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C^*-ALGEBRAS OF INFINITE REAL RANK

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We introduce the notion of weakly (strongly) infinite real rank for unital C^*-algebras. It is shown that a compact space X is weakly (strongly) infine-dimensional if and only if $C(X)$ has weakly (strongly) infinite real rank. Some other properties of this concept are also investigated. In particular, we show that the group C^*-algebra $C^*(\mathbb{F}_\infty)$ of the free group on countable number of generators has strongly infinite real rank.

It is clear that some C^*-algebras of infinite real rank have infinite rank in a very strong sense of this word, while others do not. In order to formally distinguish these types of infinite ranks from each other we introduce the concept of weakly (strongly) infinite real rank. Proposition 1.1 characterises usual real rank in terms of *infinite* sequences of self-adjoint elements and serves as a basis of our definition 2.1. We completely settle the commutative case by proving (Theorem 2.9) that the algebra $C(X)$ has weakly infinite real rank if and only if X is a weakly infinite dimensional compactum. As expected, the group C^*-algebra $C^*(\mathbb{F}_\infty)$ of the free group on countable number of generators has strongly infinite real rank (Corollary 2.10).

1. PRELIMINARIES

All C^*-algebras below are assumed to be unital. The set of all self-adjoint elements of a C^*-algebra X is denoted by X_{sa}.

The real rank of a unital C^*-algebra X, denoted by $\mathrm{rr}(X)$, is defined as follows ([**2**]). We say that $\mathrm{rr}(X) \leqslant n$ if for each $(n+1)$-tuple $(x_1, \dots, x_{n+1})$ of self-adjoint elements in X and every $\varepsilon > 0$, there exists an $(n+1)$-tuple $(y_1, \dots, y_{n+1})$ in X_{sa} such that $\sum\limits_{k=1}^{n+1} y_k^2$ is invertible and $\left\| \sum\limits_{k=1}^{n+1} (x_k - y_k)^2 \right\| < \varepsilon$.

1.1. ALTERNATIVE DEFINITIONS OF THE REAL RANK It is interesting that the real rank can be equivalently defined in terms of infinite sequences.

PROPOSITION 1.1. *Let X be a unital C^*-algebra. Then the following conditions are equivalent:*

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(i) $\mathrm{rr}(X) \leqslant n$.

(ii) *for each $(n+1)$-tuple $(x_1, \dots, x_{n+1})$ in X_{sa} and for each $\varepsilon > 0$, there exists an $(n+1)$-tuple $(y_1, \dots, y_{n+1})$ in X_{sa} such that $\sum_{k=1}^{n+1} y_k^2$ is invertible and $\|x_k - y_k\| < \varepsilon$ for each $k = 1, 2, \dots, n+1$.*

(iii) *for any sequence of self-adjoint elements $\{x_i : i \in N\} \subseteq X_{sa}$ and for any sequence of positive real numbers $\{\varepsilon_i : i \in \mathbb{N}\}$ there exists a sequence $\{y_i : i \in \mathbb{N}\} \subseteq X_{sa}$ such that*

  (a) *$\|x_i - y_i\| < \varepsilon_i$, for each $i \in \mathbb{N}$,*

  (b) *for any subset $D \subseteq \mathbb{N}$, with $|D| = n+1$, the element $\sum_{i \in D} y_i^2$ is invertible.*

(iv) *for any sequence of self-adjoint elements $\{x_i : i \in \mathbb{N}\} \subseteq X_{sa}$ and for any $\varepsilon > 0$ there exists a sequence $\{y_i : i \in \mathbb{N}\} \subseteq X_{sa}$ such that*

  (a) *$\|x_i - y_i\| < \varepsilon$, for each $i \in \mathbb{N}$,*

  (b) *for any subset $D \subseteq \mathbb{N}$, with $|D| = n+1$, the element $\sum_{i \in D} y_i^2$ is invertible.*

(v) *for any sequence of self-adjoint elements $\{x_i : i \in \mathbb{N}\} \subseteq X_{sa}$ such that $\|x_i\| = 1$ for each $i \in \mathbb{N}$ and for any $\varepsilon > 0$ there exists a sequence $\{y_i : i \in \mathbb{N}\} \subseteq X_{sa}$ such that*

