§ 4. EXCENTRES.

The internal bisector of any angle of a triangle and the external bisectors of the two other angles are concurrent.

The following demonstration is different from the usual one.

FIGURE 27.

Let AL be the internal bisector of $\angle$ A, and let the external bisector of $\angle$ B cut it at I_1.

Then
$$AI_1 : LI_1 = BA : BL$$
$$= CA : CL;$$

therefore the external bisector of $\angle$ C passes through I_1.

Hence also the internal bisector of $\angle$ B and the external bisectors of $\angle$ C and $\angle$ A are concurrent at I_2; the internal bisector of $\angle$ C and the external bisectors of $\angle$ A and $\angle$ B are concurrent at I_3.

FIGURE 28.

The points of concurrency, which will be denoted by I_1, I_2, I_3, are the centres of the circles escribed * to ABC.

These circles are often called the *excircles*,† and the centres of them the excentres.†

The radii of the excircles are denoted sometimes by r_1, r_2, r_3, sometimes by r_a, r_b, r_c.

(1) The points A, I, I_1; B, I, I_2; C, I, I_3 are collinear.
So are I_2, A, I_3; I_3, B, I_1; I_1, C, I_2.

These results expressed in words are:

The six bisectors of the interior and the exterior angles of a triangle meet three and three in four points which are the centres of the four circles touching the sides of the triangle. Or

The six straight lines joining two and two the centres of the four circles which touch the sides of a triangle pass each through one of the vertices of the triangle.

---

* This expression in its French form (*exinscrit*) was first used by Simon Lhuilier. See his *Élémens d'Analyse*, p. 198 (1809). If the term *escribed* was not introduced by T. S. Davies, currency at least was given to it by him. See *Ladies' Diary* for 1835, p. 50.

† See the note on p. 32.

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(2) The points $\quad I, \quad I_1, \quad I_2, \quad I_3$
are the respective orthocentres of the triangles

$$I_1I_2I_3, \; II_3I_2, \; I_3II_1, \; I_2I_1I.$$

Attention should be called to the order of the subscripts in the naming of the triangles. See § 5, (2).

(3) When the diagram of a triangle with its incentres and excentres has to be drawn, instead of beginning with the triangle ABC and determining I, I_1, I_2, I_3 by the bisection of certain angles, it is easier to begin with the triangle $I_1I_2I_3$. The feet of the perpendiculars of $I_1I_2I_3$ will then be A, B, C, and I will be the point of intersection of the perpendiculars. The only instrument therefore which is necessary to determine these points is a draughtsman's square.

A circle may be escribed to a given triangle by a method exactly analogous to that on p. 39 for inscribing a circle.

FIGURE 24.

The only difference in the construction is that CQ is cut off equal to CB not in the same direction as CP, but in the opposite direction.

(4) The area of a triangle is equal to the rectangle under the excess of the semiperimeter above any side and the radius of the excircle which touches the other two sides produced.

This is expressed, $\quad \Delta = s_1r_1 = s_2r_2 = s_3r_3$
where $\quad s_1 = \frac{1}{2}(-a+b+c), \; s_2 = \frac{1}{2}(a-b+c), \; s_3 = \frac{1}{2}(a+b-c).$

If $\quad \frac{1}{2}(a+b+c) = s,$
then $\quad \frac{1}{2}(-a+b+c) = s-a, \; \frac{1}{2}(a-b+c) = s-b, \; \frac{1}{2}(a+b-c) = s-c.$

The expressions $s-a$, $s-b$, $s-c$ will be denoted by s_1, s_2, s_3, a notation introduced by Thomas Weddle, who in a letter to T. S. Davies, dated March 31st, 1842, and printed in the *Lady's and Gentleman's Diary* for 1843, p. 78, remarks that s, s_1, s_2, s_3 are the lengths of the segments of the sides made by the four circles of contact, and that the change from $s-a$ to s_1, etc., will be justified by observing how much more symmetrical many theorems appear under the new notation than the old.

