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ON CENTRAL Ω-KRULL RINGS AND THEIR CLASS GROUPS

E. JESPERS AND P. WAUTERS

0. Introduction. The aim of this note is to study the class group of a central Ω-Krull ring and to determine in some cases whether a twisted (semi) group ring is a central Ω-Krull ring. In **[8]** we defined an Ω-Krull ring as a generalization of a commutative Krull domain. In the commutative theory, the class group plays an important role. In the second and third section, we generalize some results to the noncommutative case, in particular the relation between the class group of a central Ω-Krull ring and the class group of a localization. Some results are obtained in case the ring is graded. Theorem 3.2 establishes the relation between the class group and the graded class group. In particular, in the P.I. case we obtain that the class group is equal to the graded class group. As a consequence of a result on direct limits of Ω-Krull rings, we are able to derive a necessary and sufficient condition in order that a polynomial ring $R[(X_i)_{i\in I}]$ (I may be infinite) is a central Ω-Krull ring. We also have
RRR

$$\mathrm{Cl}\,(R) \cong \mathrm{Cl}\,(R[(X_i)_{i\in I}]).$$

In the final two sections we study twisted (semi) group rings. In this case, we deal with torsion free abelian (cancellative) (semi) groups. We obtain necessary and sufficient conditions for a twisted (semi) group ring to be a symmetric maximal order. Moreover, if G is a torsion free abelian group and $R^t[G]$ is a P.I. ring (cf. Proposition 5.7), then $R^t[G]$ is a P.I. Ω-Krull ring if and only if R is a P.I. Ω-Krull ring and G satisfies the ascending chain condition on cyclic subgroups (Theorem 5.9). In the final section we obtain some results on twisted semigroup rings generalizing results of **[1]**.

1. Preliminaries. Throughout this note, R will be a prime ring satisfying Formanek's condition, i.e., every nonzero ideal of R has a nontrivial intersection with C, the center of R. In this case,

$$Q_{\mathrm{sym}}(R) = \{c^{-1}r = rc^{-1} \mid\; 0 \neq c \in C, r \in R\}$$

is a simple ring. If $\mathscr{L}^2(\sigma)$ is a multiplicatively closed filter, then

$$Q_\sigma(R) = \{\alpha \in Q_{\mathrm{sym}}(R) | I\alpha \subset R \text{ and } \alpha\mathrm{I} \subset R \text{ for some } I \in \mathscr{L}^2(\sigma)\}$$

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is a subring of R. Note that $Q_\sigma(R)$ need not be a localization of R, since it can occur that σ is not an idempotent kernel functor.

In [**8**] we defined an *Ω-Krull ring* R to be a prime, Formanek ring such that

(1) $R = \bigcap_{p_i \in X^1(R)} Q_{R \smallsetminus P_i}(R)$ (write $Q_{R \smallsetminus P_i}(R) = R_i$);

(2) each ring R_i is a quasi-local Ω-ring, i.e., every nonzero ideal of R_i is a power of the unique maximal ideal of R_i;

(3) for all i and for all $I \in \mathscr{L}^2(R \smallsetminus P)$, $IR_i = R_iI = R_i$;

(4) for all $r \in R$ there are only finitely many indices i such that $RrR \in \mathscr{L}^2(R \smallsetminus P_i)$.

We say that R is a *geometric* (resp. *central*) Ω-Krull ring if every kernel functor σ_i is geometric (resp. central) (cf. [**7**]). In case R is a central Ω-Krull ring, we have

$$Q_{R \smallsetminus P_i}(R) = Q_{C \smallsetminus P_i}(R) = \{c^{-1}r | c \in C \smallsetminus p_i, r \in R\}$$

where $p_i = P_i \cap C$ ([**9**]). R is said to be an Ω-ring if every ideal is invertible. In particular, an Ω-ring is an Ω-Krull ring such that all prime ideals are maximal (cf. [**10**]). The interested reader is referred to [**18**] and [**20**] for details about localization.

Recall that a ring S is said to be an *extension* of a subring R if $S = R.\ Z_S(R)$ where

$$Z_S(R) = \{s \in S | \forall\, r \in R, sr = rs\}.$$

In particular $Z(R) \subset Z(S)$ and $SI = IS$ for all ideals of R.

A ring R is called a *symmetric maximal order* if each ring S between R and $Q_{\text{sym}}(R)$ such that $cS \subset R$ for some nonzero $c \in C$ equals R. Recall that a fractional R-ideal I is a twosided R-submodule of $Q_{\text{sym}}(R)$ such that $cI \subset R$ for some nonzero $c \in C$. If A and B are subsets of $Q_{\text{sym}}(R)$, we define

$$(A :_l B) = \{q \in Q_{\text{sym}}(R) | qB \subset A\} \quad \text{and}$$

$$(A :_r B) = \{q \in Q_{\text{sym}}(R) | Bq \subset A\}.$$

LEMMA 1.1. *The following conditions are equivalent*:

(1) *R is a symmetric maximal order*;

(2) *for any ideal I of R, $(I :_l I) = (I :_r I) = R$*;

(3) *for any fractional R-ideal I, $(I :_l I) = I :_r I) = R$*;

(4) *let $\alpha \in Q_{\text{sym}}(R)$; suppose that there exists an element $c \in C$ such that for all $n \in \mathbf{N}_0$, $c(R\alpha R)^n \subset R$, then $\alpha \in R$.*

Proof. The equivalences (1) to (3) have been proved in [**9**]. (3) $\Leftrightarrow$ (4) may be proved in a way similar to the commutative case (cf. [**6**]).

If $R \subset S \subset Q_{\mathrm{sym}}(R)$, we say that S is completely integral over R if and only if whenever $\alpha \in S$, suppose there exists an element $0 \neq c \in Z(R)$ such that for all $n \in \mathbf{N}_0$, $c(R\alpha R)^n \subset R$, then $\alpha \in R$. (Compare to Lemma 1.1 (4) and the commutative case (cf. [**6**]).)

Several authors have tried to generalize the notion of a Krull domain to the noncommutative case. In particular, Marubayashi and Chamarie have worked on this problem. We refer to [**11**], [**12**], [**3**] and [**4**] for their definitions. The main difference between these concepts and ours is that the Krull rings considered in loc. cit. are prime Goldie rings and we deal with prime Formanek rings. In case R is a P.I. ring, we prove that these three definitions coincide.

PROPOSITION 1.2. *Let R be a prime* P.I. *ring. Then R is an Ω-Krull ring if and only if R is Marubayashi-Krull if and only if R is Chamarie-Krull.*

Proof. Since R is a prime P.I. ring, R is Goldie and bounded. In this case, Chamarie has proved ([**3**]) that his definition and the one of Marubayashi coincide. Suppose R is a P.I. Ω-Krull ring, then all $R_i = Q_{R\setminus P_i}(R)$ $(P_i \in X^1(R))$ are P.I. rings since $R_i \subset Q_{\mathrm{sym}}(R)$. Since each R_i has ACC on twosided ideals, R_i is also left and right Noetherian. This follows from a theorem of Cauchon (see e.g. [**16**]). Since R is also a prime Goldie ring, we conclude that R is Marubayashi-Krull ([**11**]). Conversely, if R is a P.I. Chamarie-Krull ring, then

$$R = \bigcap_{p \in X^1(C)} Q_{C\setminus p}(R), \quad Q_{C\setminus p}(R) = R_{C(P)}$$

(cf. [**4**]) and therefore R is a central Ω-Krull ring.

In the sequel we will sometimes use the following proposition derived from a result due to Chamarie (cf. [**4**]).

PROPOSITION 1.3. *A* P.I.*-ring R is an Ω-Krull ring if and only if R is a symmetric maximal order and $Z(R)$ is a Krull domain.*

Note that from the proof of Proposition 1.2 we have that a P.I. Ω-Krull ring is a central Ω-Krull ring.

Let S be a semigroup with 1. A ring R is said to be an S-graded ring if there exist additive subgroups R_s indexed by the elements of S such that

$$R = \bigoplus_{s \in S} R_s \quad \text{and} \quad R_sR_t \subset R_{st} \quad \text{for all } s, t \in S.$$

Throughout this note, S will be abelian, cancellative and torsion free, i.e., if $s, t \in S$ and $s^n = t^n$ for some $n \in \mathbf{N}_0$, then $s = t$. In particular, S is embeddable in its quotient group G which is torsion free abelian. Since S is abelian, the center C of R is also an S-graded ring. The elements $h(R) = \cup_{s \in S} R_s$ are called the homogeneous elements of R. If I is an ideal of R, denote by I^g the ideal of R generated by the homogeneous elements in I. I

is said to be graded if $I = I^g$. Since S is a torsion free abelian cancellative semigroup, S is an ordered semigroup (because G is ordered). Then it is easy to check that if P is a prime ideal of R, P^g is also a prime ideal of R. Therefore, if P is a prime ideal of R of height one, then $P^g = 0$ or $P = P^g$.

2. Class group and Picard group of a central Ω-Krull ring. Let R be a central Ω-Krull ring. We recall some definitions and notations (cf. [**9**]). $\mathbf{F}(R)$ is the set of nonzero fractional R-ideals. $\mathbf{D}(R)$ is the group of divisorial ideals, i.e., those fractional R-ideals I such that $R:(R:I) = I$. Recall that $\mathbf{D}(R)$ is a free abelian group generated by the prime ideals of height one. $\mathbf{I}(R)$ is the group of invertible ideals, i.e., those fractional R-ideals I such that there exists a fractional R-ideal J and $I.J = J.I = R$. The group of principal ideals, i.e., those fractional R-ideals I such that $I = Rc$ for some $c \in K\backslash\{0\}$ (K is the field of fractions of $C = Z(R)$), is denoted by $\mathbf{P}(R)$. The *class group* Cl (R) of R is defined by $\mathbf{D}(R)_{/\mathbf{P}(R)}$ and the *Picard group* Pic $(R) = \mathbf{I}(R)_{/\mathbf{P}(R)}$.

Let A and B be central Ω-Krull rings, A a subring of B and B an extension of A. Write

$$A = \bigcap_i A_i,\ A_i = A_{A\backslash P_i}(A) \quad \text{and } p_i \in X^1(A),$$

$$B = \bigcap_i B_i,\ B_i = Q_{B\backslash P_i}(B) \quad \text{and } P_i \in X^1(B).$$

Denote by p_i' (resp. P_i') the unique maximal ideal of A_i (resp. B_i). Suppose p (resp. P) is a prime ideal of A (resp. B). We say that P lies over p if $P \cap A = p$ and in this case we write $P|p$. Define ϕ: $\mathbf{D}(A) \to \mathbf{D}(B)$ by sending $p \in X^1(A)$ to

$$\phi(p) = \prod_{\substack{P_i|p \\ P_i \in X^1(B)}} P_i^{e(P_i,p)}$$

and $e(P_i, p)$ is the natural number such that

$$B_i p = (P_i)^{e(P_i,p)}.$$

Extend ϕ to $\mathbf{D}(A)$ by linearity. Define

$$\psi: \mathbf{D}(A) \to \mathbf{D}(B): I \mapsto (BI)^* = B:(B:BI).$$

The next proposition shows when $\psi = \phi$. Analogous to the commutative case, we say that the condition (PDE) is satisfied if and only if for all $P \in X^1(B)$,

$$\text{ht } (P \cap A) \leqq 1.$$

PROPOSITION 2.1. *With the foregoing notations, the following three conditions are equivalent*:

(1) $\phi = \psi$;

(2) $\forall\, x \in Z(A)$, $\phi(Ax) = Bx$;

(3) *condition* (PDE) *is satisfied.*

Proof. (1) $\Rightarrow$ (2): This is obvious since

$$\phi(Ax) = \psi(Ax) = (Bx)^* = Bx.$$

(2) $\Rightarrow$ (3): Suppose $P \in X^1(B)$ and ht $(P \cap A) > 1$. Choose a nonzero element $x \in P \cap Z(A)$. Then

$$Bx = \phi(Ax) = P_1^{n_1} * \ldots * P_k^{n_k}$$

and for all $i \in \{1, \ldots, k\}$, $P_i \neq P$ since ht $(P \cap A) > 1$. On the other hand

$$Bx = \psi(Ax) = P_1^{m_1} * \ldots * P_k^{m_k}$$

and here P will occur since $Bx \subset P$ and ht $(P) = 1$. This is a contradiction.

