# THE KERNEL OF THE GENERAL-SUM FOUR-PERSON GAME

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**1. Introduction.** In this paper we apply various results and methods of previous papers on the kernel to four-person games.

Section 2 contains the basic definitions needed. In §3 we prove that the kernel of the general-sum four-person game consists of a line segment (which may shrink to a point). A method for classifying games according to their kernels is suggested in §4 and is used there to characterize all four-person games whose kernel consists of a non-degenerate interval. In the last section, §5, we offer a bargaining procedure, based on principles established in (1), which leads to the kernel in the case of a non-degenerate interval.

**2. Definitions.** Let $N = \{1, 2, 3, 4\}$ be a set with four elements. A *characteristic function* is a non-negative real function $v$ defined on the subsets of $N$ satisfying

(2.1) $$v(\{i\}) = 0, \qquad i = 1, 2, 3, 4.$$

The pair $(N; v)$ is a *four-person game*. The members of $N$ are called *players*. Subsets of $N$ are called *coalitions*.

Let $(N; v)$ be a four-person game. A *coalition structure* is a partition of $N$. An *individually rational payoff configuration* is a pair $(x; \mathfrak{B})$, where $\mathfrak{B}$ is a coalition structure and $x = (x_1, x_2, x_3, x_4)$, the *payoff vector*, is a quadruple of real numbers that satisfies

(2.2) $$x_i \geqslant 0, \qquad i = 1, 2, 3, 4 \text{ (individual rationality)},$$

and

(2.3) $$\sum_{i \in B} x_i = v(B) \qquad \text{for all } B \in \mathfrak{B}.$$

An individually rational payoff configuration $(x; \mathfrak{B})$ represents a *possible outcome* of $(N; v)$.

Let $i$ and $j$ be two distinct players. We denote by $T_{ij}$ the set of all the coalitions that contain player $i$ but do not contain player $j$; thus

(2.4) $$T_{ij} = \{B : B \subset N, i \in B \text{ and } j \notin B\}.$$

Let $(x; \mathfrak{B})$ be an individually rational payoff configuration and let $D$ be an arbitrary coalition. The *excess* of $D$ with respect to $(x; \mathfrak{B})$ is

(2.5) $$e(x, D) = v(D) - \sum_{h \in D} x_h.$$

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The *maximum surplus* of $i$ over $j$ with respect to $(x; \mathfrak{B})$ is

$$s_{ij}(x) = \max\{e(x, D) : D \in T_{ij}\}. \tag{2.6}$$

If $i$ and $j$ both belong to the *same* coalition $B \in \mathfrak{B}$, then $i$ is said to *outweigh* $j$ with respect to $x$ if $s_{ij}(x) > s_{ji}(x)$ and $x_j > 0$. $x$ is *balanced* if there exists no pair of players $h$ and $k$ such that $h$ outweighs $k$. The *kernel* (of the coalition structure $\mathfrak{B}$) is the set of all balanced payoff vectors.

It was shown in (**1**, §8) that if $\mathfrak{B}$ contains a one-person coalition, then the kernel of $\mathfrak{B}$ consists of a unique payoff vector, which is determined by the kernel of a related three-person game. Thus we can restrict our attention to coalition structures of the following forms:

$$\{1, 2, 3, 4\} \quad \text{or} \quad \{\{i, j\}, \{k, l\}\}. \tag{2.7}$$

**3. The geometrical structure of the kernel.** Let $(N, v)$ be a four-person game. By $\mathfrak{B}$ we shall denote a coalition structure that appears in (2.7).