(5) The distances from the vertices and from each other of some of the points of inscribed and escribed contact are given in the following expressions* :

* Compare the subscripts in the values of s, s_1, s_2, s_3 with the subscripts in § 4 (2).

FIGURE 28.

$$
\begin{aligned}
s &= AE_1 = AF_1 = BF_2 = BD_2 = CD_3 = CE_3 \\
s_1 &= AE = AF = BF_3 = BD_3 = CD_2 = CE_2 \\
s_2 &= AE_3 = AF_3 = BF = BD = CD_1 = CE_1 \\
s_3 &= AE_2 = AF_2 = BF_1 = BD_1 = CD = CE
\end{aligned}
$$

$$
\begin{aligned}
a &= EE_1 = E_2E_3 = FF_1 = F_3F_2 \\
b &= FF_2 = F_3F_1 = DD_2 = D_1D_3 \\
c &= DD_3 = D_1D_2 = EE_3 = E_2E_1
\end{aligned}
$$

$$
\begin{aligned}
b + c &= D_2D_3 \quad & b \sim c &= DD_1 \\
c + a &= E_3E_1 \quad & c \sim a &= EE_2 \\
a + b &= F_1F_2 \quad & a \sim b &= FF_3
\end{aligned}
$$

(6)
$$
\begin{aligned}
a + b + c &= 2s = s + s_1 + s_2 + s_3 \\
&= AE + AE_1 + AE_2 + AE_3 = \text{etc.}
\end{aligned}
$$

(7) Because $BD = s_2 = CD_1$

therefore D and D_1 are equidistant from the mid point of BC.

Because $BD_3 = s_1 = CD_2$

therefore D_2 and D_3 are equidistant from the mid point of BC.

Similarly for the E points and the F points.

(8)
$$
\begin{aligned}
& AD^2 + AD_1^2 + AD_2^2 + AD_3^2 \\
& + BE^2 + BE_1^2 + BE_2^2 + BE_3^2 \\
& + CF^2 + CF_1^2 + CF_2^2 + CF_3^2 \\
& = 5(a^2 + b^2 + c^2).
\end{aligned}
$$

FIGURE 28.

Let A′ be the mid point of BC;

then A′ is the mid point of DD_1 and of D_2D_3.

Now since $D_2D_3 = b + c \qquad DD_1 = b \sim c$,

therefore $2A'D_2 = b + c \qquad 2A'D = b \sim c$.

But
$$
\begin{aligned}
AD^2 + AD_1^2 &= 2A'D^2 + 2A'A^2 \\
AD_2^2 + AD_3^2 &= 2A'D_2^2 + 2A'A^2;
\end{aligned}
$$

therefore $\Sigma(AD^2) = 2A'D^2 + 2A'D_2^2 + 4A'A^2$.

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Again $2AB^2+2AC^2=4A'B^2+4A'A^2$

$$= BC^2+4A'A^2;$$

therefore $2c^2+2b^2-a^2=4A'A^2$.

Hence $\Sigma(AD^2)=\frac{1}{2}\{(b\sim c)^2+(b+c)^2\}+2c^2+2b^2-a^2$

$$=3\ (b^2+c^2)-a^2.$$

Similarly $\Sigma(BE^2)=3\ (c^2+a^2)-b^2$

and $\Sigma(CF^2)=3\ (a^2+b^2)-c^2$.

Consequently the sum of the squares on the twelve specified lines

$$=5\ (a^2+b^2+c^2).$$

(8) is stated by W. H. Levy of Shalbourne in the *Lady's and Gentleman's Diary* for 1852, p. 71, and proved in 1853, pp. 52-3.

(9) The angles of triangle DEF expressed in terms of A, B, C are

$$\angle D=\tfrac{1}{2}(B+C)=90^\circ-\tfrac{1}{2}A$$
$$\angle E=\tfrac{1}{2}(C+A)=90^\circ-\tfrac{1}{2}B$$
$$\angle F=\tfrac{1}{2}(A+B)=90^\circ-\tfrac{1}{2}C.$$

Hence whatever be the size of the angles A, B, C the triangle DEF is always acute-angled.*

(10) *If ABC be a triangle, DEF the triangle formed by joining the inscribed points of contact of ABC; $D_1E_1F_1$ the triangle formed by joining the inscribed points of contact of DEF; $D_2E_2F_2$ the triangle formed by joining the inscribed points of contact of $D_1E_1F_1$; and this process of construction be continued, the successive triangles will approximate to an equilateral triangle.*†

Suppose $\angle A$ greater than $\angle B$, and $\angle B$ greater than $\angle C$.

