitude would be compromised, also compromising the success of the trip.

Aware of these difficulties, several authors searched and have developed methods to determine the latitude using two altitudes of the Sun at two different times of the day. The first method found in the literature was proposed by Pedro Nunes in 1537. It was a mechanical method since all the calculations had to be done using an auxiliary instrument, a globe, designed and built by Nunes (1537).[1] In the 17th century, another mechanical method was proposed by Estancel (1658). All these mechanical methods used an auxiliary instrument to facilitate the required complex calculations that involved products and divisions of trigonometric functions. All these calculations were time-consuming and very prone to

[1]A modern edition of this work can be found in Nunes (2002).

errors, also the use of complex instruments was not easily done on board. Therefore, these methods were impractical and hard to use on board of a ship where the objective would be to determine the latitude through quick and expeditious processes. The finding of the logarithmic function, in the 17th century, by John Napier, allowed the simplification of all the complex calculations necessary to determine the latitude, allowing for the appearance of algorithmic methods whose calculations were, clearly, more simple and easily done on board.

The first algorithmic method was proposed in Dutch by Cornelis Douwes, in approximately 1740.[2] His method requires the knowledge of two altitudes of the Sun, the time elapsed between observations, the latitude by account and the Sun's declination at the time when the greater altitude was observed. This method was disseminated in English through the translation of Harrison (1759) and can be found in texts by Norie (1835a) and Bowditch (1837), the latter two being the internationally most known navigation manuals at the time.

Douwes' method depended on the knowledge of the latitude by account. To avoid this drawback, in 1821, James Ivory proposed a new method that was independent of the latitude by account. The process proposed by Ivory was improved and simplified by Edward Riddle in 1822. Riddle's proposal served as the basis for numerous authors in the 19th century who were looking for quick and reliable methods to determine latitude using two altitudes of the Sun. These methods are described in texts by Norie (1835a), Bowditch (1837) and Leitão (1865), among others. At the end of the 19th century, Marcq Saint-Hilaire proposed a method that is used until today. This is a completely different method and not comparable with Douwes' and Riddle's methods, since it permits to determine the latitude and longitude simultaneously. Marcq Saint-Hilaire's method can be used at any time of the day or the night, while Douwes' and Riddle's methods can only be used during the day since they require Sun observations.

In this work, we will study the methods proposed by Douwes and Riddle to determine the latitude by two altitudes of the Sun. Both methods will be applied to real observations made during the years 2021 and 2022 on trips in the Atlantic ocean and the results obtained will be compared with the GPS (Global Positioning System) latitude to assess the reliability of each method.

2. Latitude by two altitudes of the Sun – Douwes' Method

To determine the latitude, the navigators used a method based on the observation of the Sun at its meridian passage. To apply this method, the Sun must be visible at that exact moment, which might not always be the case, making it impossible to determine latitude. In 1740, Cornelis Douwes presented a method to determine the latitude using two altitudes of the Sun. This is considered to be the first algorithmic method and the first method to use the logarithmic function. It is also considered the precursor of all other methods that followed. The method proposed by Douwes requires the knowledge of two altitudes of the Sun, the time elapsed between observations, the Sun's declination at the time when the greater altitude was observed and the latitude by account. The method is iterative, that is, at the end, the obtained latitude must be compared with the latitude by account, evaluating the difference. If the difference is large, the process is repeated, using the latitude obtained in the previous step as the latitude by account. This method, originally written in Dutch, was disseminated in English through the translation of Harrison (1759). In this work, our main goal is to apply Douwes' method to real observations to assess its reliability. Even though there are several manuscripts where this method can be found, we will use the 11th edition of the book *A New and Complete Epitome of Practical Navigation* written by J. W. Norie in 1835.[3]

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[2] There are some doubts about the date of the first Douwes' manuscript. Robertson (1772, p. 324) says 'At about the year 1740, Mr. John Douwes, a mathematician at Amsterdam, [. . .] first communicated, to the sea officers of his nation, a solution of this problem, by the help of a set of solar tables which he had fitted for that purpose.' According to *The Oxford Encyclopedia of Maritime History*, Douwes wrote the first manuscript with the description of his method in 1749. However, the first official publication dates from 1754 (Douwes 1754), even though it did not present the tables necessary for the complete resolution of the method. The first complete edition was published in 1760 (Douwes, 1760).

[3] The book is available for free download at the website www.archive.org through the link https://archive.org/details/acompleteepitom00norigoog/mode/2up.

https://doi.org/10.1017/S0373463324000043 Published online by Cambridge University Press

150 TABLE XXV.

LOGARITHMIC SINES, TANGENTS, AND SECANTS.

34 Degrees.

M	Sine.	Diff.	Co-sine.	D.	Tang.	Diff.	Co-tang.	Secant.	Co-sec.	M
0	9.747562	312	9.918574	142	9.828987	454	10.171013	10.081426	10.252438	60
1	9.747749	312	9.918489	142	9.829260	454	10.170740	10.081511	10.252251	59
2	9.747936	312	9.918404	142	9.829532	454	10.170468	10.081596	10.252064	58

Figure 1. *Value of the artificial sine of* 34° *in the logarithmic tables (Norie, 1828, p. 150).*

Prior to the explanation of Douwes' method, we must make some notes and clarify a few concepts and notation.

First of all, we shall note that the observed Sun's altitudes have to be corrected for index error (if necessary), dip, refraction and parallax. Also, the value of the Sun's declination has to be reduced to the meridian of the ship, this is called the *reduced declination*. All the Sun's altitudes that appear in this work have been corrected and all the Sun's declinations are in fact reduced declinations, however, and for the sake of simplicity, we will simple say Sun's declination instead of Sun's reduced declination. The way these corrections are done is out of the scope of this paper; however, the interested reader can see a detailed explanation on how to do these corrections in Norie (1835a, p. 180).