  (a) *$\|x_i - y_i\| < \varepsilon$, for each $i \in \mathbb{N}$,*

  (b) *for any subset $D \subseteq \mathbb{N}$, with $|D| = n+1$, the element $\sum_{i \in D} y_i^2$ is invertible.*

PROOF: (i) $\Longrightarrow$ (ii). Let $(x_1, \dots, x_{n+1})$ be an $(n+1)$-tuple in X_{sa} and $\varepsilon > 0$. By (i), there exists an $(n+1)$-tuple $(y_1, \dots, y_{n+1})$ in X_{sa} such that $\sum_{k=1}^{n+1} y_k^2$ is invertible and $\left\| \sum_{k=1}^{n+1} (x_k - y_k)^2 \right\| < \varepsilon^2$.

Since $x_k - y_k \in X_{sa}$, it follows ([**4**, 2.2.4 Theorem]) that $(x_k - y_k)^2 \geqslant 0$ for each $k = 1, \dots, n+1$. Then, by [**4**, 2.2.3 Lemma], $\sum_{k=1}^{n+1} (x_k - y_k)^2 \geqslant 0$. Note also that $\sum_{i=1}^{n+1} (x_i - y_i)^2 - (x_k - y_k)^2 = \sum_{i=1, i \neq k}^{n+1} (x_i - y_i)^2 \geqslant 0$, $k = 1, \dots, n+1$, which guarantees that $(x_k - y_k)^2 \leqslant \sum_{i=1}^{n+1} (x_i - y_i)^2$ for each $k = 1, \dots, n+1$. By [**4**, 2.2.5 Theorem], $\|x_k - y_k\|^2 = \left\|(x_k - y_k)^2\right\| \leqslant \left\| \sum_{i=1}^{n+1} (x_i - y_i)^2 \right\| < \varepsilon^2$. Consequently, $\|x_k - y_k\| < \varepsilon$, $k = 1, \dots, n+1$. This shows that condition (ii) is satisfied.

(ii) $\Longrightarrow$ (iii). Suppose (ii) and let $\{x_i : i \in \mathbb{N}\} \subset X_{sa}$ and $\{\varepsilon_i : i \in \mathbb{N}\}$ be sequences of self-adjoint elements of X and positive real numbers, respectively. Denote by $\mathbb{N}_{n+1}$ the family of all subsets of $\mathbb{N}$ of cardinality $n+1$. For every $i \in \mathbb{N}$ and $D \in \mathbb{N}_{n+1}$ let $H_i = \left\{x \in X_{sa} : \|x - x_i\| \leqslant 2^{-1}\varepsilon_i\right\}$ and $H_D = \prod\{H_i : i \in D\}$ be the topological product of all H_i, $i \in D$. We also consider the topological product $H = \prod\{H_i : i \in \mathbb{N}\}$

and the natural projections $\pi_D \colon H \to H_D$. Define the continuous maps $\phi_D \colon X_{sa}^D \to X$, $\phi_D(z_{i_1}, z_{i_2}, .., z_{i_{n+1}}) = \sum_{j=1}^{n+1} z_{i_j}^2$, $D \in \mathbb{N}_{n+1}$. Condition (ii) yields that $\phi_D^{-1}(G)$ is dense (and, obviously, open) in X_{sa}^D for every $D \in \mathbb{N}_{n+1}$, where G is the set of all invertible elements of X. The last observation implies that each set $G_D = \phi_D^{-1}(G) \cap H_D$ is open and dense in H_D. Consequently, each $U_D = \pi_D^{-1}(G_D)$ is open and dense in H because the projections π_D are continuous and open maps. Finally, using that H (as a product of countably many complete metric spaces) has the Baire property, we conclude that the intersection U of all U_D is non-empty. Take any point (y_i) from U. Then $y_i \in H_i$, so $y_i \in X_{sa}$ and $\|x_i - y_i\| \leqslant 2^{-1}\varepsilon_i < \varepsilon_i$ for every $i \in \mathbb{N}$. Moreover, for any $D \in \mathbb{N}_{n+1}$ the point $y_D = (y_i)_{i \in D}$ belongs to G_D, hence $\sum\{y_i^2 : i \in D\}$ is invertible.

Implications (iii) $\Longrightarrow$ (iv) and (iv) $\Longrightarrow$ (v) are trivial.