Then $D=\frac{1}{2}(B+C)$, $E=\frac{1}{2}(C+A)$, $F=\frac{1}{2}(A+B)$;

therefore $E-D=\frac{1}{2}(A-B)$, $F-E=\frac{1}{2}(B-C)$, $F-D=\frac{1}{2}(A-C)$.

Now $D_1=\frac{1}{2}(E+F)$, $E_1=\frac{1}{2}(F+D)$, $F_1=\frac{1}{2}(D+E)$;

therefore $D_1-E_1=\frac{1}{2}(E-D)$, $E_1-F_1=\frac{1}{2}(F-E)$, $D_1-F_1=\frac{1}{2}(F-D)$,

$=\frac{1}{4}(A-B)$, $=\frac{1}{4}(B-C)$, $=\frac{1}{4}(A-C)$;

and so on.

* Feuerbach, *Eigenschaften...des...Dreiecks*, § 66 (1822).

† Todhunter's *Plane Trigonometry*, Chapter XVI. Example 16 (1859).

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Therefore the differences between the angles of the successive triangles become always a smaller and smaller fraction of the differences between the angles of the fundamental triangle; and hence the successive triangles approximate to an equilateral triangle.

The properties (11)–(14), (16)–(18) have reference to Figure 28.

(11) EF, E_1F_1 are parallel to I_2I_3

and EF_1, E_1F intersect on AI; and so on.

(12) E_2F_2, E_3F_3 are parallel to AI

and E_2F_3, E_3F_2 intersect on I_2I_3; and so on.

(13) The angles of triangles $D_1E_1F_1$, $D_2E_2F_2$, $D_3E_3F_3$ expressed in terms of A, B, C are

$$\begin{array}{lll} \angle D_1 = 90^\circ + \frac{1}{2}A, & \angle E_1 = \frac{1}{2}B, & \angle F_1 = \frac{1}{2}C \\ \angle D_2 = \frac{1}{2}A, & \angle E_2 = 90^\circ + \frac{1}{2}B, & \angle F_2 = \frac{1}{2}C \\ \angle D_3 = \frac{1}{2}A, & \angle E_3 = \frac{1}{2}B, & \angle F_3 = 90^\circ + \frac{1}{2}C. \end{array}$$

Hence whatever be the size of the angles A, B, C these three triangles are always obtuse-angled.*

(14) The angles of triangle $I_1I_2I_3$ expressed in terms of A, B, C are

$$\begin{aligned} \angle I_1 &= \tfrac{1}{2}(B + C) = 90^\circ - \tfrac{1}{2}A \\ \angle I_2 &= \tfrac{1}{2}(C + A) = 90^\circ - \tfrac{1}{2}B \\ \angle I_3 &= \tfrac{1}{2}(A + B) = 90^\circ - \tfrac{1}{2}C \end{aligned}$$

Hence whatever be the size of the angles A, B, C the triangle $I_1I_2I_3$ is always acute-angled.

(15) If ABC be a triangle, $A_1B_1C_1$ the triangle formed by joining the excentres of ABC; $A_2B_2C_2$ the triangle formed by joining the excentres of $A_1B_1C_1$; and this process of construction be continued, the successive triangles will approximate to an equilateral triangle.†

* Feuerbach, *Eigenschaften...des...Dreiecks*, § 66 (1822).

† Mr R. Tucker in *Mathematical Questions from the Educational Times*, XV. 103-4 (1871).