LEMMA 3.1. *The kernel of* $\mathfrak{B}$ *is convex.*

*Proof.* Let $x$ and $y$ be two different vector payoffs in the kernel. As in (**1**, §8), we write $\xi_i = y_i - x_i$, $i \in N$, and name the players in such a way that

$$\xi_1 \leqslant \xi_2 \leqslant \xi_3 \leqslant \xi_4,$$

and that 1 and 4 belong to the same coalition of $\mathfrak{B}$. It was shown in (**1**, §8) that

$$\xi_4 = \xi_3 = -\xi_2 = -\xi_1, \tag{3.1}$$

$$s_{14}(x) = s_{14}(y) = s_{41}(x) = s_{41}(y), \tag{3.2}$$

$$s_{14}(x) = e(x, 13) = e(y, 13) = s_{14}(y), \tag{3.3}$$

and

$$s_{41}(x) = e(x, 24) = e(y, 24) = s_{41}(y). \tag{3.4}$$

It follows from (3.2), (3.3), and (3.4) that

$$s_{14}(tx + (1 - t)y) = s_{41}(tx + (1 - t)y), \qquad 0 \leqslant t \leqslant 1. \tag{3.5}$$

(3.1) implies that (3.2), (3.3), and (3.4) hold when we replace 4 by 3 and 1 by 2.

Hence (3.5) is true after making the same replacement. If $\mathfrak{B}$ is of the type $\{\{i, j\}, \{k, l\}\}$, then the above remarks prove that the whole segment with the end points $x$ and $y$ is in the kernel. In the remaining case, $\mathfrak{B} = \{1, 2, 3, 4\}$, (3.2) (3.3), and (3.4) hold when we replace 1 by an $i \in \{1, 2\}$ and 4 by a $j \in \{3, 4\}$. If

$$s(x) = \max\{e(x, D) : D \neq N\}$$

and $s(y)$ is defined similarly, it follows that $s(x) = s(y)$ and that $s(x)$ equals either $e(x, 13)$ or $e(x, 14)$. In both cases

$$s_{ij}(x) = s_{ij}(y) = s_{ji}(x) = s_{ji}(y) = s(x) = s(y), \tag{3.6}$$

for the pairs $\{i, j\} = \{1, 2\}$ and $\{3, 4\}$.

Summing ,we have for each pair $\{i, j\}$ that $s_{ij}(x) = s_{ij}(y) = s_{ji}(x) = s_{ji}(y)$, and that $s_{ji}$ is attained by the same coalition in both $x$ and $y$. This proves that every convex combination of $x$ and $y$ is in the kernel.

THEOREM 3.2. *If $\mathfrak{B}$ is a coalition structure in a four-person game, then the kernel of $\mathfrak{B}$ consists of a line segment (which may shrink into a single point).*

*Proof.* By Lemma 3.1 and the remark at the end of §2 the kernel of $\mathfrak{B}$ is convex; by **(3**, Theorem 6.7**)**, its dimension is at most one.

COROLLARY 3.3. *Let G be a four-person game. If the core of G is not empty, then it contains the kernel.*

*Proof.* We have seen in the proof of Lemma 3.1 that if $x$ and $y$ are points in the kernel, then

$$\max\{e(x, D): D \subset N\} = \max\{e(y, D): D \subset N\};$$

see (3.6). By **(3**, Theorem 5.4**)**, the kernel intersects the core when the core is not empty. The truth of the corollary is a consequence of these remarks.

We remark that the above result cannot be generalized further. A counter-example is the following: take the constant-sum homogeneous-weighted majority game $[1, 1, 1, 2, 2]_h$ and increase $v(N)$ to 22/12. The core of the game thus obtained is not empty and its kernel, which consists of the interval

$$\left(x, x, x, \frac{11}{12} - \frac{3}{2}x, \frac{11}{12} - \frac{3}{2}x : 0 \leqslant x \leqslant \frac{22}{84}\right),$$

is not contained in the core.

**4. On the classification of games according to their kernels.** In this section we present and use a method for classifying games according to their kernels. Our starting point is **(3**, §4, Theorems 4.5 and 4.6 and Formula 4.14**)** where a detailed description of the kernel as a union of convex polyhedra is given. By "classifying games according to their kernels" we understand the determination, for each non-empty subset of convex polyhedra that appear in the representation of the kernel, of the set of all games for which these polyhedra are non-empty; cf. **(3**, (4.14) and Theorem 4.6**)**. This determination amounts to writing down the inequalities that assure us that some specific set of polyhedra is non-empty, and eliminating the payoff variables from the inequalities. The nature of the inequalities involved in the definition of the kernel enables us to use the method of **(2)** for eliminating quantified variables out of systems of linear inequalities; indeed this can be done by a computer; for details see **(2)**. We remark that the method of classification sketched above applies to games with any number of players.