(v) $\Longrightarrow$ (i). Let $(x_1, \ldots, x_{n+1})$ be an $(n+1)$-tuple of non-zero self-adjoint elements in X and $\varepsilon > 0$. Consider the sequence $\{\overline{x}_i\}$ of self-adjoint elements of X, where

$$\overline{x}_i = \begin{cases} \dfrac{x_i}{\|x_i\|}, & \text{if } i \leqslant n+1; \\ 1, & \text{if } i > n+1. \end{cases}$$

By (iv), there exists a sequence $\{\overline{y}_i \colon i \in \mathbb{N}\}$ of self-adjoint elements of X such that $\sum_{i=1}^{n+1} \overline{y}_i^2$ is invertible and $\|\overline{x}_i - \overline{y}_i\| < \Bigl(\varepsilon / \Bigl(\max\bigl\{\|x_i\| \colon i = 1, \ldots n+1\bigr\}\Bigr)\Bigr)$ for each $i \in \mathbb{N}$. Now let $y_i = \|x_i\| \cdot \overline{y}_i$, $i \in \mathbb{N}$. Then for every $i \leqslant n+1$ we have

$$\begin{aligned} \|x_i - y_i\| = \bigl\| \|x_i\| \cdot \overline{x}_i - \|x_i\| \cdot \overline{y}_i \bigr\| &= \|x_i\| \cdot \|\overline{x}_i - \overline{y}_i\| \\ &< \|x_i\| \cdot \frac{\varepsilon \cdot}{\max\bigl\{\|x_i\| \colon i = 1, \ldots n+1\bigr\}} \leqslant \varepsilon. \end{aligned}$$

The invertibility of $\sum_{i=1}^{n+1} \overline{y}_i^2$ is equivalent to the validity of the equation $1 = \sum_{i=1}^{n+1} z_i \overline{y}_i$ for a suitable $(n+1)$-tuple $(z_1, \ldots, z_{n+1})$. Clearly $1 = \sum_{i=1}^{n+1} \dfrac{z_i}{\|x_i\|} \cdot \|x_i\| \overline{y}_i = \sum_{i=1}^{n+1} \dfrac{z_i}{\|x_i\|} \cdot y_i$ which in turn implies the invertibility of $\sum_{i=1}^{n+1} y_i^2$. ∎

1.2. Bounded rank For the readers convenience below we present definitions and couple of results related to the bounded rank. Details can be found in [3].

Definition 1.2: Let $K > 0$. We say that an m-tuple $(y_1, \ldots, y_m)$ of self-adjoint elements of a unital C^*-algebra X is *K-unessential* if for every rational $\delta > 0$ there exists an m-tuple $(z_1, \ldots, z_m)$ of self-adjoint elements of X satisfying the following conditions:

(a) $\|y_k - z_k\| \leqslant \delta$ for each $k = 1, \ldots, m$,

(b) The element $\sum_{k=1}^{m} z_k^2$ is invertible and $\left\| \left(\sum_{k=1}^{m} z_k^2 \right)^{-1} \right\| \leqslant \dfrac{1}{K \cdot \delta^2}$.

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1-unessential tuples are referred as *unessential.*

DEFINITION 1.3: Let $K > 0$. We say that the *bounded rank* of a unital C^*-algebra X with respect to K does not exceed n (notation: $\mathrm{br}_K(X) \leqslant n$) if for any $(n+1)$-tuple $(x_1, \dots, x_{n+1})$ of self-adjoint elements of X and for any $\varepsilon > 0$ there exists a K-unessential $(n+1)$-tuple $(y_1, \dots, y_{n+1})$ in X such that $\|x_k - y_k\| < \varepsilon$ for each $k = 1, \dots, n+1$. For simplicity $\mathrm{br}_1(X)$ is denoted by $\mathrm{br}(X)$ and it is called a *bounded rank.*

PROPOSITION 1.4. *Let $(y_1, \dots, y_m)$ be a commuting m-tuple of self-adjoint elements of the unital C^*-algebra X. If $\sum\limits_{i=1}^{m} y_i^2$ is invertible, then $(y_1, \dots, y_m)$ is K-unessential for any positive $K \leqslant 1$.*

COROLLARY 1.5. *Let X be a commutative unital C^*-algebra and $0 < K \leqslant 1$. Then $\mathrm{br}_K(X) = \mathrm{rr}(X) = \dim \Omega(X)$, where $\Omega(X)$ is the spectrum of X.*

2. INFINITE RANK

We begin by presenting the definition of weakly infinite real rank.