(3) $\Rightarrow$ (1): Suppose

$$I = p_1^{n_1} * \ldots * p_k^{n_k} \in \mathbf{D}(A).$$

Then $A_iI = p_i'^{n_i}$. We compute

$$\psi(I) = \bigcap_i B_iI.$$

Consider such a ring $B_i = Q_{B\setminus P_i}(B)$ and $P_i \in X^1(B)$. Suppose first that $P_i \cap A = p_i \in X^1(A)$. Then

$$A_i = Q_{A\setminus p_i}(A) \subset B_i.$$

Therefore

$$B_iI = B_iA_iI = B_ip_i'^{n_i} = (P_i'^{e(P_i, p_i)})^{n_i}.$$

In the other case, $P_i \cap A = 0$. Then $BI \not\subset P_i$ and hence $BI \in \mathscr{L}^2(B\setminus P_i)$ and $B_iI = B_i$.

$$\phi(I) = \phi(p_1)^{n_1} * \ldots * \phi(p_k)^{n_k} = \prod_{P_i|p_1} (P_i^{e(P_i, p_1)})^{n_1} * \ldots$$

yielding that $(BI)^* = \phi(I)$.

If condition (PDE) is satisfied, then ψ is a homomorphism. Note that in this case $\psi(\mathbf{P}(A)) \subset \mathbf{P}(B)$ and $\bar{\psi}$:Cl $(A) \to$ Cl (B) is a homomorphism. In general, ψ need not be a homomorphism, even in the commutative case (cf. [**1**]). However, if we restrict ψ to $\mathbf{I}(A)$, then this restriction is always a homomorphism from $\mathbf{I}(A)$ to $\mathbf{I}(B)$.

We now give a few special cases in which the homomorphism ϕ is interesting. Let $A = C = Z(R) \subset B = R$ and R a central Ω-Krull ring. For all $P \in X^1(R)$, ht $(P \cap A) = 1$ and moreover, if $p \in X^1(C)$ there is exactly one $P \in X^1(R)$ lying over p. Therefore

$$\phi:\mathbf{D}(C) \to \mathbf{D}(R):I = p_1^{n_1} * \ldots * p_k^{n_k} \mapsto (RI)^* = P_1^{e(P_1, p_1)n_1} * \ldots * P_k^{e(P_k, p_k)n_k}.$$

We claim that the homomorphism

$$\bar{\phi}:\mathrm{Cl}\,(C) \to \mathrm{Cl}\,(R):[I] \mapsto [\,(RI)^*]$$

is injective. Let $[I] \in \mathrm{Cl}\,(C)$ and suppose $\bar{\phi}(\,[I]\,) = [R]$, i.e., $(RI)^* = Rx$ for some $x \in K$. Hence $R_iI = R_i$ for all $i \in \wedge$. But each C_i is a discrete valuation ring yielding that $C_iI = C_ia_i$ for some $a_i \in K$. Therefore

$$C_iI = C_ia_i = C_ix \quad \text{and} \quad I = \bigcap_i C_iI = Cx.$$

This proves that Cl (C) is embedded in Cl (R).

Another important example is the case where R is a central Ω-Krull ring and B is a subintersection of R. Recall from [7] that B is a subintersection of R if and only if $B = Q_\sigma(R)$ and σ is a multiplicatively closed symmetric filter; if $R = \cap_{i \in \wedge} R_i$, then

$$B = \bigcap_{i \in \wedge_0} R_i \quad \text{for some } \wedge_0 \subset \wedge.$$

In particular, if $P \in X^1(B)$, then $P = P_i' \cap B$ where $i \in \wedge_0$ and P_i' is the unique maximal ideal of R_i. Therefore

$$P \cap R = P_i' \cap R \in X^1(R)$$

and condition (PDE) is satisfied. Hence

$$\psi:\mathbf{D}(R) \to \mathbf{D}(B):I = \bigcap_{i \in \wedge} R_iI \mapsto (BI)^* = \bigcap_{i \in \wedge_0} R_iI.$$

In this case, we can compute the kernel of ψ explicitly. Let $P \in X^1(R)$, we consider two cases: if $i \in \wedge_0$, then

$$\psi(P) = \bigcap_{j \in \wedge_0} R_jP = B \cap P_i' \in X^1(B);$$

if $i \in \wedge \setminus \wedge_0$, then

$$\psi(P) = \bigcap_{j \in \wedge_0} R_jP = B.$$

Hence, if $P \in X^1(R)$, then $\psi(P)$ is either B or a height one prime ideal of B. If

$$I = P_1^{n_1} * \ldots * P_k^{n_k} \in \ker \psi,$$

then

$$\psi(I) = \psi(P_1)^{n_1} * \ldots * \psi(P_k)^{n_k} = B.$$

Since $\mathbf{D}(B)$ is free on $X^1(B)$, all $\psi(P_i) = B$. Therefore $\ker \psi$ is the free group generated by the prime ideals P_i with $i \in \wedge \setminus \wedge_0$. Note that ψ is a surjection in this case.

THEOREM 2.2. *Suppose that R is a central Ω-Krull ring and let B be a subintersection of R and an extension of R, say*

$$R = \bigcap_{i \in \wedge} R_i \quad \text{and} \quad B = \bigcap_{i \in \wedge_0} R_i (\wedge_0 \subset \wedge).$$

Then $\bar{\psi}$:Cl $(R) \to$ Cl (B) is onto and ker $\bar{\psi}$ *is generated by the classes of prime ideals $P_i \in X^1(R)$ such that $i \in \wedge \setminus \wedge_0$.*

Proof. Since ψ is onto, the same is true for $\bar{\psi}$. For the second assertion it suffices to prove that

$$\ker \bar{\psi} = \frac{\ker \psi^* \mathbf{P}(R)}{\mathbf{P}(R)}.$$

Let $[I] \in \ker \bar{\psi}$, then $\psi(I) = Bc$ for some $c \in K$ (K is the field of fractions of $C = Z(R)$). But $\psi(Ac) = Bc$ and hence

$$\psi(I^*Ac^{-1}) = B.$$

Therefore $I^*Ac^{-1} \in \ker \psi$ and $[I] = [I^*Ac^{-1}]$.

COROLLARY 2.3. *Let R be a central Ω-Krull ring and σ a kernelfunctor satisfying property* (T) *such that Q_σ is an extension of R. Put*

$$\bar{\psi}\text{:Cl } (R) \to \text{Cl } (Q_\sigma(R)).$$

Then ker $\bar{\psi}$ *is generated by the classes of those prime ideals $P_i \in X^1(R)$ such that $P \in \mathscr{L}^2(\sigma)$.*

Proof. In view of the preceding theorem (and with the same notations) it suffices to prove that $P \in \mathscr{L}^2(\sigma)$ if and only if $i \in \wedge \setminus \wedge_0$. If $P \in \mathscr{L}^2(\sigma)$, then $BP = B$ because σ has property (T). Therefore $\psi (P) = (BP)^* = B$ and $P \in \ker \psi$ entailing that $i \notin \wedge_0$. If $P \notin \mathscr{L}^2(\sigma)$, we claim that

$$B \subset Q_{R \setminus P} (R) = R'.$$

Indeed, if $x \in B$, then $Ix \subset R$ for some $I \in \mathscr{L}^2(\sigma)$. So $I \not\subset P_i$, for if $I \subset P_i$, then $P_i \in \mathscr{L}^2(\sigma)$, a contradiction. Hence $x \in R'$. Since

$$B = \bigcap_{i \in \wedge_0} R_i$$

R' has to be a ring of type R_i (cf. [7]) and $i \in \wedge_0$.

A sufficient condition for having (PDE) is given in

PROPOSITION 2.4. *Let $A \hookrightarrow B$ be an extension of central Ω-Krull rings. Then* (PDE) *holds if B is a left or right flat A-module.*

Proof. This is as in the commutative case (cf. [**17**], Theorem 6.2).

3. Class group of graded Ω-Krull rings. Let R be an S-graded ring and let R also be a central Ω-Krull ring. Recall that we only consider semigroups S with 1 which are cancellative, abelian and torsion free. Consider the symmetric filter of ideals $\mathscr{L}^2(\sigma) = \{I|I$ an ideal of R containing a nonzero homogeneous central element.$\}$ It is clear that $\mathscr{L}^2(\sigma)$ is multiplicatively closed. Then

$$\begin{aligned} Q^g_\sigma(R) &= Q_\sigma(R) \\ &= \{\alpha \in Q_{\text{sym}}(R)|I\alpha \subset R \text{ for some } I \in \mathscr{L}^2(\sigma)\} \\ &= \{c^{-1}\, r|r \in R, 0 \neq c \in h(C)\} \subset Q_{\text{sym}}(R) \end{aligned}$$

and we denote this ring $Q_\sigma(R)$ by $Q^g_{\text{sym}}(R) = Q^g$. Remark that $Q^g_{\text{sym}}(R)$ is a G-graded ring, where G is the quotient group of S. In particular, Q^g is a subintersection of R. For more details about graded localization, we refer to [**14**].

LEMMA 3.1.

$$Q^g_{\text{sym}}(R) = \bigcap_{\substack{P_i \in X^1(R) \\ P_i^g = 0}} Q_{R\setminus P_i}(R).$$

Proof. Recall from [**7**] that we have to prove that

$$Q^g \subset Q_{R\setminus P_i}(R) \text{ if and only if } P_i^g = 0.$$

First, let $P_i^g = 0$, then each element $c^{-1}r \in Q^g$ belongs to $Q_{R\setminus P_i}(R)$ since $Rc \subsetneq P_i$ (remember that c is homogeneous). Conversely, suppose $P_i^g \neq 0$, choose a nonzero element $c \in P_i^g \cap h(C)$. We claim that

$$c^{-1} \notin Q_{R\setminus P_i}(R).$$

Suppose $c^{-1} \in Q_{R\setminus P_i}(R)$, then $Ic^{-1} \subset R$ for an ideal I of R with $I \subsetneq P_i$. But $I \subset Rc \subset P_i$, a contradiction. Hence

$$Q^g \subsetneq Q_{RP_i}(R) \quad \text{if } P_i^g = 0.$$

Note that Q^g is a graded simple ring, i.e., the only graded ideals of Q^g are 0 and Q^g itself. If I is a graded ideal of R, then $(R{:}I)$ is a graded fractional R-ideal. Hence, if I is a graded ideal, I^* is graded too. The graded class group $\text{Cl}_g\,(R)$ is defined by $\mathbf{D}_g(R)_{/\mathbf{P}_g(R)}$ where $\mathbf{D}_g(R)$ is the subgroup of $\mathbf{D}_g(R)$ of the graded divisorial ideals of R and

$$\mathbf{P}_g(R) = \{Rc|c \in h(C)^{-1}C\}.$$

It is easy to see that $\mathbf{D}_g(R)$ is the subgroup of $\mathbf{D}(R)$ generated by the graded prime ideals of height of one of R. Similarly

$$\mathrm{Pic}_g\,(R) = \mathbf{I}_g(R)_{/\mathbf{P}_g(R)} \quad \text{and} \quad \mathbf{I}_g(R) = \mathbf{D}_g(R) \cap \mathbf{I}(R).$$

THEOREM 3.2. *Suppose that R is an S-graded ring and R is a central Ω-Krull ring. The following sequences are exact*:

(1) $1 \to \mathrm{Cl}_g\,(R) \to \mathrm{Cl}\,(R) \to \mathrm{Cl}\,(Q^g) \to 1$

(2) $1 \to \mathrm{Pic}_g\,(R) \to \mathrm{Pic}\,(R) \to \mathrm{Pic}\,(Q^g)$.