With the appearance and dissemination of the logarithmic function, two ways of mentioning the trigonometric functions have appeared, namely the name of the function or the name of the function accompanied by the term *artificial*, and the name of the function accompanied by the term *natural*. When associated with the term *natural*, it is meant the value of the function itself. If the name of the function appears alone or with the term *artificial*, then it is meant the value of the logarithm of that function.[4] For example, the sine of 30° corresponds to $\log(\sin(30°))$, while the natural sine of 30° corresponds to $\sin(30°)$.

The values of these functions were obtained from a set of tables, namely the Table of Logarithmic Sines, Tangents and Secants and the Table of Natural Sines.[5] To simplify the calculations, the number 10 (in some cases the number 5) has been added to all the values of the trigonometric functions that appear in the tables. This little trick had the objective to transform all negative numbers into positive numbers, which made all the calculations much easier. For example, if we search for the value of the artificial sine of 34° in the logarithmic tables, we will find the number 9.747562, which is exactly the value of $\log(\sin(34°)) + 10$ (see Figure 1). The addition of a positive integer was common at that time and was used by several authors to simplify the calculations that had to be done.

To simplify his method and all the calculations that had to be done, Douwes introduced some auxiliary functions, namely the functions *rising*, *half elapsed time* and *middle time*. For all these functions, he has also created the tables with the necessary values needed for the resolution of his method.

The function *rising* is defined as $\text{rising}(x) = 1 - \cos(x)$ and its values can be found in the table Logarithms for Rising. The parameter *elapsed time*, denoted by t, represents the time between observations, that is, $t = |t_2 - t_1|$ where t_1 and t_2 denote the times when the two Sun's altitudes where observed. The parameter *middle time*, denoted by mt, corresponds to the value $(t_2 + t_1)/2$. For these functions, the values necessary for the resolution of the method are $-\log\sin(t/2)$ and $\log(2\sin(mt))$. Therefore, the table Logarithms for Half Elapsed Time lists the values of $-\log\sin(t/2)$ and the Table for Middle Time contains the values of $\log(2\sin(mt)) + 5$.

Before we state all the steps of Douwes' method, we must introduce the starting variables. Let a_1 and a_2 denote the two altitudes of the Sun, with the assumption that $a_1 > a_2$; let t_1 and t_2 denote the time of the observations of a_1 and a_2, respectively; let δ denote the Sun's declination at the time when the greater altitude was taken, that is, the Sun's declination at the instant t_1, and let φ_a denote the latitude

[4] All logarithms considered in this work are base 10.

[5] The functions co-sine, co-tangent and co-secant are complements of the functions sine, tangent and secant, respectively, therefore, their values can also be obtained from these tables.

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by account. We will now describe the seven steps of this method as described by Norie (1835a, p. 191). For each step, we will also include the respective mathematical formulation.

1. To the log secant of the latitude by account, add the log secant of the Sun's declination; their sum, rejecting 20 from the index, call the *log.ratio*:

$$\log\sec\varphi_a + \log\sec\delta = \log(ratio)$$

2. From the natural sine of the greater altitude, subtract the natural sine of the less altitude and set the logarithm of their difference under the log.ratio:

$$\log(\sin(a_1) - \sin(a_2))$$

3. Take out the logarithm answering to half the elapsed time and set it likewise under the log.ratio:

$$-\log\sin\left(\frac{t}{2}\right)$$

4. Add these three logarithms together and find the middle time (mt) corresponding to their sum, the difference between which and the half elapsed time, will be the time from noon when the greater altitude was observed:

$$\log(ratio) + \log(\sin(a_1) - \sin(a_2)) + \log\sin\left(\frac{t}{2}\right) = \log(2\sin(mt))$$

5. From the log.rising, answering to this time, subtract the log.ratio and the remainder will be the logarithm of a natural number (n); which being found and added to the natural sine of the greater altitude, their sum will be the natural co-sine of the meridian zenith distance (mzd):

$$\log\text{rising}\left(\frac{t}{2} - mt\right) - \log(ratio) = \log(n); \qquad n + \sin(a_1) = \cos(mzd)$$

6. Having found the meridian zenith distance, apply to it the declination and the result will be the latitude, φ, at the time of taking the greater altitude:

$$\text{If } mzd \text{ and } \delta \text{ are both North or South, then } \varphi = mzd + \delta \text{ else } \varphi = |mzd - \delta|.$$

7. If the latitude thus found should differ considerably from the latitude by account, the operation is to be repeated, using the computed latitude instead of that by account, until the latitude last found agree nearly with the latitude used in the computation.

In the method listed above, it is assumed that both Sun's altitudes were taken at the same place; however, this rarely happens if the ship is navigating. Therefore,

> [...] it will be necessary when the ship is making much way, and the elapsed time is considerable to correct the less altitude, in order to find what it would have been, had it been taken at the place where the greater altitude was observed. (Norie, 1835a, p. 194)

According to Douwes (Norie, 1835a, p. 194), to correct the lesser altitude, we have to proceed as follows.

(a) With the compass, find the number of points between the Sun's azimuth at the time when the lesser altitude was observed and the ship's course. Subtract from 16 if the value is greater than 8 points.
(b) Compute the distance sailed during the elapsed time.
(c) Entering this number of points and the distance sailed in Table I,[6] and obtain the corresponding difference of latitude. This value is the correction value to be applied to the lesser altitude.

[6]Table I can be found in Norie (1835a).

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EXAMPLE III.

November 10, 1829, latitude by account 32° 30′ N. at 9h. 30m. the altitude of the Sun's lower limb was 28° 14′, the bearing of its center by compass being S. E. ½ E.; and at 11h. 17m. 42s. the altitude of the upper limb was 39° 8′; the height of the eye being 18 feet, and the ship's course between the observations S. b. E. running 7 knots per hour; required the latitude of the ship at the time of the latter observation.