Let $(N, v)$ be a four-person game. From the results of §3 (see also **4**) it follows that the kernel of $N$ consists of a non-degenerate interval if and only if, for a suitable naming of the players, the following sentence is true:

$$(4.1)\qquad \exists x_1 \exists x_2 \exists x_3 \exists x_4 (x_1 > 0 \wedge x_2 > 0 \wedge x_3 > 0 \wedge x_4 > 0 \wedge x_1 + x_2 + x_3 + x_4 = v(N) \wedge e(x, 12) = e(x, 34) \wedge e(x, 14) = e(x, 23) \wedge e(x, 12) > F(x_1, x_2, x_3, x_4) \wedge e(x, 14) > F(x_1, x_2, x_3, x_4)),$$

where

$$F(x_1, x_2, x_3, x_4) = \max(-x_1, -x_2, -x_3, -x_4, e(x, 13), e(x, 24), e(x, 123), e(x, 124), e(x, 134), e(x, 234)).$$

Using the method of **(2)**, we have found that (4.1) is equivalent to

$$(4.2)\qquad \min\{v(N) - f_{12}, v(N) - f_{23}, v(12) - f_{12} + f_{14}, v(34), \tfrac{1}{2}[f_{23} + v(14) + v(34) - v(23) - v(13)], v(12) - v(123) + f_{23} - f_{12} + f_{14}, f_{23} + v(14) + v(34) - v(134) - v(23), v(14), v(23) + f_{14} - f_{12}, \tfrac{1}{2}[v(14) - v(13) + f_{23} - f_{12} + f_{14}], v(14) - v(134) + f_{23} - f_{12} + f_{14}, v(14) + f_{23} - v(123)\}$$
$$> \max\{0, f_{14} - f_{12}, v(N) - v(12) - f_{23}, f_{12} - v(12), \tfrac{1}{2}[v(24) - f_{23} + v(N) - v(12)], v(124) - v(12), v(234) + \tfrac{1}{2}[v(14) - v(23) - v(12) - v(34)], f_{14} - f_{12} - f_{23} + v(N) - v(14), v(N) - f_{23} - v(14), \tfrac{1}{2}[v(24) - f_{23} + f_{14} - f_{12} + v(N) - v(14)], v(124) - v(14) - f_{12} + f_{14}, v(234) - v(23)\}$$

where $f_{ij} = \frac{1}{2}[v(ij) + v(N) - v(N - \{i, j\})]$. Thus (4.2) characterizes all four-person games for which the kernel of $N$ consists of a non-degenerate interval.

**5. An example.** We shall now discuss a typical example of a kernel consisting of a non-degenerate interval. Let us consider the game $(N, v)$ defined by:

$$v(12) = v(23) = v(34) = v(14) = 1, \quad v(N) = 2, \quad \text{and } v(B) = 0 \text{ otherwise.}$$

(This example has been suggested by M. Maschler.) The kernel of the coalition structure $N$ is $(x, 1 - x, x, 1 - x)$, $0 \leqslant x \leqslant 1$. We shall now generalize the arguments given in the analysis of the three-person game **(1**, §4**)** to describe a bargaining procedure leading to this result. To make the discussion more vivid let us call players 1 and 3 "workers," and players 2 and 4 "employers."

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In the first round of the negotiations each "employer" chooses a "worker"; let us assume that 1 joins 2, and 3 joins 4. The joint profit to each pair is 1, and the question that remains to be decided is the level of "wages." This question is answered by solving the following two *interconnected* pure (**1**, Definition 3.1) bargaining 2-person games (described graphically):


The interpretation is clear: The basis for 1's claims is the maximum "salary" he will be able to get from 4, etc. Now determining the payoffs using (**1**, (3.4)), we obtain the kernel. The same procedure leads to the kernels of all the games that satisfy (4.2).

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