(16) Triangles DEF, $I_1I_2I_3$ are similar and similarly situated,* and their homothetic centre is the point of concurrency of the triad

$$I_1D,\ I_2E,\ I_3F.$$

(17) Triangles $D_1E_1F_1$, II_3I_2 are similar and similarly situated; so are $D_2E_2F_2$, I_3II_1; and $D_3E_3F_3$, I_2I_1I; and their homothetic centres are the points of concurrency of the triads

$$ID_1,\ I_3E_1,\ I_2F_1\text{; and so on.}$$

(18) *The quadrilaterals AFIE, BDIF, CEID are such that circles may be inscribed in them.*

For a circle may be inscribed in a quadrilateral when the sum of one pair of opposite sides is equal to the sum of the other pair.

Now $\qquad AF = AE$ and $IE = IF$.

(19) *If the radii of the circles inscribed in the quadrilaterals* † *AFIE, BDIF, CEID* be denoted by ρ_1, ρ_2, ρ_3,

$$\left(\frac{1}{\rho_2}-\frac{1}{r}\right)\left(\frac{1}{\rho_3}-\frac{1}{r}\right)+\left(\frac{1}{\rho_3}-\frac{1}{r}\right)\left(\frac{1}{\rho_1}-\frac{1}{r}\right)+\left(\frac{1}{\rho_1}-\frac{1}{r}\right)\left(\frac{1}{\rho_2}-\frac{1}{r}\right)=\frac{1}{r^2}.$$

FIGURE 29.

Bisect $\angle$ AFI by FM meeting AI at M; and draw MN perpendicular to AF.

Then M is the centre of the circle inscribed in AFIE, MN is the radius of it, and MN = FN.

From the similar triangles AFI, ANM

$$AF : IF = AN : MN$$

that is, $\qquad s_1 : r = s_1 - \rho_1 : \rho_1$;

therefore $\qquad s_1 : r = \rho_1 : r - \rho_1$;

therefore $\qquad s_1 = \dfrac{r\rho_1}{r-\rho_1}$;

therefore $\qquad \dfrac{1}{s_1} = \dfrac{1}{\rho_1} - \dfrac{1}{r}.$

Similarly $\qquad \dfrac{1}{s_2} = \dfrac{1}{\rho_2} - \dfrac{1}{r},\quad \dfrac{1}{s_3} = \dfrac{1}{\rho_3} - \dfrac{1}{r}.$

Hence given expression $\qquad = \dfrac{1}{s_2s_3} + \dfrac{1}{s_3s_1} + \dfrac{1}{s_1s_2} = \dfrac{1}{r^2}.$

---

* Feuerbach, *Eigenschaften ... des ... Dreiecks*, § 61 (1822).

† *The Museum*, III. 269-70 and 342 (1866).

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(20) $$\frac{\rho_1}{r-\rho_1}+\frac{\rho_2}{r-\rho_2}+\frac{\rho_3}{r-\rho_3} = \frac{\rho_1}{r-\rho_1}\cdot\frac{\rho_2}{r-\rho_2}\cdot\frac{\rho_3}{r-\rho_3}.$$

For $$\frac{r\rho_1}{r-\rho_1}=s_1;$$

therefore $$\frac{\rho_1}{r-\rho_1}=\frac{s_1}{r}.$$

Similarly $$\frac{\rho_2}{r-\rho_2}=\frac{s_2}{r},\quad \frac{\rho_3}{r-\rho_3}=\frac{s_3}{r};$$

therefore $$\frac{\rho_1}{r-\rho_1}+\frac{\rho_2}{r-\rho_2}+\frac{\rho_3}{r-\rho_3}=\frac{s_1+s_2+s_3}{r}=\frac{s}{r}$$

and $$\frac{\rho_1}{r-\rho}\cdot\frac{\rho_2}{r-\rho_2},\frac{\rho_3}{r-\rho_3}=\frac{s_1s_2s_3}{r^3}=\frac{s}{r}.$$

(21) *The quadrilaterals* $AF_1I_1E_1$, $BD_1I_1F_1$, $CE_1I_1D_1$ *are such that circles may be inscribed in them.*

FIGURE 28.

For $AF_1 = AE_1$ and $I_1E_1 = I_1F_1$.