The elapsed time between the observations is nearly 1 hour 48 minutes, and her rate of sailing 7 miles per hour; therefore, as 1 hour : 7 miles :: 1h. 48m. : 12 miles, the distance run between the observations.

Sun's bearing at first observation S. E. ½ E. or S. 4½ Pts. E.
Ship's course during the elapsed time S.b.E. or S. 1 Pt. E.

Angle between them . 3½ Pts. which being taken as a course, and the distance run during the elapsed time 12 miles, as a distance, give in the latitude column 9 miles nearly.

Obs. alt. Sun's lower limb	28° 14′		Obs. alt. Sun's upper limb	39° 8′
Semidiameter . + 16′ / Dip . − 4	+ 12		Semidiameter . − 16′ / Dip. . − 4	− 20
	28 26			38 48
Refraction	− 2		Refraction	− 1
True altitude	28 24		True altitude	38 47
Corr. for Ship's way	+ 9			
Reduced Altitude	28 33			

Times.	Altitudes.	Nat. Sines.		
9h 30m 0s	28° 33′	47793	Lat. by acc. 32° 30′	Sec. 0.07397
11 17 42	38 47	62638	Declination 17 10	Sec. 0.01979
			Log. ratio	0.09376
1 47 42 Elapsed time		Diff. 14845	Log.	4.17158
0 53 51 Half elapsed time			Log.	0.63299
1 33 13 Middle time			Log.	4.89833
0 39 22 Time from noon			Log. rising	3.16784
			Log. ratio	0.09376
Natural number		1186	Log.	3.07408
Natural sine of greater alt.		62638		
Nat. co. sine of mer. zen. dist.		63824 = 50° 20′ N.		
	Declination	17 10 S.		Sec. 0.01979
	Latitude	33 10 N.		Sec. 0.07723
			Log. ratio	0.09702
h m s Diff. Nat. Sines		14845	Log.	4.17158
0 53 51 Half elapsed time			Log.	0.63299
1 33 58 Middle time			Log.	4.90159
0 40 7 Time from noon			Log. rising	3.18411
			Log. ratio	0.09702
Natural number		1222	Log.	3.08709
Natural sine of greater alt.		62638		
Nat. co. sine of mer. zen. dist.		63860 = 50° 19′ N.		
	Declination	17 10 S.		
	Latitude	33 9 N.		

Figure 2. *Example of Douwes' method (Norie, 1828, p. 191).*

(i) If the lesser altitude was observed in the morning, then add the correction if the angle determined above is less than 8 points, otherwise subtract the correction.

(ii) If the lesser altitude was observed in the afternoon, then subtract the correction if the angle determined above is less than 8 points, otherwise add the correction.

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In Figure 2, we can see an application of this method. In the first block, we have the procedure to determine the Sun's true altitudes. In the second block, we have the first iteration of the method and finally, in the last block, a second iteration of the method.

To conclude the description of the method, there are some remarks concerning the times of the observations.

> As the above method is only an approximation to the truth, it must be used under the following restrictions. (Norie, 1835a, p. 192)

According to Douwes, the hours of the Sun observations must take into account the following restrictions (Norie, 1835a, p. 192):

(a) observations must be carried out between 9 h and 15 h;
(b) if both observations are in the morning or both in the afternoon, the time elapsed between observations must be greater than the time elapsed between noon and the time of greatest altitude;
(c) if the observations are one in the morning and one in the afternoon, the time elapsed between observations must not exceed 4 h 30;
(d) in all cases, the closer the greater altitude is to noon, the better.

In Section 4, we will implement Douwes' method in an Excel spreadsheet and use it to determine the latitude using real observations of the Sun. The values of the latitude obtained will be compared with the GPS latitude to determine the error and evaluate the reliability of the method.

3. Latitude by two altitudes of the Sun – Ivory and Riddle's method

The method proposed by Douwes to determine the latitude depended on the knowledge of the latitude by account. This was considered to be an inconvenience since it required the regular estimation of the latitude, which was a time-consuming process. To avoid this inconvenience, in 1821, James Ivory (Ivory, 1821) proposed a new method that was independent of the latitude by account.

> This method of finding the Latitude by two Altitudes of the Sun, which is much simpler and more general than the former, and independent of the Latitude by Account, was proposed by Mr. James Ivory, who has given an ingenious Solution of it in the Philosophical Magazine for August 1821. Mr. Riddle, of the Royal Naval Asylum, Greenwich, has since considerably improved Mr. I.'s Solution, and given a Rule, similar to the above, in the same work for September 1822. (Norie, 1835a, p. 198)

In 1822, Edward Riddle published a paper in the *Philosophical Magazine* (Riddle, 1822) where he proposes some simplifications on Ivory's method. The simplifications proposed by Riddle transforms Ivory's method into a simpler and more accessible procedure.

> By a very simple trigonometrical transformation, he adapted Ivory's solution to logarithmic computation, and gave to it that practical working form now in use. (Report 1855, p. 113)

As before, we shall use as reference the 11th edition of the book *A New and Complete Epitome of Practical Navigation* written by J. W Norie in 1835. This method considers the true altitudes of the Sun, that is, the Sun's altitudes corrected for index error (if necessary), dip, refraction and parallax. As it was noted before, the interested reader can see all the rules to perform this correction in Norie (1835a, p. 180). The lesser altitude has to be corrected for the time when the greater altitude was taken. This procedure is done as it was explained before in Douwes' method.

Prior to the description of the steps of Riddle's method, we have to introduce the initial variables. Let t_1 and t_2 represent the time, in hours, when the greater and lesser altitudes were observed, respectively. The Sun's greater and lesser altitudes are denoted by the variables a_1 and a_2, respectively, and let δ denote the Sun's declination for the time t_1, that is, for the time when the greater altitude was taken. The elapsed time, denoted by t, represents the time between observations, that is, $t = |t_1 - t_2|$. Finally, we shall observe that all the values of the trigonometric functions mentioned in the method are actually the values of their logarithms (base 10) and for the sake of simplicity, we shall omit the log function. For example, when it is mentioned, the secant of the declination, mathematically speaking, it is meant the $\log \sec(\delta)$. We will now describe the eight steps of this method as described by Norie (1835a, p. 198). For each step, we will also include the respective mathematical formulation.