Proof. (1) Consider the homomorphism

$$\psi{:}\mathbf{D}(R) \to \mathbf{D}(Q^g){:}I = \bigcap_i R_iI \mapsto \bigcap_{P_i^g=0} R_iI.$$

In the preceding paragraph we computed $\ker \psi$. In this case, $\ker \psi$ is generated by the homogeneous prime ideals of height one of R. Let

$$\overline{\psi}{:}\ \mathrm{Cl}\,(R) \to \mathrm{Cl}\,(Q^g).$$

Since $\psi(\mathbf{P}(R)) = \mathbf{P}(Q^g)$, we obtain

$$\ker \overline{\psi} = \frac{\ker \psi^*\mathbf{P}(R)}{\mathbf{P}(R)} \cong \frac{\ker \psi}{\ker \psi \cap \mathbf{P}(R)}.$$

Let $A \in \ker \psi \cap \mathbf{P}(R)$. Then $A = Rx(x \in K)$ and A is homogeneous. Hence $Q^gA = Q^gx = Q^g$ since A is homogeneous. Therefore x is invertible in Q^g. Write

$$x = \sum_{i=1}^{n} x_{g_i}$$

and all x_g homogeneous. Then there exists an element $y \in Q^g$, $y = \sum_{j=1}^{m} y_{h_j}$ such that $xy = 1$. Since G is torsion free abelian, G is orderable. Suppose that $g_1 < \ldots < g_n$ and $h_1 < \ldots < h_m$. From $xy = 1$ we then deduce that x is homogeneous. Hence

$$\ker \psi \cap \mathbf{P}(R) = \mathbf{P}_g(R) \quad \text{and}$$

$$\ker \overline{\psi} \cong \mathbf{D}_g(R)_{/\mathbf{P}_g(R)} = \mathrm{Cl}_g\,(R).$$

(2) Restrict ψ to $\mathbf{I}(R)$. Then

$$\psi' = \psi\,|{:}\mathbf{I}(R) \to \mathbf{I}\,(Q^g){:}I \mapsto Q^gI$$

is a homomorphism,

$$\ker \psi' = \mathbf{D}_g(R) \cap \mathbf{I}(R) = \mathbf{I}_g(R) \quad \text{and}$$

$$\ker \psi' \cap \mathbf{P}(R) = \mathbf{P}_g(R).$$

Therefore

$$\ker \bar{\psi}' \cong \mathrm{Pic}_g (R).$$

If R is a P.I. Ω-Krull ring, then $\mathrm{Cl}(Q^g)$ is always trivial. To establish this we need a proposition proved by F. Van Oystaeyen ([**19**], [**20**] p. 113) in the case of $\mathbf{Z}$-graded rings. But in fact the proof only uses the fact that the ring is graded by a (torsion free) abelian group.

PROPOSITION 3.3. *If R is a graded* P.I. *ring satisfying the identities of nxn matrices (e.g. R is a prime* P.I. *ring) and such that the center of R is a graded field, i.e, every homogeneous element is invertible, then R is an Azumaya algebra over its center.*

LEMMA 3.4. *If R is a graded simple, prime* P.I.*-ring, then R is a symmetric maximal order.*

Proof. Since C is a graded field, R is an Azumaya algebra by Proposition 3.3. If I is a fractional R-ideal, then $I = R(I \cap K)$ where K is the field of fractions of C (cf. [**5**]). Let $\alpha \in Q_{\mathrm{sym}}(R)$ and $\alpha I \subset I$. Then

$$(R\alpha R)I \subset I \quad \text{and} \quad ((R\alpha R) \cap K)(I \cap K) \subset (I \cap K).$$

Therefore $(R\alpha R) \cap K \subset C$ because C is completely integrally closed (cf. [**1**]). Hence

$$(R\alpha R) = R((R\alpha R) \cap K) \subset R \quad \text{and} \quad \alpha \in R.$$

This proves the result.

In particular, if R is a graded prime P.I.-ring, then $Q^g_{\mathrm{sym}}(R)$ is a symmetric maximal order.

THEOREM 3.5. *Let R be an S-graded ring and R a* P.I. *Ω-Krull ring. Then* $\mathrm{Cl}_g (R) = \mathrm{Cl}(R)$ *and* $\mathrm{Pic}_g (R) = \mathrm{Pic}(R)$.

Proof. In view of Theorem 3.2 we only need to prove that $\mathrm{Cl}(Q^g) = 1$. Since R is an Ω-Krull ring, the same is true for Q^g and therefore $Z(Q^g)$ is a Krull domain. Because $Z(Q^g)$ is a graded field (Q^g is graded simple) we have

$$\mathrm{Cl}(Z(Q^g)) = 1.$$

This is a result of Anderson ([**1**]). In particular, if $p \in X^1(Z(Q^g))$ we have

$$p = Z(Q^g)\alpha \quad \text{for some } \alpha \in Z(Q^g).$$

Now all ideals of Q^g are generated by the center of Q^g because Q^g is an Azumaya algebra (Proposition 3.3). Therefore if $P \in X^1(Q^g)$,

$$P = Q^g(P \cap Z(Q^g)) = Q^g\alpha$$

since

$$P \cap Z(Q^g) \in X^1(Z(Q^g)).$$

This proves the result.

LEMMA 3.6. *Let* $R = \bigoplus_{s \in S} R_s$ *be an* S*-graded ring,* R *an extension of* R_1 *and* R_1 *a ring satisfying Formanek's condition. If* R *is a* P.I. Ω*-Krull ring, then* R_1 *is* P.I. Ω*-Krull.*

Proof. Since R_1 is a P.I. ring, we only need to prove that R_1 is a symmetric maximal order and $Z(R_1)$ is a Krull domain (cf. Proposition 1.3). Now $Z(R)$ is a Krull domain and hence $Z(R_1) = (Z(R))_1$ is a Krull domain (cf. [**1**]). Let R_1' be a ring such that

$$R_1 \subset R_1' \subset Q_{\text{sym}}(R_1)$$

and $cR_1' \subset R_1$ for some $c \in Z(R_1)$. Since R is an extension of R_1, we have $RR_1' = R_1'R$ and hence RR_1' is a ring such that $R \subset RR_1'$ and $cRR_1' \subset R$. Therefore $RR_1' = R$, $R_1' \subset R$ and hence $R_1 = R_1'$.

PROPOSITION 3.7. *Let* R *be an* S*-graded ring,* R *an extension of* R_1 *and* S *a semigroup no element of which has an inverse. Suppose* R *and* R_1 *are central* Ω*-Krull rings. Then* $\bar{\psi}$: $\mathrm{Cl}\,(R_1) \to \mathrm{Cl}\,(R)$ *is injective.*

Proof. In view of Theorem 3.2. it suffices to prove that

$$\bar{\psi}: \mathrm{Cl}\,(R_1) \to \mathrm{Cl}_g\,(R)$$

is an injection. Note that in general $\bar{\psi}$ need not be a homomorphism. If I is a graded ideal of R_1, then it is easy to check that

$$(R:RI) \cap Q_{\text{sym}}(R_1) = (R_1:I).$$

It follows that if I is a divisorial R_1-ideal, then

$$(RI)^* \cap Q_{\text{sym}}(R_1) = I.$$

Now let $[I], [J] \in \mathrm{Cl}\,(R_1)$ and $\bar{\psi}([I]) = \bar{\psi}([J])$. We may suppose $I, J \subset R_1$. Hence

$$(RI)^*x = (RJ)^*y \quad \text{for some } x, y \in h(C).$$

We can write

$$(RI)^* = (RJ)^*(yx^{-1}).$$

Suppose yx^{-1} has degree t. Take an element a of degree 1 in $(RI)^*$, then $a = b.(yx^{-1})$ and $b \in (RJ)^* \subset R$. Therefore b has degree t^{-1}, $b \in R$ and hence $t^{-1} \in S$. We also have

$$(RJ)^* = (RI)^*(xy^{-1})$$

and xy^{-1} has degree t^{-1}. A similar reasoning as above yields that $t \in S$.

The hypothesis on S yields that $t = 1$. From this it is easy to conclude that $I = Jyx^{-1}$ and therefore $[I] = [J]$ in Cl (R_1).

Assume $R_1 \subset R = \bigoplus_{s \in S} R_s$ satisfies (PDE) and R is an extension of R_1. Let $\mathscr{L}^2_{R_1}(\sigma)$ be a symmetric T-functor of ideals of R_1. To $\mathscr{L}^2_{R_1}(\sigma)$ is associated a filter

$$\mathscr{L}^2(\sigma') = \{I \mid I \text{ an ideal of } R \text{ and } I_1 \subset I \text{ for some } I_1 \in \mathscr{L}^2_{R_1}(\sigma')\}.$$

Clearly $Q_\sigma(R_1) \subset Q_{\sigma'}(R)$. If $I \in \mathscr{L}^2(\sigma')$ and $I_1 \subset I$ with $I_1 \in \mathscr{L}^2_{R_1}(\sigma')$, then

$$Q_{\sigma'}(R)I \supset Q_{\sigma'}(R)Q_\sigma(R_1)I_1 \supset Q_{\sigma'}(R).$$

Hence $\mathscr{L}^2(\sigma')$ is a T-functor. In particular, σ' is an idempotent kernel functor (cf. [**8**]).

Proposition 3.8. *Let R be an S-graded ring, S a semigroup no element of which has an inverse, R an extension of R_1 satisfying* (PDE). *Suppose R and R_1 are central Ω-Krull rings. For each $P \in X^1(R_1)$, assume that $(RP)^*$ is a prime divisorial ideal of R. Let $\mathscr{L}^2_{R_1}(\sigma)$ be a symmetric T-functor in R_1. Then we have the following exact sequence*

$$1 \to A \to \mathrm{Cl}\,(R) \to \mathrm{Cl}\,(Q_{\sigma'}(R)) \to 1$$

where A is the subgroup of Cl (R_1) *generated by the prime ideals P of height one of R such that $P \in \mathscr{L}^2_{R_1}(\sigma)$.*

Proof. We have the following commutative diagram

$$\begin{tikzcd}
 & 1 \arrow[d] & & & \\
 & \mathbf{D}(R_1) = D_1 \times D_2 \arrow[d, shift right] \arrow[ddr] & & & \\
 & D_2 \arrow[u, bend right, shift right] \arrow[d] \arrow[dr, "\mu"] & & & \\
1 \arrow[r] & \ker\lambda \arrow[r] & \mathbf{D}(R) \arrow[r, "\lambda"] & \mathbf{D}(Q_{\sigma'}(R)) \arrow[r] & 1
\end{tikzcd}$$

where D_1(resp. D_2) is the subgroup of $\mathbf{D}(R_1)$ generated by the prime ideals $P \in X^1(R_1)$ such that $P \notin \mathscr{L}^2_{R_1}(\sigma)$ (resp. $P \in \mathscr{L}^2_{R_1}(\sigma)$). Ker λ is the subgroup of $\mathbf{D}(R)$ generated by the prime ideals $P \in X^1(R)$ such that $P \in \mathscr{L}^2(\sigma')$. Recall that the homomorphism μ from D_2 to $\mathbf{D}(R)$ is defined by sending I to $(RI)^*$. Note that by the construction of the filter $\mathscr{L}^2(\sigma')$, μ

sends D_2 to ker λ. If $P \in X^1(R)$ and $P \in \mathscr{L}^2(\sigma')$ (i.e., $P \in \ker \lambda$), then $P \cap R_1 \neq 0$. Therefore ht $(P \cap R_1) = 1$ since (PDE) is satisfied. Moreover we have

$$P = (R(P \cap R_1))^*$$

yielding that

$$D_2 \xrightarrow{\mu} \ker \lambda$$

is an isomorphism. We can now write a commutative diagram concerning the class group.

$$\begin{tikzcd}
 & \dfrac{D_2^*\mathbf{P}(R_1)}{\mathbf{P}(R_1)} \arrow[d, "\bar{\mu}"'] & & & \\
1 \arrow[r] & \ker \bar{\lambda} \arrow[r] \arrow[d, equal] & \mathrm{Cl}\,(R) \arrow[r, "\lambda"] & \mathrm{Cl}\,(Q_{\sigma'}(R)) \arrow[r] & 1 \\
 & \dfrac{\ker \lambda^*\mathbf{P}(R)}{\mathbf{P}(R)} & & &
\end{tikzcd}$$

Since $\mu(\mathbf{P}(R_1)) \subset \mathbf{P}(R)$, $\bar{\mu}$ is a well defined homomorphism. Note that $\bar{\mu}$ is a surjection, since μ is a surjection. The fact that μ is injective is proved in a similar way as in Proposition 3.6. For this purpose the condition that S does not contain inverse elements is needed.