2.1. WEAKLY (STRONGLY) INFINITE REAL AND BOUNDED RANKS

DEFINITION 2.1: We say that a unital C^* algebra X has a weakly infinite real rank if for any sequence of self-adjoint elements $\{x_i : i \in \mathbb{N}\} \subset X_{sa}$ and any $\varepsilon > 0$ there is a sequence $\{y_i : i \in \mathbb{N}\} \subset X_{sa}$ such that $\|x_i - y_i\| < \varepsilon$ for every $i \in \mathbb{N}$ and the element $\sum\limits_{i \in D} y_i^2$ is invertible for some finite set D of indices. If X does not have weakly infinite real rank, then we say that X has strongly infinite real rank.

The bounded version can be defined similarly.

DEFINITION 2.2: Let $K > 0$. A sequence of self-adjoint elements of a unital C^*-algebra is *K-unessential* if it contains a finite K-unessential (in the sense of Definition 1.2) subset.

DEFINITION 2.3: Let $K > 0$. We say that a unital C^* algebra X has a weakly infinite bounded rank with respect to K if for any sequence of self-adjoint elements $\{x_i : i \in \mathbb{N}\} \subset X_{sa}$ and any $\varepsilon > 0$ there is a K-unessential sequence $\{y_i : i \in \mathbb{N}\} \subset X_{sa}$ such that $\|x_i - y_i\| < \varepsilon$ for every $i \in \mathbb{N}$. If X does not have weakly infinite bounded rank, then we say that X has strongly infinite bounded rank.

For future references we record the following statement.

PROPOSITION 2.4. *Every unital C^*-algebra of a finite real rank has weakly infinite real rank.*

PROOF: Apply Proposition 1.1. ∎

Note that, as it follows from Proposition 2.12, the converse of Proposition 2.4 is not true.

PROPOSITION 2.5. *Let $f: X \to Y$ be a surjective $*$-homomorphism of unital C^*-algebras. If X has weakly infinite real rank, then so does Y.*

PROOF: For any sequence of self-adjoint elements $\{y_i: i \in \mathbb{N}\} \subset Y_{sa}$ and for any $\varepsilon > 0$ we need to find a sequence $\{z_i: i \in \mathbb{N}\} \subset Y_{sa}$ such that

(i)$_Y$ $\|y_i - z_i\| < \varepsilon$ for every $i \in \mathbb{N}$,

(ii)$_Y$ for some $k \geqslant 1$ the element $\sum_{i=1}^{k} z_i^2$ is invertible.

For every $i \in \mathbb{N}$ let x_i be a self-adjoint element in X such that $f(x_i) = y_i$. Since X has weakly infinite real rank, there exists a sequence $\{w_i: i \in \mathbb{N}\}$ of self-adjoint elements in X such that

(i)$_X$ $\|x_i - w_i\| < \varepsilon$ for every $i \in \mathbb{N}$,

(ii)$_X$ the element $\sum_{i=1}^{k} w_i^2$ is invertible for some $k \geqslant 1$.

Let $z_i = f(w_i)$, $i \in \mathbb{N}$. Clearly $z_i \in Y_{sa}$ and $\|y_i - z_i\| = \|f(x_i) - f(w_i)\| = \|f(x_i - w_i)\| \leqslant \|x_i - w_i\| < \varepsilon$ for each $i \in \mathbb{N}$. By (ii)$_X$, there exists an element $a \in X$ such that $a \cdot \sum_{i=1}^{k} w_i^2 = 1$. Clearly $f(a) \cdot \sum_{i=1}^{k} z_i^2 = f(a) \cdot \sum_{i=1}^{k} f(w_i)^2 = f(a) \cdot f\left(\sum_{i=1}^{k} w_i^2\right) = f\left(a \cdot \sum_{i=1}^{k} w_i^2\right) = f(1) = 1$ which shows that $\sum_{i=1}^{k} z_i^2$ is invertible. ∎

PROPOSITION 2.6. *Let $K > 0$ and $f: X \to Y$ be a surjective $*$-homomorphism of unital C^*-algebras. If X has weakly infinite bounded rank with respect to K, then so does Y.*