1. Add together the true altitudes (found as before) and take half their sum; subtract the lesser altitude from the greater and take half their difference:

$$\frac{a_1 + a_2}{2}; \quad \frac{a_1 - a_2}{2}$$

2. Find the interval between the times of observing the two altitudes, called *elapsed time*, take half the elapsed time and reduce it to degrees:[7]

$$\frac{t}{2} = \frac{|t_1 - t_2|}{2} * 15$$

3. Add together the co-secant of half the elapsed time and the secant of the declination; their sum will be the co-secant of *arc first*:

$$\csc\left(\frac{t}{2}\right) + \sec(\delta) = \csc(arc\ first)$$

4. Add together the co-secant of arc first, the co-sine of half the sum of the altitudes and the sine of half their difference: the sum of these logarithms will be the sine of *arc second*:

$$\csc(arc\ first) + \cos\left(\frac{a_1 + a_2}{2}\right) + \sin\left(\frac{a_1 - a_2}{2}\right) = \sin(arc\ second)$$

5. Add together the secant of arc first, the sine of half the sum of the altitudes, the co-sine of half their difference and the secant of arc second; their sum will be the co-sine of *arc third*:

$$\sec(arc\ first) + \sin\left(\frac{a_1 + a_2}{2}\right) + \cos\left(\frac{a_1 - a_2}{2}\right) + \sec(arc\ second) = \cos(arc\ third)$$

6. Add together the secant of arc first and the sine of the declination; their sum will be the co-sine of *arc fourth*, when latitude and declination are of the same name; but when they are of contrary names, take the supplement for *arc fourth*:[8]

$$\sec(arc\ first) + \sin(\delta) = \cos(arc\ fourth)$$

7. Take the sum or difference of arcs third and fourth, for *arc fifth* (see Note):

$$arc\ third \pm arc\ fourth = arc\ fifth$$

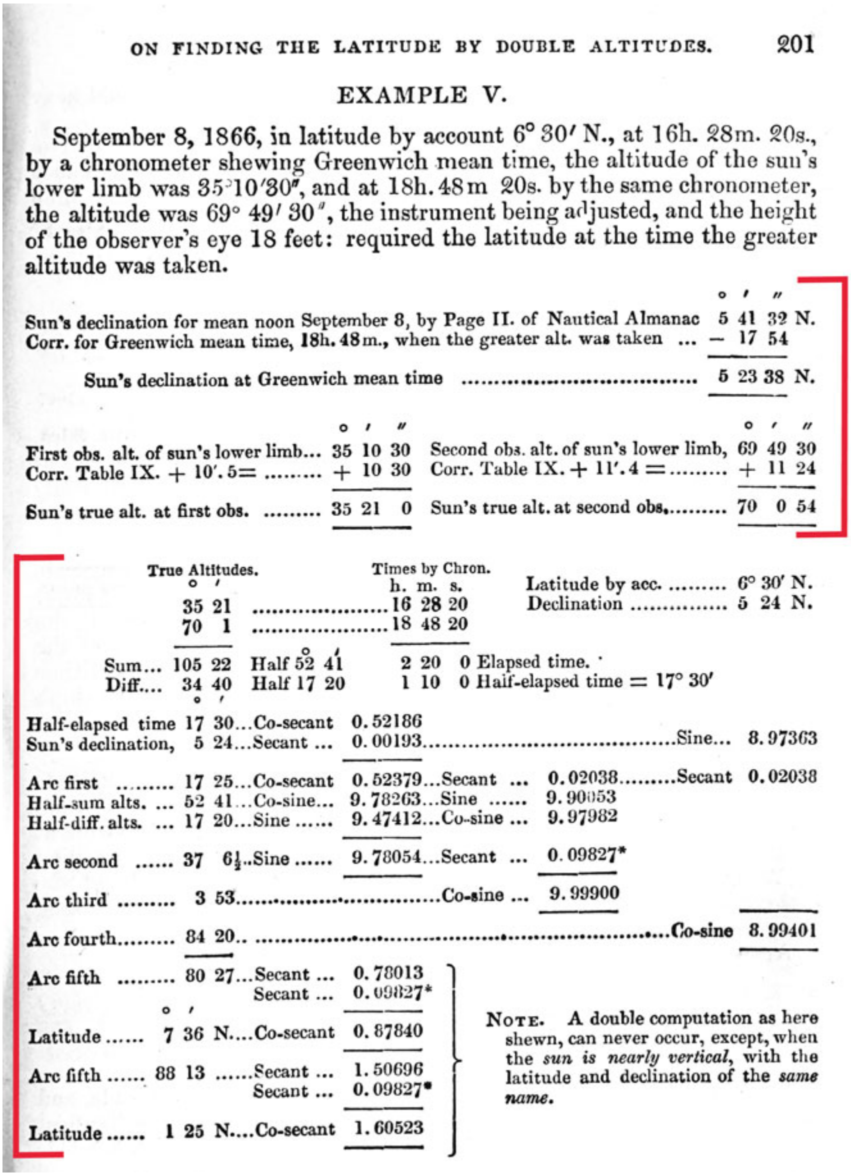
8. Add together the secants of arc second and arc fifth; their sum will be the co-secant of the *Latitude*, denoted by φ:

$$\sec(arc\ second) + \sec(arc\ fifth) = \csc(\varphi)$$

9. Note: When the sum of arcs third and fourth is equal to or greater than 90°, their difference is always arc fifth; but when their sum is less than 90°, it may doubtful whether their sum or difference ought

[7]The reduction to degrees is done using the conversion rate that 1 h is equal to 15°.
[8]The latitude and declination are said to be the same name if they are both North or both South, they are said to have contrary names if one is North and the other is South. The supplement of the angle α is equal to $180° - \alpha$.

https://doi.org/10.1017/S0373463324000043 Published online by Cambridge University Press



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EXAMPLE V.