COROLLARY 3.9. *Under the same conditions as in the preceding proposition, except that* $\mathscr{L}^2_{R_1}(\sigma)$ *is the filter of all ideals of* R_1 *in this case, we have an exact sequence*

$$1 \rightarrow \mathrm{Cl}\,(R_1) \rightarrow \mathrm{Cl}\,(R) \rightarrow \mathrm{Cl}\,(Q_{\sigma'}(R)) \rightarrow 1.$$

In particular, $\mathrm{Cl}\,(R_1) \cong \mathrm{Cl}\,(R)$ *if and only if* $\mathrm{Cl}\,(Q_{\sigma'}(R)) = 1$.

4. Direct limits of central Ω-Krull rings. The first proposition of this section is a generalization of a commutative result of [**6**].

PROPOSITION 4.1. *Let* $\{R_\alpha\}_{\alpha \in I}$ *be a filtered family of central Ω-Krull rings such that, for* $\alpha \leqq \beta$, *the embedding* $R_\alpha \hookrightarrow R_\beta$ *is an extension which satisfies* (PDE) *and* $(R_\beta P)^*$ *is a prime divisorial ideal for each* P *in* $X^1(R_\alpha)$. *Then* $\cup_{\alpha \in I} R_\alpha$ *is a central Ω-Krull ring. Furthermore,* $X^1(\cup_{\alpha \in I} R_\alpha)$ *is the direct limit of the system*

$$\{X^1(R_\alpha),\, i_{\alpha\beta}{:}X^1(R_\alpha) \rightarrow X^1(R_\beta){:}P \mapsto (R_\beta P)^*;\ \alpha, \beta \in I \textit{ and } \alpha \leqq \beta\},$$

notation $\varinjlim_{\alpha} X^1(R_\alpha)$.

Proof. Since $Z(R_\alpha) \subset Z(R_\beta)$ if $\alpha \leqq \beta$, it is clear that

$$Q_{\mathrm{sym}}\Big(\bigcup_{\alpha \in I} R_\alpha\Big) = \bigcup_{\alpha \in I} Q_{\mathrm{sym}}(R_\alpha).$$

Let $S = \cup_{\alpha \in I} R_\alpha$. We first prove that S satisfies Formanek's condition. Let J be a nonzero ideal of S, then $J \cap R_\alpha$ is a nonzero ideal of R_α, for some $\alpha \in J$. Therefore

$$\{0\} \neq J \cap Z(R_\alpha) \subset J \cap Z(S).$$

So it is also clear that S is a prime ring.

For a fixed α in I, let

$$I_\alpha = \{\beta \in I | \beta \geqq \alpha\}.$$

Let S_α denote the ring $\cup_{\alpha \in I} R_\beta$. Then $S = S_\alpha$ for each α. Suppose $P \in X^1(R_\alpha)$. For each $\beta \geqq \alpha$, there is a unique prime ideal P_β in $X^1(R_\beta)$ such that $P_\beta \cap R_\alpha = P$, namely $P_\beta = (R_\beta P)^*$. So, since

$$Q_{\mathrm{sym}}(R_\alpha) \subset Q_{\mathrm{sym}}(R_\beta) \quad \text{for } \alpha \leqq \beta,$$

it follows easily that

$$(R_\alpha)_{R_\alpha \setminus P} \subset (R_\beta)_{R_\beta \setminus P_\beta}.$$

(We denote by $(R_\alpha)_{R_\alpha \setminus P}$ the quotient ring $Q_{R_\alpha \setminus P}(R_\alpha)$.) Let

$$L_P = \bigcup_{\beta \geqq \alpha} (R_\beta)_{R_\beta \setminus P_\beta}.$$

We will now prove that L_P is a quasi-local Ω-ring. To this end consider a nonzero ideal J of L_P, then

$$J = \bigcup_{\beta \geqq \alpha} \left((R_\beta)_{R_\beta \setminus P_\beta} \cap J \right),$$

where $(R_\beta)_{R_\beta \setminus P_\beta} \cap J$ is an ideal (eventually zero) of $(R_\beta)_{R_\beta \setminus P_\beta}$ and

$$(R_\beta)_{R_\beta \setminus P_\beta} P_\beta = P_\beta (R_\beta)_{R_\beta \setminus P_\beta} = P'_\beta$$

is the unique maximal ideal of $(R_\beta)_{R_\beta \setminus P_\beta}$. Because $P_\beta = (R_\beta P)^*$ is divisorial, we obtain that

$$P'_\beta = (R_\beta)_{R_\beta \setminus P_\beta} P = (R_\beta)_{R_\beta \setminus P_\beta} P (R_\beta)_{R_\beta \setminus P_\beta}.$$

Thus,

$$J = \bigcup_{\beta \geqq \alpha} \left((R_\beta)_{R_\beta \setminus P_\beta} P \right)^{n_\beta},$$

where n_β is a natural number for $\beta \geqq \alpha$. Let

$$N_\alpha = \inf \{n_\beta | \beta \geqq \alpha\},$$

then

$$J = \bigcup_{\beta \geqq \alpha} (R_\beta)_{R_\beta \setminus P_\beta} P^{N_\alpha} = \left[\bigcup_{\beta \geqq \alpha} (R_\beta)_{R_\beta \setminus P_\beta} P \right]^{N_\alpha} = (L_P P)^{N_\alpha}.$$

It follows that L_P is a quasi-local Ω-ring. Because $L_P P \cap R_\alpha = P$ it is clear that

$$X^1(S) = \lim_{\substack{\rightarrow \\ \alpha}} X^1(R_\alpha)$$

in case S is a central Ω-Krull ring.

Secondly, we prove that L_P is a central localization of S. Indeed, let

$$T_\beta = Z(R_\beta) \setminus P_\beta$$

and let

$$T = \bigcup_{\beta \geqq \alpha} T_\beta.$$

Since $(R_\beta)_{R_\beta \setminus P_\beta} = T_\beta^{-1} R_\beta$ and since $T_{\beta'} \subset T_\beta$ for $\beta' \leqq \beta$, it is clear that $L_P = T^{-1}S$.

We show now that

$$\bigcap_{\substack{\alpha \in I \\ P \in X^1(R_\alpha)}} L_P = S.$$

Certainly $S \subset \cap L_P$. On the other hand, if $x \in \cap L_P$, then

$$x \in Q_{\text{sym}}(R_\beta) \cap (\cap L_P) \quad \text{for some } \beta \text{ in } I.$$

Thus $x = c^{-1} r$, with $c \in Z(R_\beta)$, $r \in R_\beta$. Now $Z(R_\beta)c \not\subset P$ for almost all $P \in X^1(R_\beta)$. So $x = c^{-1}r \in (R_\beta)_{R_\beta \setminus P}$ for almost all $P \in X^1(R_\beta)$. Let $\{P_1, \ldots, P_n\}$ be the finite subset of $X^1(R_\beta)$ such that $c \in P_i$, $1 \leqq i \leqq n$. Because $x \in \cap L_P$ there is a $\gamma \in I$, $\gamma \geqq \beta$, such that

$$x \in \bigcap_{P \in X^1(R_\beta)} (R_\gamma)_{R_\gamma \setminus P_\gamma}.$$

If P_γ is another minimal prime of R_γ then PDE yields

$$P_\gamma \cap Z(R_\beta) = \{0\};$$

therefore $c \in Z(R_\gamma) \setminus P_\gamma$. Thus

$$x \in \bigcap_{P \in X^1(R_\gamma)} (R_\gamma)_{R_\gamma \setminus P} = R_\gamma,$$

i.e., $x \in S$. So

$$\bigcap_{\substack{\alpha \in I \\ P \in X^1(R_\alpha)}} L_P = S.$$

It remains to prove that

$$\bigcap_{\substack{\alpha \in I \\ P \in X^1(R_\alpha)}} L_P = S$$

has the finite character property. Let $x \in S$, i.e., $x \in R_\beta$ for some $\beta \in I$. Because R_β is a central Ω-Krull ring, $R_\beta x R_\beta \not\subset P$ for all $P \in Y$, where $Y \subset X^1(R_\beta)$ and $X^1(R_\beta)\backslash Y$ is a finite set. So $L_P(SxS) = L_P$, for all $P \in X^1(R_\gamma)$ with $\gamma \geqq \beta$ and $P \cap R_\beta \in Y$. If $\gamma \geqq \beta$ and $Q \in X^1(R_\gamma)$ such that $Q \cap R_\beta = P \in X^1(R_\beta)$, resp. $Q \cap R_\beta = \{0\}$, then $L_P = L_Q$, resp. $L_Q(SxS) = L_Q$ because in this case

$$Z(R_\beta)^{-1}Z(R_\beta) \subset (R_\gamma)_{R_\gamma \backslash Q}$$

and because

$$SxS \cap Z(R_\beta) \neq \{0\}.$$

So it follows immediately that S satisfies the finite character property.

With the notations and assumptions as in Proposition 4.1 we have

COROLLARY 4.2.

(1) $\displaystyle \lim_{\substack{\longrightarrow \\ \alpha}} \mathrm{Cl}\,(R_\alpha) = \mathrm{Cl}\,(\bigcup_{\alpha \in I} R_\alpha)$

(2) $\displaystyle \lim_{\substack{\longrightarrow \\ \alpha}} \mathrm{Pic}\,(R) = \mathrm{Pic}\,(\bigcup_{\alpha \in I} R_\alpha).$

Proof. (1) As follows from the proof of the proposition, the minimal nonzero prime ideals of $S = \cup_{\alpha \in I} R_\alpha$ are of the form

$$\mathbf{P}_\alpha = \bigcup_{\beta \geqq \alpha} P_\beta,$$

where $P_\beta \in X^1(R_\beta)$ and $P_\beta \cap R_\alpha = R_\alpha$. Since

$$L_{P_\alpha}\mathbf{P}_\alpha = (SP_\alpha)_{S\backslash \mathbf{P}_\alpha}$$

we have $e(\mathbf{P}_\alpha, P_\alpha) = 1$. So it is clear that the extension

$$R_\alpha \hookrightarrow \bigcup_{\alpha \in I} R_\alpha$$

satisfies condition (PDE). Consequently, by Proposition 2.1, we obtain that

$$\lim_{\substack{\longrightarrow \\ \alpha}} \mathrm{Cl}\,(R_\alpha) = \mathrm{Cl}\,(\bigcup_{\alpha \in I} R_\alpha).$$

(2) Because of the assumptions on the extensions $R_\alpha \hookrightarrow R_\beta$, $\alpha \leqq \beta$, there is a bijection between $X^1(R_\alpha)$ and a subset Y of $X^1(R_\beta)$. If $P_\beta \in X^1(R_\beta)\setminus Y$, i.e., $P_\beta \cap R_\alpha = \{0\}$, then

$$Z(R_\alpha)\setminus\{0\} \subset Z(R_\beta)\setminus P_\beta.$$

Therefore, if X is an ideal of R_α, then

$$(R_\alpha X)^* \subset (R_\beta X)^* \text{ for } \alpha \leqq \beta.$$

To prove statement (2) we first need the following. Let $X \subset S$ be a divisorial S-ideal, so $X = \mathbf{P}_1 * \ldots * \mathbf{P}_k$, where $\mathbf{P}_i \in X^1(S)$ for $1 \leqq i \leqq k$ (we allow $\mathbf{P}_i = \mathbf{P}_j$ for $i \neq j$). Let $P_1, \ldots, P_k \in X^1(R_\alpha)$ such that

$$\mathbf{P}_i = \bigcup_{\beta \geqq \alpha} P^{(i)}$$

and where the union is taken over all $P_\beta^{(i)} \in X^1(R_\beta)$ with

$$P_\beta^{(i)} \cap R_d = P_i.$$

If $X' = P_1 * \ldots * P_k \in \mathbf{D}(R_\alpha)$, then $X = (SX')^*$. We claim:

$$X = \bigcup_{\beta \geqq \alpha} (R_\beta X')^*.$$

Indeed,

$$\begin{aligned} X = \mathbf{P}_1 * \ldots * \mathbf{P}_k &= \bigcup_{\beta \geqq \alpha} (R_\beta P_1)^* * \ldots * \bigcup_{\beta \geqq \alpha} (R_\beta P_k)^* \\ &= \Big(\bigcup_{\beta \geqq \alpha} (R_\beta P_1)^* \cdot \ldots \cdot \bigcup_{\beta \geqq \alpha} (R_\beta P_k)^*\Big)^* \\ &= \Big(\bigcup_{\beta \geqq \alpha} [(R_\beta P_1)^* \cdot \ldots \cdot (R_\beta P_k)^*]\Big)^* \\ &= \bigcap_{\substack{P \in X^1(R_\delta) \\ \delta \geqq \alpha}} \Big(\bigcup_{\substack{\beta \geqq \delta \\ P_\beta | P}} (R_\beta X')^*_{Z(R_\beta)\setminus P_\beta}\Big) \\ &= \bigcap_{\substack{P \in X^1(R_\delta) \\ \delta \geqq \alpha}} \Big(\bigcup_{\beta \geqq \delta} (R_\beta X')^*\Big)\Big(\bigcup_{\beta \geqq \delta} Z(R_\beta)\setminus P_\beta\Big)^{-1}. \end{aligned}$$