PROOF: For a sequence $\{y_i: i \in \mathbb{N}\}$ of self-adjoint elements in Y and $\varepsilon > 0$ choose a sequence $\{x_i: i \in \mathbb{N}\} \subseteq X_{sa}$ such that $f(x_i) = y_i$ for each $i \in \mathbb{N}$. Since X has weakly infinite bounded rank with respect to K, there exists a K-unessential sequence $\{w_i: i \in \mathbb{N}\} \subseteq X$ such that $\|x_i - w_i\| < \varepsilon$ for each $i \in \mathbb{N}$. We claim that $\{z_i = f(w_i): i \in \mathbb{N}\}$ is a K-unessential sequence in Y. Indeed, let $\delta > 0$ be a rational number. Since $\{w_i: i \in \mathbb{N}\}$ is K-unessential in X, there exists a finite subset $D \subseteq \mathbb{N}$ and a D-tuple $(s_i)_{i\in D}$ such that

(i)$_X$ $\|w_i - s_i\| \leqslant \delta$ for each $i \in D$;

(ii)$_X$ $\left\|\left(\sum_{i\in D} s_i^2\right)^{-1}\right\| \leqslant \dfrac{1}{K \cdot \delta^2}$.

Now consider the D-tuple $(r_i = f(s_i))_{i\in D}$. Clearly

(i)$_Y$ $\|z_i - r_i\| \leqslant \|f(w_i) - f(s_i)\| \leqslant \|f(w_i - s_i)\| \leqslant \|w_i - s_i\| \leqslant \delta$ for each $i \in D$;

(ii)$_Y$ $\left\|\left(\sum_{i\in D} r_i^2\right)^{-1}\right\| = \left\|f\left(\sum_{i\in D} s_i^2\right)^{-1}\right\| \leqslant \left\|\left(\sum_{i\in D} s_i^2\right)^{-1}\right\| \leqslant \dfrac{1}{K \cdot \delta^2}$.

Proof is completed. ∎

PROPOSITION 2.7. *Let $K > 0$. If the unital C^*-algebra X has weakly infinite bounded rank with respect to K, then it has weakly infinite real rank.*

PROOF: Let $\{x_i : i \in \mathbb{N}\}$ be a sequence of self-adjoint elements in X and let $\varepsilon > 0$. Take a K-unessential sequence $\{y_i : i \in \mathbb{N}\}$ in X such that $\|x_i - y_i\| < \varepsilon/2$ for each $i \in \mathbb{N}$. Since $\{y_i : i \in \mathbb{N}\}$ is K-unessential, it contains a finite K-unessential subset $\{y_i : i \in D\}$, $D \subseteq \mathbb{N}$. As in the proof of Proposition 4.6, there exists a D-tuple $(z_i)_{i \in D}$ such that

(i) $\|y_i - z_i\| \leqslant \delta$ for each $i \in D$,

(ii) $\sum_{i \in D} z_i^2$ is invertible,

Clearly, $\|x_i - z_i\| \leqslant \|x_i - y_i\| + \|y_i - z_i\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, $i \in D$. According to (ii), $\sum_{i \in D} z_i^2$ is invertible, which shows that X has weakly infinite real rank. ∎

COROLLARY 2.8. *If a unital C^*-algebra has strongly infinite real rank, then it has strongly infinite bounded rank with respect to any positive constant.*

2.2. THE COMMUTATIVE CASE If X is a finite-dimensional compact space, then, according to Corollary 1.5, $\mathrm{rr}\big(C(X)\big) = \mathrm{br}_1\big(C(X)\big) = \dim X$ for any positive $K \leqslant 1$. Our next goal is to extend this result to the infinite-dimensional situation.

First, recall that a compact Hausdorff space X is called *weakly infinite-dimensional* [**1**] if for any sequence $\big\{(F_i, H_i) : i \in \mathbb{N}\big\}$ of pairs of closed disjoint subsets of X there are partitions L_i between F_i and H_i such that $\bigcap_{i=1}^{\infty} L_i = \emptyset$. Here, $L_i \subset X$ is called a partition between F_i and H_i if L_i is closed in X and $X \backslash L_i$ is decomposed as the union $U_i \cup V_i$ of disjoint open sets with $F_i \subset U_i$ and $H_i \subset V_i$. Since X is compact, $\bigcap_{i=1}^{\infty} L_i = \emptyset$ is equivalent to $\bigcap_{i=1}^{k} L_i = \emptyset$ for some $k \in \mathbb{N}$. If X is not weakly infinite-dimensional, then it is *strongly infinite-dimensional.*

A standard example of a weakly infinite dimensional, but not finite-dimensional, metrisable compactum can be obtained by taking the one-point compactification $\alpha\big(\oplus\{\mathbb{I}^n : n \in \mathbb{N}\}\big)$ of the discrete union of increasing dimensional cubes. The Hilbert cube $\mathbb{Q}$ is, of course, strongly infinite-dimensional.