September 8, 1866, in latitude by account 6° 30′ N., at 16h. 28m. 20s., by a chronometer shewing Greenwich mean time, the altitude of the sun's lower limb was 35°10′30″, and at 18h. 48m 20s. by the same chronometer, the altitude was 69° 49′ 30″, the instrument being adjusted, and the height of the observer's eye 18 feet: required the latitude at the time the greater altitude was taken.

	° ′ ″
Sun's declination for mean noon September 8, by Page II. of Nautical Almanac	5 41 32 N.
Corr. for Greenwich mean time, 18h. 48m., when the greater alt. was taken ...	− 17 54
Sun's declination at Greenwich mean time	5 23 38 N.

	° ′ ″		° ′ ″
First obs. alt. of sun's lower limb...	35 10 30	Second obs. alt. of sun's lower limb,	69 49 30
Corr. Table IX. + 10′.5=	+ 10 30	Corr. Table IX. + 11′.4 = ...	+ 11 24
Sun's true alt. at first obs.	35 21 0	Sun's true alt. at second obs.........	70 0 54

True Altitudes. ° ′ — Times by Chron. h. m. s.

35 21 ....16 28 20
70 1 ....18 48 20

Latitude by acc. .... 6° 30′ N.
Declination .... 5 24 N.

Sum... 105 22 Half 52 41 — 2 20 0 Elapsed time.
Diff.... 34 40 Half 17 20 — 1 10 0 Half-elapsed time = 17° 30′

	° ′						
Half-elapsed time	17 30	Co-secant	0.52186				
Sun's declination,	5 24	Secant ...	0.00193			Sine...	8.97363
Arc first	17 25	Co-secant	0.52379	Secant ...	0.02038	Secant	0.02038
Half-sum alts. ...	52 41	Co-sine...	9.78263	Sine	9.90053		
Half-diff. alts. ...	17 20	Sine	9.47412	Co-sine ...	9.97982		
Arc second	37 6½	Sine	9.78054	Secant ...	0.09827*		
Arc third	3 53			Co-sine ...	9.99900		
Arc fourth.........	84 20					Co-sine	8.99401
Arc fifth	80 27	Secant ...	0.78013				
		Secant ...	0.09827*				
Latitude	7 36 N.	Co-secant	0.87840				
Arc fifth	88 13	Secant ...	1.50696				
		Secant ...	0.09827*				
Latitude	1 25 N.	Co-secant	1.60523				

NOTE. A double computation as here shewn, can never occur, except, when the *sun is nearly vertical*, with the latitude and declination of the *same name*.

Figure 3. *Example of Riddle's method (Norie, 1835b, 18th ed., p. 201).*

to be taken for arc fifth. Do both cases and one of the results must certainly be the required latitude, and the latitude by account will generally be sufficient to determine which of them ought to be taken.

As in the previous method, there are some remarks concerning the times of the observations.

In this method the observations should, if possible, be taken under the same limitations as directed in the former. (Norie, 1835a, p. 198)

In Figure 3, we can see an application of this method. In the first block, it is determined the Sun's reduced declination and the Sun's true altitudes, and in the last part, we have the various steps of the algorithm to determine the latitude.

Riddle's method was implemented in an Excel spreadsheet and used with real observations of the Sun to determine the latitude. The value of the latitude obtained was compared with the GPS latitude to evaluate the error and assess the reliability of the method. A complete and detailed analysis of the results will be presented in Section 4.

4. Real world application of the methods

In this section, our objective is to see how the methods proposed by Douwes and Riddle behave in practice. For that purpose, we will apply both methods to determine the latitude using real observations and the values obtained will be compared with the GPS latitude.

In 2021, between the 5th and the 23rd of August, during a round trip on the tall ship *Sagres* between Lisbon and Azores, we collected 41 Sun altitudes. The altitudes were taken using a sextant and were corrected for index error, dip, refraction and parallax. The times of the observations were also registered as well as the GPS position. The values of the Sun's declination for the times of the observations were obtained from the Nautical Almanac for 2021 (Nautical Almanac, 2020).

This procedure was repeated in 2022 on a trip on the tall ship *Sagres* from Lisbon to Brazil. Between the 27th of July and the 5th of August, we collected 34 Sun altitudes, the times of the observations and the GPS position. The Sun's declinations were obtained from the Nautical Almanac for 2022 (Nautical Almanac, 2021).

Recall that to apply both methods, we need to use a pair of altitudes within the same day. Thus, for each day of observations, we built all possible pairs making a total of 129 pairs of observations, 61 pairs in 2021 and 68 pairs in 2022. For each of these pairs, the lesser altitude was corrected for the instant when the greater altitude was observed using the procedure described in Section 2.

Both methods were implemented in an Excel spreadsheet. For each pair, the latitude was determined using Douwes' and Riddle's methods and the values obtained were compared with the real GPS latitude. Finally, a careful analysis of the results was carried out to assess the reliability of the proposed methods. We will also analyse the behaviour of the two algorithms when we travel in different directions. In 2021, the ship travelled East/West, where the latitude varies slowly with time, while in 2022, the ship sailed South, where the variation in latitude is more significant over time.

4.1. Douwes' method

In this section, we will present all the steps needed to implement Douwes' method in an Excel spreadsheet. Before we explain each step, we need to define the initial variables and make some general remarks. For practical reasons, all times and angular values were converted to a decimal format. The decimal angles where later converted to radians since the arguments of all trigonometric functions in Excel are in radians.