Similarly, circles may be inscribed in the quadrilaterals

$AF_2I_2E_2$, $BD_2I_2F_2$, $CE_2I_2D_2$
$AF_3I_3E_3$, $BD_3I_3F_3$, $CE_3I_3D_3$.

(22) If the radii of the circles inscribed in the first three of these quadrilaterals be denoted by ρ_1', ρ_2', ρ_3' then

$$\left(\frac{1}{\rho_2'}-\frac{1}{r_1}\right)\left(\frac{1}{\rho_3'}-\frac{1}{r_1}\right)-\left(\frac{1}{\rho_3'}-\frac{1}{r_1}\right)\left(\frac{1}{\rho_1'}-\frac{1}{r_1}\right)-\left(\frac{1}{\rho_1'}-\frac{1}{r_1}\right)\left(\frac{1}{\rho_2'}-\frac{1}{r_1}\right)=\frac{1}{r_1^2};$$

and similarly for the others.

(23) $$\frac{\rho_1'}{r_1-\rho_1'}-\frac{\rho_2'}{r_1-\rho_2'}-\frac{\rho_3'}{r_1-\rho_3'}=\frac{\rho_1'}{r_1-\rho_1'}\cdot\frac{\rho_2'}{r_1-\rho_2'}\cdot\frac{\rho_3'}{r_1-\rho_3'}$$

For $$s=\frac{r_1\rho_1'}{r_1-\rho_1'},\quad s_3=\frac{r_1\rho_2'}{r_1-\rho_2'},\quad s_2=\frac{r_1\rho_3'}{r_1-\rho_3'}.$$

(24) The following relation * exists between the radii of the circles inscribed in the quadrilaterals

$AF_1I_1E_1$, $BD_2I_2F_2$, $CE_3I_3D_3$.

* Mr R. E. Anderson in *Proceedings of the Edinburgh Mathematical Society*, X. 79 (1892).

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If these radii be denoted by v_1, v_2, v_3

$$\frac{1}{v_1}+\frac{1}{v_2}+\frac{1}{v_3}=\frac{3}{s}+\frac{1}{r}\cdot$$

(25) *If the exterior angles of ABC be bisected by straight lines which meet the circumcircle at U′, V′, W′, the sides of U′V′W′ are also perpendicular to AI, BI, CI, and U′V′W′ is congruent and symmetrically situated to UVW with respect to the circumcentre of ABC.*

FIGURE 25.

For $\angle$ UAU′ is right;
therefore UU′ is a diameter of the circumcircle,
and U′ is symmetrical to U with respect to O.

(26) If from the six points U, V, W, U′, V′, W′ all the UVW triangles be formed, it will be found that there are four pairs

UVW, UV′W′, U′VW′, U′V′W
U′V′W′, U′VW, UV′W, UVW′.

These pairs of triangles are congruent and symmetrically situated with respect to O, and their sides are either perpendicular or parallel to AI, BI, CI.

(27) The angles of these triangles can be expressed in terms of A, B, C.

Take, for example, triangle U′VW from the second pair.

$$\begin{aligned}\text{WU'V} &= 180^\circ - \text{WUV}\\ &= 180^\circ - \tfrac{1}{2}(\text{B}+\text{C}) = 90^\circ + \tfrac{1}{2}\text{A}.\end{aligned}$$

Since AU′, VW are both perpendicular to AI,

therefore arc U′W = arc AV
therefore $\angle$ U′VW = $\angle$ ABV = $\frac{1}{2}$B,
and $\angle$ U′WV = $\angle$ ACW = $\frac{1}{2}$C.

The angles of the third and fourth pairs of triangles are respectively equal to

$$\begin{array}{lll}\tfrac{1}{2}\text{A}, & 90^\circ+\tfrac{1}{2}\text{B}, & \tfrac{1}{2}\text{C}\\ \tfrac{1}{2}\text{A}, & \tfrac{1}{2}\text{B}, & 90^\circ+\tfrac{1}{2}\text{C}.\end{array}$$

Hence whatever be the size of the angles A, B, C the second, third, and fourth pairs of triangles are always obtuse-angled.