It follows:

$$\bigcup_{\beta \geqq \alpha} (R_\beta X')^* \subset X.$$

Conversely, let $x \in X$. Then, there is a $\beta \in I$ such that

$$x \in (R_\beta X')^*{}_{Z(R_\beta)\setminus P_\beta}, \quad P_\beta \in X^1(R_\beta);$$

i.e., $x = c^{-1}r$ with $r \in (R_\beta X')^*$ and $c \in Z(R_\beta)\setminus P_\beta$. Let $\{P_1, \ldots, P_n\}$ be the finite subset of $X^1(R_\beta)$ such that $c \in P_i$, $1 \leqq i \leqq n$. From the above equality it follows that there are $\gamma_i \in I$, $1 \leqq i \leqq n$, $\gamma_i \leqq \beta$, such that

$$x \in (R_{\gamma_i} X')^*{}_{Z(R_{\gamma_i})\setminus P_{\gamma_i}} \quad \text{and} \quad P_{\gamma_i} \cap P_\beta = P_i.$$

We obtain that for some $\delta \geqq \gamma_i$, $1 \leqq i \leqq n$,

$$x \in (R_\delta X')^*{}_{Z(R_\delta)\setminus P_\gamma} \quad \text{for all } P_\delta \in X^1(R_\delta)$$

with $P_\delta \cap R_\beta \in X^1(R_\beta)$. As before we obtain

$$x \in \bigcap_{P_\delta \in X^1(R_\delta)} (R_\delta X')_{Z(R_\delta)\setminus P_\delta} = (R_\delta X')^*.$$

This shows

$$X = \bigcup_{\beta \geqq \alpha} (R_\beta X')^*;$$

then also

$$X = \bigcup_{\beta \geqq \alpha} (R_\beta X')^*$$

for any divisorial S-ideal X, when X' is such that $X = (SX')^*$.

If X' is an invertible ideal in R, for some $\alpha \in I$, then SX' is an invertible ideal of S. Therefore, by (1) $\varinjlim_\alpha \text{Pic}\,(R_\alpha)$ is embedded in $\text{Pic}\,(\cup_{\alpha \in I} R_\alpha)$.

On the other hand if X is an invertible ideal of S, then $X \cdot Y = S$ for some fractional S-ideal Y of S. By the foregoing we have

$$\bigcup_{\beta \geqq \alpha} (R_\beta X')^* \cdot \bigcup_{\beta \geqq \alpha} (R_\beta Y')^* = S,$$

where X' and Y' are such that $X = (SX')^*$ and $Y = (SY')^*$. So there is a $\beta \geqq \alpha$ such that

$$1 = \sum_{i=1}^{n} x_i y_i,$$

where $x_i \in (R_\beta X')^*$ and $y_i \in (R_\beta Y')^*$ for $1 \leqq i \leqq n$. It follows that

$$R_\beta \subset (R_\beta X')^*(R_\beta Y')^* \subset (R_\beta X'Y')^* \subset R_\beta,$$

i.e.,

$$R_\beta = (R_\beta X')^* \cdot (R_\beta Y')^* \quad \text{and} \quad (R_\beta X')^* \in \text{Pic}\, R_\beta.$$

From $X = (S(R_\beta X')^*)^*$ we obtain

$$\varinjlim_\alpha \text{Pic}\,(R_\alpha) = \text{Pic}\,(\bigcup_{\alpha \in I} R_\alpha).$$

It is known (cf. [9]) that a polynomial ring $R[X_1, \ldots, X_n]$ is a central Ω-Krull ring if and only if R is a central Ω-Krull ring such that for all $P \in X^1(R)Q(Z(R/P))$ is an algebraic field extension of $Q(Z(R)/Z(R) \cap P)$, where $Q(Z(R/P))$, resp. $Q(Z(R)/Z(R) \cap P)$, denotes the field of fractions of $Z(R/P)$, resp. $Z(R)/Z(R) \cap P$. From Proposition 4.1 it follows that we can extend this result to polynomial rings over an arbitrary set of indeterminates, say $X = \{X_i\}_{i \in I}$.

COROLLARY 4.3. *A polynomial ring $R[X]$ is a central Ω-Krull ring if and only if R is a central Ω-Krull ring such that $Q(Z(R/P))$ is an algebraic field extension of $Q(Z(R)/Z(R) \cap P)$. In particular, if R is a* P.I. *ring, then $R[X]$ is a central Ω-Krull ring if and only if R is such a ring.*

Proof. If $R[X]$ is a central Ω-Krull ring then it is proved as in Proposition 3.8 of [9] that R is a central Ω-Krull ring with the properties listed in the statement. The converse follows immediately from Proposition 4.1. We obtain the last statement by combining Theorem 5.4 with [11], Theorem 3.7 in [9] and Proposition 1.2.

If R satisfies a polynomial identity, this result coincides with Theorem 3.7 in [13].

PROPOSITION 4.4. *Suppose R and $R[X]$ are central Ω-Krull rings. Then* $\mathrm{Cl}\,(R) \cong \mathrm{Cl}\,(R[X])$ *and* $\mathrm{Pic}\,(R) \cong \mathrm{Pic}\,(R[X])$.

Proof. First we prove that $\mathrm{Cl}\,(R) \cong \mathrm{Cl}\,(R[X])$. Consider the morphism

$$\mathbf{D}(R[X]) \xrightarrow{\phi} \mathbf{D}(Q[X])$$

mentioned in Section 2. Note that Q means $Q_{\mathrm{sym}}(R)$. In Section 2 we computed that $\ker \phi$ is the group freely generated by the prime ideals $P \in X^1(R[X])$ such that $P \cap R \neq \{0\}$. Therefore

$$P = (P \cap R)[X] \quad \text{and} \quad \ker \phi \cong \mathbf{P}(R).$$

We also have

$$1 \to \ker \phi \to \mathrm{Cl}\,(R[X]) \to \mathrm{Cl}\,(Q[X]) \to 1 \quad \text{and}$$

$$\ker \bar{\phi} = \ker \phi^* \mathbf{P}(R[X])/\mathbf{P}(R[X]) \cong \ker \phi / \ker \phi \cap \mathbf{P}(R[X]).$$

Here we needed that $\mathbf{P}(R[X]) \to \mathbf{P}(Q[X])$ is a surjection. This is satisfied since $\mathbf{D}(Q[X]) = \mathbf{P}(Q[X])$ (recall that all ideals of $Q[X]$ are generated by a central element). Suppose now

$$I[X] \in \ker \phi \cap \mathbf{P}[R[X]).$$

Then

$$I[X] = R[X]\alpha, \quad \alpha \in Z(Q[X]).$$

Choose $0 \neq c \in I \cap Z(R)$ and $c = \beta\alpha$, $\beta \in R[X]$. Since α is a central element, all coefficients of α are central and hence non zero divisors. By writing out the equation $c = \beta\alpha$ and looking at the terms of highest and lowest degree, one concludes that α is a homogeneous element belonging to R, yielding that

$$I = R\alpha \quad \text{and} \quad \ker \phi \cap \mathbf{P}(R[X]) \cong \mathbf{P}(R).$$

Therefore $\ker \phi \cong \mathrm{Cl}\,(R)$ and $\mathrm{Cl}(R) \cong \mathrm{Cl}\,(R[X])$ since $\mathrm{Cl}\,(Q[X]) = 1$.

So φ: Cl $(R \to$ Cl $(R[X])$ with $\varphi([I]) = [I[X]]$, $[I] \in \mathrm{Cl}\,(R)$, is an isomorphism. Restrict φ to Pic (R). Then

$$\varphi_1: \mathrm{Pic}\,(R) \to \mathrm{Pic}\,(R[X])$$

is clearly an injection. It remains to prove that φ_1 is surjective. Let $[A] \in \mathrm{Pic}\,(R[X])$, then there is exactly one $[B]$ in $\mathrm{Cl}\,(R)$ such that $[A] = [B[X]]$, and hence

$$A = B[X]R[X]\alpha$$

for some central element $\alpha \in K(X)$. Therefore $B[X]$ is invertible since the same is true for A and $R[X]\alpha$. We can write

$$C \,.\, B[X] = B[X] \,.\, C = R[X],$$

where C is an invertible $R[X]$-ideal. We can conclude that

$$C = (R[X]:B[X]) = (R:B)[X]$$

so that $B(R:B) = (R:B)B = R$ and $[B] \in \mathrm{Pic}\,(R)$.

COROLLARY 4.5. *Suppose R and $R[X]$ are central Ω-Krull rings. Then* $\mathrm{Cl}\,(R) \cong \mathrm{Cl}\,(R[X])$ *and* $\mathrm{Pic}\,(R) \cong \mathrm{Pic}\,(R[X])$.

Proof. $R[X]$ is a central Ω-Krull ring by Corollary 4.3. The rest follows from Corollary 4.2.

Consider the following category C: the objects are prime rings satisfying Formanek's condition; $f \in \mathrm{Hom}\,(R, S)(R, S \in \mathrm{Ob}\, C)$ if $f:R \to S$ is a ring homomorphism, $f(Z(R)) \subset Z(S)$ and $Sf(I) = f(I)S$ for all ideals I of R. For example, when $R \subset S$ is a ring-extension, the inclusion map is such a homomorphism. It is easy to check that this is indeed a category. In Section 2 the Picard group Pic (R) of an Ω-Krull ring was defined. It is clear from the construction that Pic (R) can be defined in the same way for prime, Formanek rings: $\mathbf{I}(R)$ is the group of fractional R-ideals which are invertible. $\mathbf{P}(R)$ is the group of principal R-ideals (i.e., Rx for some $x \in K$ (K is the field of fractions of $Z(R)$)). Then

$$\mathbf{P}(R) \lhd \mathbf{I}(R) \quad \text{and} \quad \mathrm{Pic}\,(R) = \mathbf{I}(R)/\mathbf{P}(R).$$

Moreover, Pic is a functor from the category C to the category of groups.

The proof of the following proposition is the same as in the Proposition 6.1 of [**1**].

PROPOSITION 4.6. *Let S be a semigroup no element of which has an inverse. Let $R = \bigoplus_{s \in S} R_s$ be an S-graded ring and F a functor on rings. If $R \hookrightarrow R[S]$ induces an isomorphism $F(R) \to F(R[S])$, then $R_1 \hookrightarrow R$ induces an isomorphism $F(R_1) \to F(R)$.*

Note that all subsemigroups of free semigroups satisfy the condition that no element has an inverse.

PROPOSITION 4.7. *Suppose S is a subsemigroup (with 1) of a free semigroup T, R an S-graded ring, R an extension of R_1, R and R_1 central Ω-Krull rings and for all $P \in X^1(R)Z(R/P)$ is algebraic over $C/(P \cap C)$. Then*

$$\text{Pic}\,(R_1) \cong \text{Pic}\,(R).$$

Proof. It is clear that R is also a T-graded ring by taking $R_t = R_s$ if $t = s \in S$ and $R_t = 0$ if $t \in T \backslash S$. Since T is a free semigroup

$$R[T] \cong R[(X_i)_{i \in I}]$$

for some index set I. By the hypothesis on $X^1(R)$, $R[T]$ is an Ω-Krull ring (Corollary 4.3). Since

$$\text{Pic}\,(R) \cong \text{Pic}\,(R[T])$$

(Corollary 4.5), the preceding proposition yields the desired result.