Let a_1 and a_2 denote the two altitudes of the Sun with the assumption that $a_1 > a_2$. Recall that a_2 denotes the corrected value, that is, the value of the lesser altitude corrected to the place and time where the greater altitude was observed. This correction was done using the procedure described in Section 2. Let t_1 and t_2 denote the observation times of a_1 and a_2, respectively. Let δ denote the Sun's declination at the time t_1 and finally, let φ_a denote the latitude by account, φ the latitude determined by algorithm and φ_{GPS} the GPS latitude.

The first thing that has to be done is to determine the time between observations, called the *elapsed time* and denoted by t. Since all the times were converted to a decimal value, the value of t is easily obtained by the formula $t = |t_1 - t_2|$. Then, we need to convert its value to degrees. This is done by multiplying the decimal value by 15, since 1 h corresponds to 15°. For example, if the elapsed time is 1 h and 30 min, then in decimal format, we have $t = 1.5$ h or $t = 22.5°$. Recall that $\text{rising}(x) = 1 - \cos(x)$.

We will now list all the steps that we have implemented in the Excel spreadsheet.

Step 1 $\log(ratio) = \log\sec\varphi_a + \log\sec\delta$.
Step 2 $x = \log(\sin(a_1) - \sin(a_2))$.
Step 3 $y = |\log\sin(t/2)|$.

https://doi.org/10.1017/S0373463324000043 Published online by Cambridge University Press

Table 1. *Results obtained by Douwes' method for the year 2021.*

	$\varphi_a = \varphi_{GPS}$				$\varphi_a = \varphi_{error}$			
	1st iteration		2nd iteration		1st iteration		2nd iteration	
error ≤ 20	59	97%	61	100%	58	95%	61	100%
$20.1 \leq$ error ≤ 40	2				3			
Total pairs	61	100%			61	100%		

Step 4 $\quad mt = \arcsin\left(\frac{10^{\log(ratio)+x+y}}{2}\right)$.
Step 5 $\quad mdz = \arccos(10^{\log(1-\cos(t/2-mt))-\log(ratio)} + \sin(a_1))$.
Step 6 $\quad$ If mzd and δ are both North or South, then $\varphi = mzd + \delta$ else $\varphi = |mzd - \delta|$.
Step 7 $\quad$ If $|\varphi - \varphi_{GPS}| > 20'$, repeat the process with $\varphi_a = \varphi$.

Observe that Douwes' method determines the latitude at the time when the Sun's greater altitude was observed, that is, at the time t_1. Furthermore, the method uses the latitude by account as a starting value. Nowadays, due to the evolution of the navigation methods, the latitude by account is no longer computed during the trip. To overcome this difficulty and be able to implement Douwes' method, we have taken the latitude by account to be the GPS latitude for the time t_1, that is, we have set $\varphi_a = \varphi_{GPS}$. However, considering that latitude by account would seldom be equal to the real latitude, we decided to analyse the behaviour of the algorithm when the latitude by account is different from the true latitude. For this purpose, we have randomly added an error term to the GPS latitude. For each pair, we have selected a random number between -0.5 and 0.5 that was added to the value of the GPS latitude in decimal degrees. Let us denote the obtained value by φ_{error}. We have run Douwes' algorithm with these two options for the latitude by account separately for the years 2021 and 2022.

For the year 2021, taking $\varphi_a = \varphi_{GPS}$ and after one iteration, we obtained 59 pairs with an error term less than 20 miles, this represents 97% of the total number of pairs considered. There are only two pairs with errors greater than 20 miles, one pair has an error of 20.3 miles, while the other has an error of 38.3 miles. Although these results are extremely good, we have performed a second iteration of the algorithm for these two pairs. After two iterations of the algorithm, all pairs have an error less than 20 miles, which shows that the algorithm is reliable.

In our second case, we have run the algorithm using $\varphi_a = \varphi_{error}$. After one iteration, only three pairs had an error greater than 20 miles, the greatest error being 34 miles. After two iterations of the algorithm, all pairs had an error less that 20 miles. A complete summary of these results can be found in Table 1.

For the year 2022, we have run Douwes' algorithm by first considering that $\varphi_a = \varphi_{GPS}$. In this case and after one iteration, we obtained 63 pairs with an error term less than 20 miles, which represents 93% of the total number of pairs considered. For the remaining five pairs, we have three pairs with errors between 20 and 40 miles, one with an error of 53.4 miles and one with an error of $1°49'$. For the latter, the reason why we have such a greater error has to do with measurement errors. In fact, we have compared the registered Sun's altitudes (see Table 2) with the Sun's calculated altitudes[9] and we found that the true altitude at the time t_1 is 66.33, while the true altitude at the time t_2 is 48.39, that is, a_1 has an error of 11.5 min, while a_2 has an error of 2.5 min. This measurement error is the reason why the latitude determined by the algorithm has such a great error.

We have performed a second iteration of the algorithm for the four pairs with errors between 20 and 60 miles. After the second iteration, we obtained two pairs with errors less than 20 miles and two pairs with errors of 29.9 and 43.8 miles. Therefore, after two iterations of the algorithm, we have obtained good latitude values for 96% of the pairs considered.

[9] With the knowledge of the latitude, longitude and time of observation, it is possible to calculate the exact altitude of the Sun that would be observed in that position.