Compare § 4, (13).

(28) The orthocentres of the first quartet of triangles are

$$I, I_1, I_2, I_3$$

and the orthocentres of the second quartet are the points symmetrical to I, I_1, I_2, I_3 with respect to O. What these points are will be seen later on.

(29)
$$\begin{aligned} UV'W' : ABC &= R : 2r_1 \\ U'VW' : ABC &= R : 2r_2 \\ U'V'W : ABC &= R : 2r_3. \end{aligned}$$

(30) If I be the incentre of ABC and about the triangles IBC, ICA, IAB circles be circumscribed, these circles will pass respectively through I_1, I_2, I_3, and their centres will be U, V, W, the points where AI, BI, CI meet the circumcircle.

FIGURE 28.

(31) If circles be circumscribed about the triangles I_1BC, I_1CA, I_1AB, these circles will pass respectively through I, I_3, I_2.

Similarly for the circumcircles of I_2BC, etc.

(32) From (30) and (31) there would seem to be twelve circles. They reduce to six, and their diameters are the six lines

$$II_1, \; II_2, \; II_3, \; I_2I_3, \; I_3I_1, \; I_1I_2.$$

(33) *In a triangle ABC, if on AB and AC as diameters circles be described and a diameter of the first circle be drawn parallel to AC, and a diameter of the second parallel to AB, one pair of extremities of these diameters will lie on the internal bisector of angle A and the other pair on the external bisector.**

FIGURE 30.

Let M and N be the centres of the circles described on AB and AC, and let EF be the diameter parallel to AC, and GH the diameter parallel to AB. Join AF.

Then $\angle MAF = \angle MFA = \angle NAF$;

therefore AF is the internal bisector of $\angle A$.

Similarly AH is the internal bisector of $\angle A$.

Join AE, and produce CA to C'.

* Mr Brocot in the *Journal de Mathématiques Élémentaires*, I. 383 (1877), II. 128 (1878).

Then $\angle MAE = \angle MEA = \angle C'AE$;
therefore AE is the external bisector of $\angle A$.
Similarly AG is the external bisector of $\angle A$.

(34) *If M, N be the feet of the bisectors of angles B and C of triangle ABC, the distance of any point P in MN from BC is equal to the sum* of its distances from CA and AB.*

FIGURE 31.

Draw MK, MK′ perpendicular to BC, AB
NL, NL′ „ „ BC, CA;
join ML meeting PR in H.

Then $PS : NL' = PM : NM$
$= PH : NL.$
But $NL' = NL$;
therefore $PS = PH$.
Again $PT : MK' = PN : MN$
$= RL : KL$
$= HR : MK.$
But $MK' = MK$;
therefore $PT = HR$;
therefore $PR = PS + PT$.

If P be situated on MN produced either way, then PR is equal to the difference between PS and PT.

The theorem may be extended to the bisectors of the exterior angles of ABC, and thus enunciated:

If M, N be the feet of the internal or external bisectors of the angles B and C, the distance of any point P in MN from BC is equal to the algebraic sum of its distances from CA and AB.

(35) If through I, I_1, I_2, I_3 parallels be drawn to BC meeting AC, AB in P, Q; P_1, Q_1; P_2, Q_2; P_3, Q_3

then $PQ = BQ + CP$, $P_1Q_1 = BQ_1 + CP_1$
$P_2Q_2 = BQ_2 - CP_2$, $P_3Q_3 = BQ_3 - CP_3$.

* Mr E. Cesaro in *Nouvelle Correspondance Mathématique*, V. 224 (1879); proof and extension of the property to the external bisectors by Mr Cauret on pp. 334-5. The proof in the text is taken from Vuibert's *Journal* IX. 72 (1885). Mr Cesaro gives the corresponding property for the tetrahedron.

(36) *If AI meet the incircle at U, and at U a tangent be drawn meeting BI and CI at P and Q, then**

$$BP = CQ = BI + CI.$$

FIGURE 32.

Let PQ meet CA, AB at S, T.