COROLLARY 4.8. *Suppose S is a subsemigroup (with 1) of a free semigroup T, R an S-graded ring such that R is an extension of R_1. If R is a* P.I. *Ω-Krull ring, then*

$$\text{Pic}\,(R_1) \cong \text{Pic}\,(R).$$

Proof. The fact that R is a P.I. Ω-Krull ring implies that R_1 is a P.I. Ω-Krull ring (Lemma 3.6) and $R[T]$ is a P.I. Ω-Krull ring by Corollary 4.3. The result is clear now from Proposition 4.7.

The following lemma generalizes Lemma 14.1 in [**6**]; the proof is similar.

LEMMA 4.9. *Let R be a prime Formanek ring and a and b central elements of R such that $aR \cap bR = abR$, thus $a^nR \cap bR = a^nbR$ for all $n \in$* **N**. *Then $R[X](bX - a)$ is a prime ideal in $R[X]$.*

PROPOSITION 4.10. *Let R be a central Ω-Krull ring such that $Q(Z(R/P))$ is algebraic over $Q(Z(R)/Z(R) \cap P)$ for all $P \in X^1(R)$. Then there is a flat ring-extension $R \hookrightarrow T$ satisfying the properties*:

(a) *T is an Ω-ring*;

(b) *T is a global Zariski-extension* (*cfr.* [**20**]) *of its center*;

(c) $\mathrm{Cl}(R) \cong \mathrm{Cl}(T)$;

(d) *$Z(T)$ is a Dedekind domain and* $\mathrm{Cl}(Z(T)) \cong \mathrm{Cl}(Z(R))$.

Proof. Since $Z(R)$ is a Krull domain (cfr. [**8**]), we know (cf. [**6**]) that there is a flat extension $Z(R) \hookrightarrow T'$, T' is a commutative ring, such that T' is a Dedekind domain and $\mathrm{Cl}(T') \cong \mathrm{Cl}(Z(R))$. If one goes through the proof of this result, then one sees that T' is obtained by taking first a polynomial extension of countable degree and then a localization to a multiplicatively closed set, say M:

$$Z(R) \hookrightarrow Z(R)[X_1, \dots, X_n, \dots] \hookrightarrow M^{-1}Z(R)[X^1, \dots, X_n, \dots].$$

Moreover by the foregoing lemma and the construction of the elements of M, M is generated by elements that generate a prime ideal of $R[X_1, \dots, X_n, \dots]$. Let

$$T = M^{-1}R[X_1, \dots, X_n, \dots],$$

then $R \hookrightarrow T$ is a flat ring-extension and T is a central Ω-Krull ring (Corollary 4.3 and Corollary 2.5 in [**7**]) the center of which is a Dedekind domain. Therefore all the minimal nonzero prime ideals of T are maximal and T is a global Zariski extension of its center. It follows (cf. [**7**]) that T has property (a). By Corollary 2.3 and Corollary 4.5 we obtain

$$\mathrm{Cl}(R) \cong \mathrm{Cl}(T), \quad \mathrm{Cl}(Z(R)) \cong \mathrm{Cl}(Z(T)).$$

5. Twisted group rings. Let R be a ring with unity, S an abelian semigroup with unity and γ a two-cocycle of S into the central invertible elements of R, i.e.,

$$\gamma \in H^2(S, \mathcal{U}(R) \cap Z(R))$$

where $\mathcal{U}(R)$ is the set of invertible elements of R. Then the *twisted semigroup ring* $R^t[S]$ (cfr. [**15**]) is the set of all formal sums

$$\alpha = \sum_{s \in S} r_s \bar{s},$$

where $r_s \in R$ and almost all r_s are zero. By supp α we denote the finite set of all $s \in S$ such that $r_s \neq 0$. Multiplication and addition are defined as follows:

$$\sum_{s\in S} a_s\bar{s} + \sum_{s\in S} b_s\bar{s} = \sum_{s\in S} (a_s + b_s)\bar{s},$$

$$\bar{s}r = r\bar{s},$$

$$\bar{s}.\bar{t} = \gamma(s, t)\overline{st},$$

where r, a_s, $b_s \in R$ and s, $t \in S$. The ring $R^t[S]$ has an identity element, namely $1 = \gamma(1, 1)^{-1}\bar{1}$ and without loss of generality we will assume throughout that $\bar{1} = 1$. Let S_f be the set of those elements $s \in S$ such that $\gamma(s, t) = \gamma(t, s)$ for all $t \in S$. Clearly S_f is a subsemigroup of S and

$$Z(R^t[S]) = \{\alpha = \sum_{s\in S} r_s\bar{s} \,|\, \text{supp } \alpha \subset S_f \text{ and } r_s \in Z(R)\}.$$

We say that a cocycle γ on S has property (SF) if for all $s \in S$ there exists a $t \in S$ such that $st \in S_f$.

PROPOSITION 5.1. *Let S be a torsion free abelian semigroup and R a ring. Then, a twisted semigroup ring $R^t[S]$ is a prime Formanek ring if and only if R is prime Formanek and γ has property* (SF). *In particular, if S is a group then $R^t[S]$ is prime Formanek if and only if R is prime Formanek.*

Proof. Suppose $R^t[S]$ is prime Formanek. Then, clearly, R is a prime Formanek ring and for all $s \in S$ the ideal $\bar{s}R^t[S]$ intersects the center non-trivially. Therefore, there is a $t \in S$ such that $st \in S_f$.

Conversely, let

$$\alpha = \sum_{i=1}^{n} a_i\bar{s}_i \quad \text{and} \quad \beta = \sum_{j=1}^{m} b_j\bar{t}_j$$

be two elements of $R^t[S]$ with $a_i \neq 0$, $b_j \neq 0$ for all $1 \leqq i \leqq n$, $1 \leqq j \leqq m$ and $\alpha R^t[S]\beta = 0$. Since S is an ordered semigroup we can assume $s_i < s_j$ and $t_k < t_\ell$ for $i < j$ and $k < \ell$. Then s_nt_m is a uniquely presented element in the set $\{s_it_j | 1 \leqq i \leqq n, 1 \leqq j \leqq m\}$, so

$$a_n\bar{s}_nRb_m\bar{t}_m = a_nRb_m\overline{s_nt_m} = 0.$$

In particular $a_nRb_m = 0$, a contradiction. Therefore $R^t[S]$ is a prime ring. Let I be a non-trivial ideal of $R^t[S]$ and let

$$\alpha = \sum_{i=1}^{n} a_i\bar{s}_i$$

be an element of minimal length in $I\backslash\{0\}$, i.e., supp α has a minimal number of elements for $0 \neq \alpha \in I$. Because R is Formanek and γ has property (SF) we may suppose that $a_n \in Z(R)$ and $s_n \in S_f$. A straightforward calculation shows that $\alpha \in Z(R^t[S])$, in particular

$$I \cap Z(R^t[S]) \neq \{0\}.$$

Therefore $R^t[S]$ is Formanek.

If R is a prime Formanek ring and σ an automorphism of R, then we denote by $R^\sigma[\mathbf{Z}]$ the skew polynomial ring $R[X, X^{-1}, \sigma]$ in the indeterminate X and its inverse X^{-1}. Of course we can extend σ to an automorphism of $Q_{\mathrm{sym}}(R) = Q$; we will also denote this automorphism by σ.

LEMMA 5.2. *Let I be an ideal of $R^\sigma[\mathbf{Z}]$, then IQ is an ideal of $Q^\sigma[\mathbf{Z}]$.*

Proof. It is enough to prove that $IQ \supset QI$. Let $\alpha \in I$ and $0 \neq c \in Z(R)$, then we must show that $c^{-1}\alpha \in IQ$. Suppose this is not the case and suppose

$$\alpha = \sum_{i=0}^{m} a_i X^i, \quad a_i \in R$$

is of minimal positive degree such that $c^{-1}\alpha \notin IQ$ for some $0 \neq c \in Z(R)$. Because $\alpha\sigma^{-m}(c) - c\alpha$ is of lower degree than α, we obtain

$$c^{-1}(\alpha\sigma^{-m}(c) - c\alpha) = \beta \in IQ.$$

Thus

$$c^{-1}\alpha = (\alpha + \beta)(\sigma^{-m}(c))^{-1} \in IQ,$$

a contradiction. So $QI \subset IQ$.

LEMMA 5.3. *Let R be a prime Formanek ring which is a symmetric maximal order. If $R^\sigma[\mathbf{Z}]$ is prime Formanek, then $R^\sigma[\mathbf{Z}]$ is a symmetric maximal order.*

Proof. We have to prove that $(I:_lI) = R^\sigma[\mathbf{Z}]$ for every ideal of $R^\sigma[\mathbf{Z}]$ (Lemma 1.1). Because $Q^\sigma[\mathbf{Z}]$ is a localization at an Ore set of $Q^\sigma[X, \sigma]$ and because in this last ring every ideal is generated by a normalizing element (a result of [**2**]) we have $IQ = Q\beta$, where β is a normalizing element of $Q^\sigma[\mathbf{Z}]$. Suppose

$$\alpha I \subset I, \quad \alpha \in Q_{\mathrm{sym}}(R^\sigma[\mathbf{Z}]),$$

then $\alpha\beta \in Q^\sigma[\mathbf{Z}]\beta$, i.e.,

$$\alpha = \sum_{i=-n}^{m} q_i X^i \in Q^\sigma[\mathbf{Z}].$$

Let $C(I)$ be the ideal of all the elements of R which occur as the leading coefficient of an element of I. Then $q_m C(I) \subset C(I)$. Since R is a symmetric maximal order, $q_m \in R$. By induction we obtain that $\alpha \in R^\sigma[\mathbf{Z}]$. This completes the proof.

We will now establish when a twisted group ring over a torsion free abelian group is a symmetric maximal order. In case the group is not abelian, this problem is open, even in the P.I. case.

LEMMA 5.4. *Let G be a finitely generated free abelian group and suppose R is a prime Formanek ring which is a symmetric maximal order. Then $R^t[G]$ is prime Formanek and a symmetric maximal order.*

Proof. It follows from Proposition 5.1 that $R^t[G]$ is prime Formanek. Let

$$G = \mathbf{Z} \times \ldots \times \mathbf{Z},$$

with n factors, and let

$$\gamma \in H^2(G, Z(R) \cap \mathcal{U}(R))$$

be the defining 2-cocycle for $R^t[G]$. We will prove by induction on n that $R^t[G]$ is a symmetric maximal order. For $n = 1$ the result follows from Lemma 5.3 because $R^t[\mathbf{Z}] \cong R[\mathbf{Z}]$. Suppose now the result is true for all $m < n$, $m \in \mathbf{N}_0$. Let

$$G^* = \mathbf{Z} \times \ldots \times \mathbf{Z},$$

$n-1$ factors, so $G = G^* \times \mathbf{Z}$. Let $R^t[G^*]$ be the twisted group ring with defining 2-cocycle γ restricted to G^*. Because, for $(0, z) \in G^* \times \mathbf{Z}$,

$$R^t[G]\overline{(0, z)} = \overline{(0, z)}R^t[G]$$

one easily checks that

$$R^t[G^*]\overline{(0, z)} = \overline{(0, z)}R^t[G^*].$$

In particular

$$\overline{(0, z)}R^t[G^*]\overline{(0, z)}^{-1} = R^t[G^*]$$

and $\overline{(0, z)}$ induces a conjugation on $R^t[G^*]$, say φ. If γ' is the 2-cocycle γ restricted to $\{0\} \times \mathbf{Z}$, it follows that

$$R^t[G] = (R^t[G^*])[\{0\} \times \mathbf{Z}, \varphi, \gamma']$$

is a crossed product (cfr. [**15**]). Because

$$H^2(\mathbf{Z}, Z(R) \cap \mathcal{U}(R)) = 0$$

we obtain that

$$R^t[G] \cong (R^t[G^*])\varphi[\mathbf{Z}],$$

i.e., a skew group ring over $\mathbf{Z}$. By the induction hypothesis $R^t[G^*]$ is a prime Formanek ring which is a symmetric maximal order. Therefore, by Lemma 5.3, $R^t[G]$ is a symmetric maximal order.