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Table 2. *Pair with large error in Douwes' method for the year 2022.*

Pair number	t_1	t_2	a_1	a_2	φ	φ_{GPS}	Error
2253	15 h 06	16 h 27	66.53	48.35	23.53	25.35	1.82

Table 3. *Results obtained by Douwes' method for the year 2022.*

	$\varphi_a = \varphi_{GPS}$				$\varphi_a = \varphi_{error}$			
	1st iteration		2nd iteration		1st iteration		2nd iteration	
error ≤ 20	63	93%	65	96%	64	94%	65	96%
20.1 ≤ error ≤ 40	3		2		1		1	
40.1 ≤ error ≤ 60	1		1		2			
error > 60	1						2	
Total pairs	68	100%			68	100%		

In a second analysis of the algorithm, we wanted to see the performance of the algorithm when we consider $\varphi_a = \varphi_{error}$. In this case, and after the first iteration, we have 64 pairs with an error less than 20 miles. As before, we have one pair with an error of $1°57'$ which corresponds to the pair with measurement errors. The remaining three pairs have errors of 33.1, 46.4 and 58.1 miles. We have performed a second iteration of the algorithm for these three pairs and have obtained one more pair with an error less than 20 miles. Overall, and after two iterations of the algorithm, we have 65 pairs for which the calculated latitude has an error less than 20 miles when compared to the GPS value, i.e. the algorithm has a success rate of 96%. A complete summary of these results can be found in Table 3.

Finally, we can easily observe that Douwes' method produced slightly better results in 2021 than in 2022. One reason for this might be the fact that in 2021, the ship sailed from Lisbon to Azores and back to Lisbon, in East/West courses, where the latitude varies slowly with time, while in 2022, the ship sailed from Lisbon to Brazil, that is, mainly South or Southwest courses in which the variation of latitude with time is more significant. However, the results are extremely satisfactory in both years.

4.2. Riddle's method

In this section, we will present all the steps needed to implement Riddle's method in an Excel spreadsheet and then we will analyse its performance using the real data collected in 2021 and 2022. In Riddle's method, the initial variables are the same as for Douwes' method except for the latitude by account. We maintain the same notation used before. In summary, the Sun's altitudes are denoted by a_1 and a_2 with the assumption that $a_1 > a_2$. The times of the observations of a_1 and a_2 are denoted by t_1 and t_2, respectively, and we denote by δ the Sun's declination at the time t_1. Let φ denote the latitude determined by algorithm and let φ_{GPS} denote the GPS latitude. As before, all times are in decimal format and angular values were converted first to a decimal format and then to radians.

The first step of the method calculates half of the sum of true altitudes and half their difference. Since we are using an Excel spreadsheet, we can skip this step. Therefore, the first step of the method becomes the determination of the parameter $t/2$ called *half elapsed time* and its conversion to degrees using the conversion rate that 1 h is equal to 15°. This is done using the formula

$$\frac{t}{2} = \frac{|t_1 - t_2|}{2} * 15.$$

https://doi.org/10.1017/S0373463324000043 Published online by Cambridge University Press

Table 4. *Results obtained by Riddle's method for the years 2021 and 2022.*

	2021		2022	
error ≤ 20	60	98%	60	88%
20.1 ≤ error ≤ 40	1		5	
error > 60	0		3	
Total pairs	61		68	

Table 5. *Pairs with large error in Riddle's method for the year 2022.*

Pair number	t_1	t_2	a_1	a_2	φ	φ_{GPS}	Error
2237	10 h 20	17 h 45	46.51	29.52	25.86	28.02	2.17
2247	16 h 27	10 h 09	48.42	42.37	26.62	25.25	1.37
2253	15 h 06	16 h 27	66.53	48.35	24.25	25.35	1.10

We will now list the procedure to determine the five arcs that are necessary to determine the latitude. A simple observation of the method shows that to determine the value of *arc first*, we need to evaluate the inverse function of the function co-secant. This function is not available in Excel as a built-in function; therefore, we first need to determine the mathematical expression of inverse function of the co-secant. By observing that $\csc(x)$ is by definition $1/\sin(x)$, we can easily conclude that the inverse function of $\csc(x)$ is defined by the expression $\arcsin(1/x)$. We will now list all the steps that we have implemented in the Excel spreadsheet.

Step 1 $1Arc = \arcsin(10^{-\log\csc(t/2)-\log\sec(\delta)})$.

Step 2 $2Arc = \arcsin\left(10^{\log\csc(1Arc)+\log\cos\frac{a_1+a_2}{2}+\log\sin\frac{a_1-a_2}{2}}\right)$.

Step 3 $3Arc = \arccos\left(10^{\log\sec(1Arc)+\log\sin\frac{a_1+a_2}{2}+\log\cos\frac{a_1-a_2}{2}+\log\sec(2Arc)}\right)$.

Step 4 $4Arc = \arccos(10^{\log\sec(1Arc)+\log\sin(\delta)})$.

Step 5 In this step, we will determine the value of the *fifth arc*, denoted by $5Arc$.

(i) If $2Arc + 4Arc \geq 90°$, then $5Arc = 4Arc - 3Arc$.
(ii) If $2Arc + 4Arc < 90°$, then $5Arc_1 = 4Arc - 3Arc$ or $5Arc_2 = 4Arc + 3Arc$.

Step 6 In this last step, we determine the latitude, denoted by φ. When case (ii) happens, we will use the two possible values for $5Arc$ and determine two options for the latitude. The final value for φ is the one that is closer to the real latitude:

$$\varphi = \arcsin(10^{-\log\sec(2Arc)-\log\sec(5Arc)})$$

Recall that the latitude obtained by Riddle's method is the latitude at the time t_1, that is, at the time when the Sun's greater altitude was observed. We have run the algorithm for the years 2021 and 2022, and have compared the value of the calculated latitude, φ, with the value of the GPS latitude for the time t_1, denoted by φ_{GPS}.

For the year 2021, we obtained 60 pairs with an error less than 20 miles and one pair with an error of 22.2 miles. That is, the algorithm has a success rate of 100%.

For the year 2022, we have 60 pairs with an error less than 20 miles, five pairs with an error between 20 and 40 miles, and three pairs with errors greater than 60 miles. A complete summary of the results obtained for the years 2021 and 2022 can be found in Table 4, and a complete list of the pairs with an error greater than 60 miles can be found in the Table 5.

https://doi.org/10.1017/S0373463324000043 Published online by Cambridge University Press

As we have seen before in Douwes' method, the pair 2253 has a measurement error in the Sun's altitudes. The error in the value of a_1 is 11.5 min, which is quite significant, hence the latitude obtained has an error greater than 60 min.