Then $\angle QTB = \angle ATS$

$= \frac{1}{2}(B + C)$;

and $\angle QIB = \angle IBC + \angle ICB$

$= \frac{1}{2}(B + C)$;

therefore the points B, I, T, Q are concyclic,

and $\angle IQU = \angle IBF$.

Hence the right-angled triangles IUQ, IFB, which have IU equal to IF, are congruent;

therefore $QI = BI$.

Similarly $PI = CI$;

therefore $BP = CQ = BI + CI$.

(37) $BT = QS$ and $CS = PT$.

For $BT = BF + TF = QU + TU = QU + SU$.

(38) If AI_1 meet the first excircle at U_1, and at U_1 a tangent be drawn meeting BI_1 and CI_1 at P_1 and Q_1, then

$$BP_1 = C\,Q_1 = BI_1 + CI_1.$$

(39) $PQ = P_1Q_1 = BC$.

(40) $BT_1 = Q_1S_1$ and $CS_1 = P_1T_1$.

(41) *If ID, the radius of the incircle, meet EF at P, then*† *P lies on the median through A.*

FIGURE 33.

Through P draw B_1C_1 parallel to BC, and join IB_1, IC_1.

Then $\angle IPB_1$ and $\angle IFB_1$ are right;

therefore the points I, P, F, B_1 are concyclic;

therefore $\angle IB_1P = \angle IFP$.

---

* This and (37) are given by William Wallace in Leybourn's *Mathematical Repository*, old series, II. 187 (1801).

† John Johnson, of Birmingham, in Leybourn's *Mathematical Repository*, old series, II. 376 (1801).

Mr E. M. Langley in the *Sixteenth Report of the Association for the Improvement of Geometrical Teaching*, pp. 35-6 (1890), gives another demonstration by means of Brianchon's theorem:

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Similarly $\angle IC_1P = \angle IE\,P$;

therefore $\angle IB_1P = \angle IC_1P$;

therefore $B_1P = C_1P$.

Hence also if I_1D_1, the radius of the first excircle, meet E_1F_1 at P_1, then P_1 lies on the median through A.

(42) *If from any point P situated on the interior or exterior bisector of the angle A of triangle ABC perpendiculars PD, PE, PF be drawn to BC, CA, AB, the point Q where PD intersects EF will lie on the median * from A.*

FIGURE 34.

Triangles FPQ, ABL are similar, since their sides are mutually perpendicular;

therefore $FQ : FP = AL : AB$.

Similarly $EQ : EP = AL : AC$;

therefore $AB \cdot FQ = AC \cdot EQ$.

Now FQ and EQ are proportional to the distances of Q from AB and AC;

therefore $ABQ = ACQ$.

But these triangles have the same base AQ;

therefore their corresponding altitudes are equal;

and hence it is easily deduced that AQ passes through the mid point of BC.

The following is another demonstration:

When the point P moves on the bisector, the point Q describes a straight line passing through A.

Place the point P at the intersection of the bisector with the circumcircle of ABC;

then the projections of P on the sides of ABC are collinear, by Wallace's theorem;

and one of these projections is the mid point of BC.

* Mr E. Cesaro in *Mathesis* I. 79 (1881). The two demonstrations are from the same volume, pp. 117-8.

(43) *The problem of finding the incentre or the excentres of a triangle is a particular case of the problem to find a point such that straight lines drawn from it to the sides shall make equal angles with the sides and shall be to each other in given ratios.**

FIGURE 35.

Let the straight lines drawn from the point to BC, CA, AB be in the ratios $d : e : f$.

Make a parallelogram having B for one of its angles, and having the sides along BA, BC in the ratio $d : f$;
let BM be one of its diagonals.

Make a parallelogram having C for one of its angles, and having the sides along CA, CB in the ratio $d : e$;
let CN be one of its diagonals.
Then BM, CN will meet at I the required point.

From I draw ID, IE, IF making with BC, CA, AB angles equal to the given angle.

The proof will offer no great difficulty if from M perpendiculars be drawn to BA, BC, from N to CA, CB, and from I to the three sides.

* Mauduit's *Leçons de Géométrie*, pp. 239-242 (1790).

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