Before proving the statement of Lemma 5.4 for arbitrary torsion free abelian groups we need the following.

LEMMA 5.5. *Let $\{R_\alpha\}_{\alpha \in I}$ be a filtered family of prime Formanek rings which are symmetric maximal orders. Let $R = \cup_{\alpha \in I} R_\alpha$ and suppose*

$$R_\alpha = R \cap Q_{\text{sym}}(R_\alpha) \quad \textit{for all } \alpha \in I.$$

If R is a Formanek ring, then R is a prime ring which is a symmetric maximal order.

Proof. Obviously, R is a prime ring and

$$Q_{\text{sym}}(R) = \bigcup_{\alpha \in I} [R_\alpha(Z(R) \cap R_\alpha)^{-1}].$$

We have to prove that $(J:_\ell J) = R$ if J is an ideal of R. Suppose $x \in (J:_\ell J)$, then, for some $\alpha \in I$,

$$x \in R_\alpha(Z(R) \cap R_\alpha)^{-1} \subset Q_{\text{sym}}(R_\alpha) \quad \text{and} \quad J \cap R_\alpha \neq \{0\}.$$

So,

$$x(J \cap R_\alpha) \subset J \cap Q_{\text{sym}}(R_\alpha) \cap R = J \cap R_\alpha.$$

Because R_α is a symmetric maximal order we obtain $x \in R_\alpha$, i.e., $x \in R$.

COROLLARY 5.6. *Let R be a prime Formanek ring which is a symmetric maximal order and let G be a torsion free abelian group. Then every twisted group ring $R^t[G]$ is a prime Formanek ring and a symmetric maximal order.*

Proof. Since

$$R^t[G] = \bigcup_{n \in \mathbf{N}_0} R^t[G_n],$$

where G_n are finitely generated free abelian groups, and because

$$R^t[G] \cap Q_{\text{sym}}(R^t[G_n]) = R^t[G_n]$$

the result follows from Proposition 5.1, Lemma 5.4 and Lemma 5.5.

In the last results of this section we consider P.I. rings.

PROPOSITION 5.7. *Let R be a ring and G a torsion free abelian group. A twisted group ring $R^t[G]$ is a prime* P.I. *ring if and only if G/G_f is finite and R is a prime* P.I. *ring.*

Proof. By Proposition 5.1 R is prime if and only if $R^t[G]$ is prime. Of course

$$R^t[G] = R \otimes_C C^t[G],$$

where $C = Z(R)$ and $C^t[G]$ is defined by the same cocycle as $R^t[G]$. Therefore $R^t[G]$ is P.I. if and only if R is P.I. and $C^t[G]$ is P.I. If K is the quotient field of C, we obtain that $R^t[G]$ is P.I. if and only if R is P.I. and $K^t[G]$ is P.I. So it remains to prove that $K^t[G]$ is P.I. if and only if G/G_f is finite in case K is a field.

Suppose $K^t[G]$ is P.I. and prime, then by Posner's theorem

$$Q = K^t[G_f]^{-1}K[G]$$

is finite dimensional over $Z(Q) = K^t[G_f]^{-1}K[G_f]$. (Note that $Z(K^t[G]) = K^t[G_f]$). Let $\alpha_1, \ldots, \alpha_n$ be a basis of Q over $Z(Q)$. Of course we may suppose that $\alpha_i \in K^t[G]$, $1 \leqq i \leq n$. Suppose

$$\alpha_i = \sum_{j=1}^{n} a_{ij}\overline{g_{ij}}, \quad g_{ij} \in G, \ a_{ij} \in K, \ n_i \in \mathbf{N}_0.$$

Let $g \in G$, then $\overline{g} = \sum \beta^{-1}\beta_i\alpha_i$, where $\beta, \beta_i \in K^t[G_f]$. Thus

$$\beta\,\overline{g} = \sum_{i,j} \beta_i a_{ij}\overline{g_{ij}},$$

in particular $G_f g = G_f g_{ij}$ for some i, j, $1 \leqq i \leqq n$, $1 \leqq j \leqq n_i$. Therefore G/G_f is finite.

Conversely, if G/G_f is finite then

$$Q_{\text{sym}}(K^t[G]) = (K^t[G_f])^{-1}K^t[G]$$

is finite dimensional over its center. So it satisfies a polynomial identity and the same is true for $K^t[G]$.

LEMMA 5.8. *Let G be an abelian group and G^* a subgroup with* ACC *on cyclic subgroups such that G/G^* is finite. Then G has* ACC *on cyclic subgroups.*

Proof. Let $\{g_i\}_{i\in\mathbf{N}_0}$ be a subset of G and suppose that

$$\langle g_1\rangle \subset \langle g_2\rangle \subset \ldots \subset \langle g_n\rangle \subset \ldots,$$

where $\langle g_i\rangle$ is the cyclic subgroup generated by g_i. Because G/G^* is finite,

$$\langle g_n\rangle G^* = \langle g_{n+i}\rangle G^*,$$

for some $n \in \mathbf{N}_0$ and for all $i \in \mathbf{N}$. Now let $a_i \in \mathbf{N}_0$ be the smallest number such that $g_i^{a_i} \in G^*$, then

$$\langle g_1{}^{a_1}\rangle \subset \langle g_2{}^{a_2}\rangle \subset \ldots \subset \langle g_n{}^{a_n}\rangle \subset \ldots.$$

Because G^* has ACC on cyclic subgroups,

$$\langle g_m\rangle \cap G^* = \langle g_m{}^{a_m}\rangle = \langle g_{m+i}^{a_{m+i}}\rangle = \langle g_{m+i}\rangle \cap G^*$$

for some $m \in \mathbf{N}_0$ and for all $i \in \mathbf{N}$. Of course we can suppose $n = m$. Now let $i \in \mathbf{N}$, then $g_{n+i} = g_n^{\alpha} \cdot x$ with $x \in G^*$ and $\alpha \in \mathbf{Z}_0$. So

$$x = g_{n+i}g_n^{-\alpha} \in \langle g_{n+i} \rangle \cap G^* = \langle g_n \rangle \cap G^*.$$

Therefore $x = g_n^b$, $b \in \mathbf{Z}_0$ and thus $g_{n+i} \in \langle g_n \rangle$. This proves that $\langle g_{n+i} \rangle = \langle g_n \rangle$ for all $i \in \mathbf{N}$.

It is easy to check that the preceding lemma still holds if G/G^* is a finitely generated (abelian) group.

THEOREM 5.9. *Let $R^t[G]$ be a prime* P.I. *ring, then $R^t[G]$ is an Ω-Krull ring if G satisfies the* ACC *on cyclic subgroups and R is an Ω-Krull ring. The converse holds if the two-cocycle is trivial.*

Proof. Suppose $R[G]$ is an Ω-Krull ring. By [**7**], $Q[G]$ is an Ω-Krull ring and hence $k[G]$ is a Krull domain ($k = Z(Q)$). Therefore G has ACC on cyclic subgroups (cf. [**1**]). Conversely, by Corollary 5.6, $R^t[G]$ is a symmetric maximal order. By [**1**], $k^t[G_f]$ is a Krull domain. Moreover $Z(R)^t[G_f]$ is a graded Krull domain in the sense of [**0**] because $Z(R)$ is a Krull domain. Hence $Z(R)^t[G_f] = Z(R^t[G])$ is a Krull domain by [**0**].

PROPOSITION 5.10. *If $R^t[G]$ is a* P.I. Ω-Krull ring, *then*

$$\mathrm{Cl}\,(R^t[G]) \cong \mathrm{Cl}\,(R) \quad \textit{and} \quad \mathrm{Pic}\,(R^t[G]) \cong \mathrm{Pic}\,(R).$$

Proof. We have

$$\mathrm{Cl}\,(R^t[G]) \cong \mathrm{Cl}_g(R^t[G])$$

by Theorem 3.4. Every homogeneous ideal I of $R^t[G]$ is of the form $A^t[G]$, where A is an ideal of $R^t[G]$. Moreover, $I = A^t[G]$ is a homogeneous divisorial (resp. homogeneous principal) if and only if A is divisorial (resp. principal). Therefore

$$\mathrm{Cl}\,(R) \cong \mathrm{Cl}_g\,(R^t[G]) \cong \mathrm{Cl}\,(R^t[G]).$$

Similarly one proves that

$$\mathrm{Pic}\,(R) \cong \mathrm{Pic}\,(R^t[G]).$$

6. Twisted semigroup rings. In the preceding paragraph twisted semigroup rings were introduced. When S is a torsion free abelian cancellative semigroup, it was proved that $R^t[S]$ is prime Formanek if and only if R is prime Formanek and for all $s \in S$ there is a $t \in S$ such that $st \in S_f$, i.e., γ has property (SF).

From now on all semigroups considered will be torsion free abelian cancellative and R is a prime ring satisfying Formanek's condition.

LEMMA 6.1. *If* γ *has property* (SF), *then* $Q^g_{\mathrm{sym}}(R^t[S])$ *is isomorphic to a twisted semigroup ring* $Q^t[\langle S\rangle]$, *where* $\langle S\rangle$ *is the quotient group of* S *and* $Q = Q_{\mathrm{sym}}(R)$. *Moreover the defining* 2-*cocycle* γ *for* $Q^t[\langle S\rangle]$ *satisfies*:

(1) $\gamma' \in H^2(\langle S\rangle, \mathscr{U}(R) \cap Z(R))$,

(2) $\gamma'_{|S\times S} = \gamma$.

In particular

$$R^t[S] \subset R^t[\langle S\rangle],$$

where $R^t[\langle S\rangle]$ *is the twisted semigroup ring defined by* γ'.

Proof. Every element of $\langle S\rangle$ can be written as $a^{-1}b$ where $a, b \in S$. In fact, each $a^{-1}b$ is an equivalence class, by the construction of $\langle S\rangle$. For every equivalence class we choose a fixed representative. Moreover, if $a^{-1}b \in \langle S\rangle$, then there exists a $t \in S$ such that $ta \in S_f$, and hence

$$a^{-1}b = (ta)^{-1}tb.$$

Therefore, we may fix a representant $a^{-1}b$ for every equivalence class such that $a \in S_f$ and if $s \in S$ then s itself is representing its equivalence class. Every element of $Q^g_{\mathrm{sym}}(R^t[S])$ is of the form $\sum q_i \,\overline{s_i}^{-1}\overline{t_i}$, where $s_i \in S_f$, all $t_i \in S$ and $q_i \in Q$, and the $s_i^{-1}t_i$ are the chosen representants. Denote $\overline{s_i}^{-1}\overline{t_i}$ by $\widetilde{s_i^{-1}t_i}$. It is clear that q_i commutes with $\widetilde{s_j^{-1}t_j}$. The only thing that remains to be proved is how $\widetilde{s_j^{-1}t_j}$ and $\widetilde{s_j^{-1}t_j}$ are multiplied. Now,

$$\widetilde{s_i^{-1}t_i}\,.\,\widetilde{s_j^{-1}t_j} = \gamma(\overline{s_j}\overline{s_i})^{-1}\overline{t_i}\overline{t_j} = \gamma(s_j, s_i)^{-1}\gamma(t_i, t_j)\overline{s_js_i}^{-1}\,\overline{t_it_j}$$

$$= \gamma(s_j, s_i)^{-1}\gamma(t_i, t_j)\gamma(s'', t_it_j)\gamma(s_js_i, t'')^{-1}\widetilde{s''^{-1}t''},$$

where the $s''^{-1}t''$ is the fixed representant of the equivalence class of $(s_js_i)^{-1}(t_it_j)$. Hence we can view $Q^g_{\mathrm{sym}}(R^t[S])$ as a twisted semigroup ring $Q^t[\langle S\rangle]$ and the defining 2-cocycle satisfies the condition of the statement.