THEOREM 2.9. *Let X be a compact Hausdorff space and $0 < K \leqslant 1$. Then the following conditions are equivalent:*

(a) *$C(X)$ has weakly infinite bounded rank with respect to K;*

(b) *$C(X)$ has weakly infinite real rank;*

(c) *X is weakly infinite-dimensional.*

PROOF: (a)$\Longrightarrow$(b). This implication follows from Proposition 2.7 (which is valid for any – not necessarily commutative – unital C^*-algebras).

(b)$\Longrightarrow$(c). Suppose that $C(X)$ has a weakly infinite real rank. Take an arbitrary sequence $\big\{(B_i, K_i) : i \in \mathbb{N}\big\}$ of pairs of disjoint closed subsets of X and define functions $f_i : X \to [-1, 1]$ such that $f_i(B_i) = -1$ and $f_i(K_i) = 1$ for every $i \in \mathbb{N}$. Then, according to our hypothesis, there is a sequence $\{g_i : i \in \mathbb{N}\} \subset C(X)$ of real-valued functions and an integer k with $\|f_i - g_i\| < 1$, $i \in \mathbb{N}$, and $\sum_{i=1}^{k} g_i^2(x) > 0$ for each $x \in X$. If C_i denotes the set $g_i^{-1}(0)$, the last inequality means that $\bigcap_{i=1}^{k} C_i = \emptyset$. Therefore, in order to prove that X is weakly infinite-dimensional, it only remains to show each C_i is a separator between B_i and K_i. To this end, we fix $i \in \mathbb{N}$ and observe that $\|f_i - g_i\| < 1$ implies the following inclusions: $g_i(B_i) \subseteq [-2, 0)$, $g_i(K_i) \subseteq (0, 2]$ and $g_i(X) \subseteq [-2, 2]$. So, $X \backslash C_i = U_i \cup V_i$, where $U_i = g_i^{-1}\big([-2,0)\big)$ and $V_i = g_i^{-1}\big((0,2]\big)$. Moreover, $B_i \subseteq U_i$ and $K_i \subseteq V_i$, that is, C_i separates B_i and K_i.

(c)$\Longrightarrow$(a). Let us show that the weak infinite-dimensionality of X forces $C(X)$ to have a weakly infinite bounded rank with respect to K. To this end, take any sequence $\{f_i \colon i \in \mathbb{N}\} \subset C(X)$ of real-valued functions and any positive number ε. It suffices to find another sequence $\{g_i \colon i \in \mathbb{N}\}$ of real-valued functions in $C(X)$ such that $\|f_i - g_i\| \leqslant \varepsilon$ for every $i \in \mathbb{N}$ and $\sum_{i=1}^{m} g_i^2(x) > 0$ for every $x \in X$ and some $m \in \mathbb{N}$. Indeed, if $\sum_{i=1}^{m} g_i^2(x) > 0$ for every $x \in X$, then the function $\sum_{i=1}^{m} g_i^2$ is invertible. This, according to Proposition 1.4, is equivalent to the K-unessentiality of the m-tuple $(g_1, \dots, g_m)$. On the other hand, $\sum_{i=1}^{m} g_i^2(x) > 0$ for each $x \in X$ if and only if $\bigcap_{i=1}^{m} g_i^{-1}(0) = \emptyset$. Further, since X is compact, the existence of $m \in \mathbb{N}$ with $\bigcap_{i=1}^{m} g_i^{-1}(0) = \emptyset$ is equivalent to $\bigcap_{i=1}^{\infty} g_i^{-1}(0) = \emptyset$. Therefore, our proof is reduced to constructing, for each $i \in \mathbb{N}$, a function g_i which is ε-close to f_i and such that the intersection of all $g_i^{-1}(0)$'s, $i \in \mathbb{N}$, is empty.