With regards to pairs 2237 and 2247, we verified that they do not respect the restrictions imposed by the method. In fact, Riddle's method has some restrictions to the times of observation.

> If one observation be taken in the forenoon, and the other in the afternoon, the elapsed time must not exceed four hours and a half; and in all cases, the nearer the greater altitude is to the noon, the better. (Norie, 1835a, p. 192)

Both pairs have one observation in the morning and the other in the afternoon. The pair 2237 has an interval between observations equal to 7 h 25, the greater altitude was observed at 10 h 20 in the morning and the local noon was at 13 h 25. As for the pair 2247, we have an interval time of 6 h 18 and the local noon was at 13 h 39, while the greater altitude was observed at 16 h 27. Therefore, both pairs fail to comply with the restrictions, which justifies the magnitude of the error obtained.

Therefore, if we exclude the pairs 2237, 2247 and 2253, we have 60 out of 65 pairs with an error less that 20 miles, which corresponds to 92% of the admissible pairs. For the remaining five pairs, the largest error is 38.1 miles. These results show that Riddle's method is reliable and produces good results.

Finally, we can easily observe that Riddle's method produced better results in 2021 than in 2022. One reason for this might be the fact that in 2021, the ship sailed from Lisbon to Azores and back to Lisbon, in East/West courses, where the latitude varies slowly with time, while in 2022, the ship sailed from Lisbon to Brazil, that is, mainly South or Southwest courses in which the variation of latitude with time is more significant. However, the results are extremely satisfactory in both years.

5. Conclusion

Until the end of the 15th century, the method to determine the ship's latitude was based on the observation of the Sun at the meridian passage. This was mathematically simple and was easily done by the sailors. However, due to meteorological conditions, it could be hard or even impossible to observe the Sun at that precise moment, making it impossible to determine the ship's latitude. To overcome this difficulty, several authors have studied and developed alternative processes to determine the ship's latitude when it is impossible to observe the Sun at the meridian passage. The first algorithmic method found in the literature was published in 1740 by Cornelius Douwes. Originally written in Dutch, this method became widely known due to the English translation published by Harrison (1759). To apply Douwes' method, it is necessary to know two altitudes of the Sun, the time elapsed between observations, the Sun's declination at the time when the greater altitude was taken and the latitude by account. The fact that this method depends on the knowledge of the latitude by account is clearly a disadvantage. Therefore, in 1821, and to overcome this difficulty, James Ivory proposed a new method that was independent of the latitude by account. Ivory's method was later improved by Edward Riddle and his proposal was widely disseminated all over Europe during the 19th century. With this work, our main objective is to see how the methods proposed by Douwes and Riddle behave in practice. For that purpose, during two trips on the tall ship *Sagres*, we collected 75 Sun altitudes, 41 in 2021 and 34 in 2022. The altitudes were taken using a sextant and were corrected for index error, dip, refraction and parallax. The times of the observations were also registered as well as the GPS position. The values of the Sun's declination for the times of the observations were obtained from the Nautical Almanac for the years 2021 and 2022. Each method requires a pair of altitudes within the same day. Thus, for each day of observations, we built all possible pairs making a total of 129 pairs of observations, 61 pairs in 2021 and 68 pairs in 2022. Douwes' and Riddle's methods were implemented in an Excel spreadsheet and used to determine the latitude for the 129 pairs that we had. The values obtained where compared with the GPS latitude to determine the magnitude of the errors obtained.

The results obtained prove that both methods produce good values for the latitude. For 2021, and after one iteration of Douwes' method, we have 95% of the pairs with a calculated latitude that has an error less than 20 miles to the GPS value, and after the second iteration, all pairs have an error less that 20 miles, which means a success rate of 100%. For 2022, from the 68 pairs considered, we have 63 with an error less than 20 miles after one iteration and 65 pairs after the second iteration, this is a success rate of 93% after one iteration and 96% after the second iteration. As for Riddle's method, the algorithm results show that the errors are less than 20 miles for 98% of the pairs obtained in 2021 and 88% of the 2022 pairs. However, for the year 2022, we have one pair with measurement errors and two pairs that do not respect the method restrictions about the times of the observations. Therefore, if we exclude these three pairs, we have 60 pairs from a total of 65 for which the calculated latitude has an error less than 20 miles, corresponding to 92% of the admissible pairs and the largest error obtained is 38.1 miles. Therefore, these results lead us to conclude that both algorithms behave extremely well in practice. Furthermore, we can also observe that we obtained better results for both algorithms in 2021 than in 2022. We are aware that the set of data collected in each year is relatively small, so it may be difficult to draw conclusions. However, one reason for this fact might be the difference in ship's course; in fact in 2021, the ship sailed from Lisbon to Azores and back to Lisbon, that is, it sailed in East/West courses, where the latitude varies slowly with time, while in 2022, the ship sailed from Lisbon to Brazil, that is, mainly South or Southwest courses in which the variation of latitude with time is more significant.

Finally, we would like to observe that, to our knowledge, the methods studied here did not have much practical use at sea despite having been widely disseminated internationally. One reason for this might be the fact that both methods are mathematically complex, require a huge amount of calculations and the use of several logarithmic tables. Furthermore, the method of calculating latitude through the meridian passage of the sun was far more simple making it the preferred choice of sailors even in the case when the weather conditions were adverse and they had to wait several days to determine the latitude.

Acknowledgements. Teresa Sousa was partially funded by national funds through the FCT - Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 and UIDP/00297/2020 (Center for Mathematics and Applications).

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Cite this article: Queirós AF, Costa Canas A, Sousa T (2024). Latitude by two altitudes of the Sun – Douwes' and Riddle's methods. *The Journal of Navigation* 1–14. https://doi.org/10.1017/S0373463324000043