Recall that

$$Z(R^t[S]) = Z(R)^t[S_f].$$

Note that we have

$$Z(R^t[S]) \subset Z(R^t[\langle S\rangle]) \subset Z(R^t[S])^{-1}Z(R^t[S])$$

and therefore $S_f \subset \langle S_f\rangle$ and $\langle S_f\rangle = \langle S\rangle_f$.

LEMMA 6.2. *Let $R^t[S]$ be a twisted semigroup ring which is prime Formanek. The following statements are equivalent*:

(1) $R^t[S]$ *is a* P.I. ring;

(2) $R^t[\langle S\rangle]$ *is a* P.I. *ring.*

Proof. Since

$$R^t[S] \subset R^t[\langle S\rangle] \subset Q_{\text{sym}}(R^t[S])$$

and since $R^t[S]$ is P.I. if and only if $Q_{\text{sym}}(R^t[S])$ is P.I., because this last ring is a central localization of $R^t[S]$, the result follows.

LEMMA 6.3. *Let $R^t[S]$ be a central Ω-Krull ring. Then R is a central Ω-Krull ring.*

Proof. This is similar to the proof of Proposition 3.10 of [**9**].

PROPOSITION 6.4. *Suppose that $R^t[S]$ is a* P.I.-*Ω-Krull ring. Then*

(1) $\mathrm{Cl}(R^t[S]) \cong \mathrm{Cl}(R) \oplus \mathrm{Cl}(Q^t[S])$

$\mathrm{Pic}(R^t[S]) \cong \mathrm{Pic}(R) \oplus \mathrm{Pic}(Q^t[S])$

(2) $\mathrm{Cl}(Q^t[S])$ *and* $\mathrm{Pic}(Q^t[S])$ *are independent of the simple ring.*

Proof. (1) By Proposition 1.2 and Lemma 6.3 R is a P.I. Ω-Krull ring. We have an exact sequence

$$1 \to \ker\phi \to \mathbf{D}(R^t[S]) \xrightarrow{\phi} \mathbf{D}(Q^t[S]) \to 1$$

(cf. Section 2) and $\ker\phi \cong \mathbf{D}(R)$ by sending $I \in \mathbf{D}(R)$ to $I^t[S]$. Moreover, a similar proof as in Proposition 4.4 yields that

$$\ker\phi \cap \mathbf{P}(R^t[S]) \cong \mathbf{P}(R)$$

and hence $\ker\bar{\phi} \cong \mathrm{Cl}(R)$. Moreover this sequence splits because

$$\mathrm{Cl}(R) \to \mathrm{Cl}(R^t[S]) \to \mathrm{Cl}(R^t[\langle S\rangle]) \to \mathrm{Cl}(R)$$

$$[I] \mapsto [I^t[S]] \mapsto [I^t[\langle S\rangle]] \mapsto [I]$$

and this last map exists because $\mathrm{Cl}(R^t[S]) \cong \mathrm{Cl}(R)$ (Proposition 5.10). Similarly we have an exact sequence

$$1 \to \mathrm{Pic}(R) \to \mathrm{Pic}(R^t[S]) \to \mathrm{Pic}(Q^t[S]).$$

But

$$\mathrm{Pic}(Q^t[S]) = \mathrm{Pic}_g(Q^t[S])$$

by Theorem 3.5. If $[I] \in \mathrm{Pic}_g(Q^t[S])$, then $I = Q^t[A]$, A an ideal of S;

$$I^{-1} = \bigoplus_{\substack{s\in G\\ sA\subset S}} Q\bar{s}.$$

Therefore $I \cap R^t[S] = R^t[A]$ and $R^t[A]$ is an invertible $R^t[S]$-ideal.

Therefore

$$1 \to \mathrm{Pic}\,(R) \to \mathrm{Pic}\,(R^t[S]) \to \mathrm{Pic}\,(Q^t[S]) \to 1$$

is exact (for more details, see **[21]**). The rest of the proof goes as before.

(2) By Theorem 3.5

$$\mathrm{Cl}\,(Q^t[S]) \cong \mathrm{Cl}_g\,(Q^t[S]).$$

A graded divisorial ideal is of the form $I = \bigoplus_{h \in H} Q\overline{h}$ and H an ideal of S. If Q' is another simple ring such that im $\gamma \subset Q'$, then we send $I = Q^t[H]$ to $Q'^t[H]$. This defines an isomorphism between $\mathrm{Cl}_g\,(Q^t[S])$ and $\mathrm{Cl}_g\,(Q'^t[S])$.

Note that when $S = \mathbf{N}$ and t is trivial we have

$$\mathrm{Cl}\,(R[X]) \cong \mathrm{Cl}\,(R) \oplus \mathrm{Cl}\,(Q[X]) = \mathrm{Cl}\,(R)$$

since $\mathrm{Cl}\,(Q[X]) = 1$.

LEMMA 6.5. *Let $R^t[S]$ be a twisted semigroup ring which is prime Formanek. If I is an ideal of $R^t[S]$, then $IR^t[\langle S \rangle] = R^t[\langle S \rangle]I$.*

Proof. We claim that

$$IR^t[\langle S \rangle] \subset R^t[\langle S \rangle]I.$$

It suffices to prove that

$$I\overline{x}^{-1} \subset R^t[\langle S \rangle]I \quad \text{for all } x \in S.$$

Let $x \in S$. There exists an element $x' \in S$ such that $x'x \in S_f$ (Proposition 5.1), in particular $\overline{x'x} \in Z(R^t[S])$. We have

$$\overline{x}^{-1} = \alpha\overline{x'x}^{-1}\,\overline{x'} \quad \text{where } \alpha \in Z(R) \cap \mathscr{U}(R).$$

Therefore

$$I\overline{x}^{-1} = I\overline{x'x}^{-1}\overline{x'} = \overline{x'x}^{-1}I\overline{x} \subset R^t[\langle S \rangle]I.$$

PROPOSITION 6.6. *A twisted semigroup ring $R^t[S]$, which is prime Formanek, is a symmetric maximal order if and only if*:

(1) *R is a symmetric maximal order*;

(2) *If $h \in \langle S \rangle$ is such that there exists an element $g \in S$ such that for all $n \in \mathbf{N}$, $gh^n \in S$, then $h \in S$.*

Proof. Suppose $R^t[S]$ is a symmetric maximal order. It follows easily that R is a symmetric maximal order. Let $h \in \langle S \rangle$, $g \in S$ such that $gh^n \in S$ for all $n \in \mathbf{N}$. Then $\overline{gh^n} \in R^t[S]$. Since $\overline{g}$ and $\overline{h}$ are normalizing elements, we have

$$\overline{g}(R^t[S]\overline{h}R^t[S])^n \subset R^t[S].$$

Now there is a $g' \in S$ such that $g'g \in S_f$, by Proposition 5.1. Therefore

$$\overline{g'g}(R^t[S]\overline{h})^n \subset R^t[S].$$

It follows from Lemma 1.1 that $h \in S$.

Conversely, suppose that I is an ideal of $R^t[S]$, $\alpha \in Q_{\text{sym}}(R^t[S])$ and $I\alpha \subset I$. By Lemma 6.5 $R^t[\langle S \rangle]I$ is an ideal of $R^t[\langle S \rangle]$ and

$$R^t[\langle S \rangle]I\alpha \subset R^t[\langle S \rangle]I.$$

Since $R^t[\langle S \rangle]$ is a symmetric maximal order (Corollary 5.6), $\alpha \in R^t[\langle S \rangle]$. Since $\langle S \rangle$ is torsion free abelian, $\langle S \rangle$ is an ordered group. Write

$$\alpha = \sum_{i=1}^{p} a_i\overline{g_i}, \quad g_i \in \langle S \rangle,$$

such that $g_1 < g_2 \ldots < g_p$. Choose

$$\beta = \sum_{j=1}^{p} b_j\overline{h_j} \in I \cap Z(R^t[S])$$

such that $h_1 < \ldots < h_q$. From $I\alpha^n \subset I \subset R^t[S]$, we deduce for all n that $\beta\alpha^n \in R^t[S]$. The fact that $\langle S \rangle$ is ordered yields $h_q g_p^n \in S$ for all n. Therefore $g_p \in S$ by the assumption on S. By induction we obtain that $\alpha \in R^t[S]$.

PROPOSITION 6.7. *Let $R^t[S]$ be a* P.I. *Ω-Krull ring, S^* a subsemigroup of S such that $S_f^* \subset S_f$. Let $R^t[S^*]$ be the twisted semigroup ring defined by the* 2-*cocycle of $R^t[S]$ restricted to S^*.*

(1) *If $S \cap \langle S^* \rangle = S^*$, then $R^t[S^*]$ is a* P.I. *Ω-Krull ring.*

(2) *If $R^t[S] \cap Q_{\text{sym}}(R^t[S^*])$ is completely integral over $R^t[S^*]$, then $R^t[S^*]$ is an Ω-Krull ring if and only if $S \cap \langle S^* \rangle = S^*$.*

Proof. (1) We have

$$R^t[S^*] = R^t[S] \cap R^t[\langle S^* \rangle].$$

First we deduce that $R^t[S^*]$ is a symmetric maximal order from Proposition 6.6. Let $h \in \langle S^* \rangle$, $g \in S^*$ and for all $n \in \mathbf{N}$, $gh^n \in S^*$. In particular $h \in \langle S \rangle$, $g \in S$ and $gh^n \in S$. Hence $h \in S \cap \langle S^* \rangle = S^*$. Because $R^t[\langle S \rangle]$ is a P.I. Ω-Krull ring and since $\langle S^* \rangle \subset \langle S \rangle$ we conclude from Theorem 5.9 that $R^t[\langle S^* \rangle]$ is an Ω-Krull ring and $Z(R^t[\langle S^* \rangle])$ is a Krull domain. Because $S_f^* \subset S_f$ we have

$$Z(R^t[S^*]) = Z(R^t[S]) \cap Z(R^t[\langle S^* \rangle])$$

yielding that $Z(R^t[S^*])$ is a Krull domain. Proposition 1.3 yields us the desired result.

(2) Let $\alpha \in R^t[S] \cap Q_{\text{sym}}(R^t[S^*])$. There exists an element $c \in Z(R^t[S^*])$ such that for all $n \in \mathbf{N}$

$$c(R^t[S^*]\alpha R^t[S^*])^n \subset R^t[S^*].$$

Since $R^t[S^*]$ is a symmetric maximal order we have that $\alpha \in R^t[S^*]$. This yields that

$$R^t[S^*] = R^t[S] \cap Q_{\text{sym}}(R^t[S^*])$$

and therefore $S \cap \langle S^* \rangle = S^*$.

PROPOSITION 6.8. *Let S be a finitely generated semigroup, such that $R^t[S]$ is a prime* P.I. *ring. The following are equivalent*:

(1) *$R^t[S]$ is a* P.I. *Ω-Krull ring*

(2) *for all $h \in \langle S \rangle$, such that there exists $g \in S$ such that for all $n \in \mathbf{N}$, $gh^n \in S$, it follows that $h \in S$.*

Proof. $R^t[S] = R^t[\langle S \rangle] \cap Q^t[S]$. Suppose $R^t[S]$ is an Ω-Krull ring. Then Proposition 6.6 yields the desired result. Conversely if (2) is satisfied, then $R^t[S]$ is a symmetric maximal order. It remains to be checked that $Z(R^t[S])$ is a Krull domain. Now

$$Z(R^t[S]) = Z(R^t[\langle S \rangle]) \cap Z(Q^t[S]), \quad \text{and}$$

$$Z(Q^t[S]) = Z(Q)^t[S_f].$$

We claim that S_f is a finitely generated semigroup. Let $s_1, \ldots, s_n$ a set of generators of S. There exists a fixed $m \in \mathbf{N}$ such that $x^m \in S_f$, $x \in S$ (Proposition 5.7 and Lemma 6.2). Therefore there will be only finitely many elements in S_f, which cannot be written as a product of $s_1^m, \ldots, s_n^m$. Hence S_f is generated by $s_1^m, \ldots, s_n^m$ plus finitely many other elements. Therefore $Z(Q)^t[S_f]$ is Noetherian and hence a Krull domain.

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Katholieke Universiteit Leuven,
Leuven, Belgium