For every $i \in \mathbb{N}$ let $c_i = \inf\big\{f_i(x) : x \in X\big\}$ and $d_i = \sup\big\{f_i(x) : x \in X\big\}$. We can suppose, without loss of generality, that each interval (c_i, d_i) is not empty and contains 0. For every i we choose $\eta_i > 0$ such that $\eta_i < \varepsilon/2$ and $L_i = [-\eta_i, \eta_i] \subset (c_i, d_i)$, $i \in \mathbb{N}$. Let $Q = \prod_{i=1}^{\infty} [c_i, d_i]$, $Q_0 = \prod_{i=1}^{\infty} L_i$ be the topological products of all $[c_i, d_i]$'s and L_i's, respectively. Consider the diagonal product $f = \Delta\{f_i \colon i \in \mathbb{N}\} \colon X \to Q$ and note that $H = \bigcap_{i=1}^{\infty} H_i$, where $H = f^{-1}(Q_0)$ and $H_i = f_i^{-1}(L_i)$ for each $i \in \mathbb{N}$. We also consider the sets

$$F_i^- = f_i^{-1}\big([c_i, -\eta_i]\big) \text{ and } F_i^+ = f_i^{-1}\big([\eta_i, d_i]\big),\ i \in \mathbb{N}.$$

Since H is weakly infinite-dimensional (as a closed subset of X), by [**1**, Theorem 19, Section 10.4], there is a continuous map $p = (p_1, p_2, \dots) \colon H \to Q_0$ and a pseudointerior point $b = \{b_i \colon i \in \mathbb{N}\} \in Q_0$ (that is, each b_i lies in the interior of the interval L_i) such that

$$b \notin p(H),\quad F_i^- \cap H \subset p_i^{-1}\big(\{-\eta_i\}\big), \text{ and } F_i^+ \cap H \subset p_i^{-1}\big(\{\eta_i\}\big),\ i \in \mathbb{N}.$$

Since each b_i is an interior point of $L_i = [-\eta_i, \eta_i]$, there exists homeomorphisms $s_i \colon L_i \to L_i$ which leaves the endpoints $-\eta_i$ and η_i fixed and such that $s_i(b_i) = 0$. Let $s = \Delta\{s_i \colon i \in \mathbb{N}\} \colon Q_0 \to Q_0$ and $q = s \circ p$. Obviously $s(b) = \mathbf{0}$ and $\mathbf{0} \notin q(H)$, where $\mathbf{0}$ denotes the point of Q_0 having all coordinates 0. Further observe that if $q_i = \pi_i \circ q$, where $q_i \colon Q_0 \to L_i$

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denotes the natural projection onto the i-th coordinate, then

$$F_i^- \cap H \subset q_i^{-1}(\{-\eta_i\}) \text{ and } F_i^+ \cap H \subset q_i^{-1}(\{\eta_i\}), \quad i \in \mathbb{N}.$$

Therefore, each q_i, $i \in \mathbb{N}$, is a function from H into L_i satisfying the following condition: $q_i(F_i^- \cap H) = f_i(F_i^- \cap H_i) = -\eta_i$ and $q_i(F_i^+ \cap H) = f_i(F_i^+ \cap H_i) = \eta_i$. Let $h_i \colon H_i \to L_i$ be an extension of q_i, $i \in \mathbb{N}$. Note that the restrictions of h_i and f_i onto the sets $F_i^- \cap H_i$ and $F_i^+ \cap H_i$ coincide. Finally, define $g_i \colon X \to [c_i, d_i]$ by letting

$$g_i(x) = \begin{cases} h_i(x), & \text{if } x \in H_i; \\ f_i(x), & \text{if } x \in X - H_i. \end{cases}$$

To finish the proof of the *if* part, we need to show that $g_i(x)$ is ε-close to $f_i(x)$ for each $i \in \mathbb{N}$ and $x \in X$, and that $\bigcap_{i=1}^{\infty} g_i^{-1}(0) = \emptyset$. Since g_i and f_i are identical outside H_i, the first condition is satisfied for $x \notin H_i$. If $x \in H_i$, then both $f_i(x)$ and $g_i(x)$ belong to L_i, so again $|f_i(x) - g_i(x)| < \varepsilon$. To prove the second condition, observe first that $x \notin H$ implies $x \notin H_j$ for some j. Hence, $g_j(x) = f_j(x) \notin L_j$, so $g_j(x) \neq 0$. If $x \in H$, then $g_i(x) = q_i(x)$ for all i and, because $\mathbf{0} \notin q(H)$, at least one $g_i(x)$ must be different from 0. Thus, $\bigcap_{i=1}^{\infty} g_i^{-1}(0) = \emptyset$. ∎

Let $C^*(\mathbb{F}_\infty)$ denote the group C^*-algebra of the free group on countable number of generators. It is clear that $\operatorname{rr}(C^*(\mathbb{F}_\infty)) > n$ for each n. Our results imply much stronger observation.

Corollary 2.10. *The group C^*-algebra $C^*(\mathbb{F}_\infty)$ of the free group on countable number of generators has strongly infinite real rank.*

Proof: It is well known that every separable unital C^*-algebra is an image of $C^*(\mathbb{F}_\infty)$ under a surjective $*$-homomorphism. In particular, there exists a surjective $*$-homomorphism $f \colon C^*(\mathbb{F}_\infty) \to C(Q)$, where Q denotes the Hilbert cube. It is well known (see, for instance, [**1**, Sections 10.5]) that the Hilbert cube Q is strongly infinite dimensional. By Theorem 2.9, $C(Q)$ has strongly infinite real rank. Finally, by Proposition 2.5, real rank of $C^*(\mathbb{F}_\infty)$ must also be strongly infinite. ∎

Proposition 2.11. *Let X and Y be unital C^*-algebras with weakly infinite real rank. Then $X \oplus Y$ also has weakly infinite real tank.*

Proof: Let $\{(x_i, y_i) \colon i \in \mathbb{N}\}$ be a sequence of self-adjoint elements of $X \oplus Y$ and $\varepsilon > 0$. Since both X and Y have weakly infinite reak rank there exist sequences $\{z_i \colon i \in \mathbb{N}\}$ and $\{w_i \colon i \in \mathbb{N}\}$ of self-adjoint elements of X and Y respectively such that

(i) $\|x_i - z_i\| < \varepsilon$ for every $i \in \mathbb{N}$;

(ii) $\|y_i - w_i\| < \varepsilon$ for every $i \in \mathbb{N}$;

(iii) for some $n \geqslant 1$ the element $\sum_{i=1}^{n} z_i^2$ is invertible;

(iv) for some $m \geqslant 1$ the element $\sum_{i=1}^{m} w_i^2$ is invertible.

Without loss of generality we may assume that $n \geqslant m$. Obviously, according to (iv), $\sum_{i=1}^{n} w_i^2$ is also invertible. Next consider the sequence $\{(z_i, w_i) \colon i \in \mathbb{N}\}$. Note that

$$\|(x_i, y_i) - (z_i, w_i)\| = \|(x_i - z_i, y_i - w_i)\| = \max\{\|x_i - z_i\|, \|y_i - w_i\|\} < \varepsilon$$

and that, according to (iii) and (iv), $\sum_{i=1}^{n}(z_i, w_i)^2 = \sum_{i=1}^{n}(z_i^2, w_i^2)$ is also invertible. ∎

Next statement provides a formal example of a unital C^*-algebra of weakly infinite, but not finite real rank.

Proposition 2.12. *Let $X = \alpha\big(\oplus\{I^n \colon n \in \mathbb{N}\}\big)$ be the one-point compactification of the discrete topological sum of increasing-dimensional cubes. In other words, $C(X) = \prod\{C(I^n) \colon n \in \mathbb{N}\}$ (here $\prod$ stands for the direct product of indicated C^*-algebras). Then $C(X)$ has weakly infinite, but not finite real rank.*

Proof: Obviously X is countably dimensional and hence, by [**1**, Corollary 1, Section 10.5], it is weakly infinite dimensional. By Theorem 2.9, $C(X)$ has weakly infinite real rank. It only remains to note that $\mathrm{rr}(X) > n$ for any $n \in \mathbb{N}$. ∎

In conclusion let us note that there exist non-commutative C^*-algebras with similar properties (compare with Corollary 2.10).

Corollary 2.13. *There exist non-commutative unital C^*-algebras of weakly infinite, but not finite real rank.*

Proof: Let X be as in Proposition 2.12 and A be a non-commutative unital C^*-algebra of a finite real rank. According to Propositions 2.4 and 2.11, the product $C(X) \oplus A$ has weakly infinite real rank, It is clear that $C(X) \oplus A$ is non-commutative and does not have a finite real rank. ∎

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