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RIGID CONTINUA WITH MANY EMBEDDINGS

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ABSTRACT. A separable metric space X is called *rigid* if the identity 1_X is the only autohomeomorphism, and *homogeneous* if, for any points x, y of X, there is an (onto) homeomorphism $h: X \longrightarrow X$ such that $h(x) = y$.

In this note, we show that this onto-ness of the homeomorphism h could not be removed in the definition of homogeneity, by constructing a continuum X which is rigid and *has many embeddings*, that is, for any two points x, y, there is an embedding (= into homeomorphism) $h: X \longrightarrow X$ such that $h(x) = y$.

All spaces in this paper are separable metric spaces.

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In our previous note [3], we showed that this onto-ness of the homeomorphism h could not be removed in the definition of homogeneity, by constructing a subspace X of the real line $\mathbb{R}$ which is rigid and *has many embeddings*, that is, for any two points x, y there is an embedding (= into homeomorphism) $h: X \longrightarrow X$ such that $h(x) = y$.

After [3] the question remained whether the situation is the same if the space X is required to be compact. In the present note we answer this question and construct a rigid continuum with many embeddings.

The Construction. For continua X, Y a map $f: X \longrightarrow Y$ is called *atomic* if any subcontinuum K of X with $|f(K)| > 1$ satisfies $f^{-1}\big(f(K)\big) = K$. It is well known that atomic maps are monotone. The following lemma is derived from [2, Theorem 15].

LEMMA. *For any continua X, Y and a point x_0 of X, there is a compactification Z of $X \setminus \{x_0\}$ such that Z is a continuum, the remainder $Z \setminus (X \setminus \{x_0\})$ is homeomorphic to Y, and there is a natural atomic map $Z \longrightarrow X$.*

In applying this Lemma in the sequel, we will say "insert Y at the point x_0 in X to obtain Z".

Now, let us call a space X *continuum-connected* if any of its two points are contained in a continuum $\subseteq X$. Obviously, path-connected spaces and continua are continuum-connected and continuum-connected spaces are connected. The relation "contained in the same continuum" is clearly an equivalence relation between two points. By this equivalence relation a space is decomposed into the disjoint union of *continuum-components*. If a continuum X is indecomposable and $p \in X$, then composants of X which do not contain

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the point p are continuum-components of $X \setminus \{p\}$. For the composant C containing p, we cannot generally say that $C \setminus \{p\}$ is continuum-connected.

We show:

THEOREM 1. *Suppose that a continuum S is given so that*

(1) every complement of a point of S is continuum-connected, and

(2) there is a countable dense subset D such that every point $\notin D$ of S is contained in a copy of S disjoint from D.

Then there is a rigid continuum Z such that every point of Z is contained in a copy $K \subseteq Z$ of S.

PROOF. Fix a point p in S. Let Y_n be a continuum obtained from the topological sum of the n mutually disjoint copies of S by identifying all points p. Let p_n stand for this identified point of Y_n. $Y_n \setminus \{p_n\}$ consists of n many continuum-components. The condition of the Theorem assures the existence of a countable dense subset $\{y_n \mid n \in \mathbb{N}\}$ of Y_1 so that every point of $Y_1 \setminus \{y_n \mid n \in \mathbb{N}\}$ is contained in a copy $K \subseteq Y_1 \setminus \{y_n \mid n \in \mathbb{N}\}$ of S.

First let $Z_1 = Y_1 = S$.

Apply Lemma to (Y_2, Z_1, y_1) and insert Y_2 at y_1 in Z_1 to obtain Z_2. We suppose that $Z_1 \setminus \{y_1\}$ is embedded in Z_2 in a natural way, again apply Lemma and insert Y_3 at y_2 in Z_2 to obtain Z_3. Generally, supposing that $Z_{n-1} \setminus \{y_{n-1}\}$ is embedded in Z_n, we insert Y_{n+1} at y_n in Z_n and obtain Z_{n+1}. Let π^n_{n-1} be the atomic map $Z_n \longrightarrow Z_{n-1}$.

We claim that $Z = \operatorname{invlim}\{Z_n, \pi^m_n\}$ is the desired continuum. It is easy to see that Z is a continuum, that each projection $\pi_n : Z \longrightarrow Z_n$ is atomic, and that every point of Z is contained in a copy $K \subseteq Z$ of S. We consider Y_n as a natural subspace of Z.

Since $|\pi_n^{-1}(p_n)| = 1$, we have that

$$Z \setminus \{p_n\} = \pi_n^{-1}(Z_n \setminus \{p_n\}) = \pi_n^{-1}\big((Z_{n-1} \setminus \{y_{n-1}\}) \cup (Y_n \setminus \{p_n\})\big).$$

Note that $Z_{n-1} \setminus \{y_{n-1}\}$ is continuum-connected by our condition (1) and that $Y_n \setminus \{p_n\}$ consists of n many continuum-components. Then the atomic-ness of π_n implies that $Z \setminus \{p_n\}$ consists of $(n+1)$ many continuum-components.

If $\pi_1(z) \notin \{y_n \mid n \in \mathbb{N}\}$, then $Z \setminus \{z\} = \pi^{-1}\big(Z_1 \setminus \{\pi_1(z)\}\big)$ is clearly continuum-connected. If $\pi_n(z) \in Y_n \setminus \{p_n\}$, then

$$Z \setminus \{z\} = \pi_n^{-1}\big(Z_n \setminus \{\pi_n(z)\}\big) = \pi_n^{-1}\Big((Z_{n-1} \setminus \{y_{n-1}\}) \cup \big(Y_n \setminus \{\pi_n(z)\}\big)\Big).$$

Here $Y_n \setminus \{\pi_n(z)\}$ is continuum-connected, and hence $Z \setminus \{z\}$ consists of two continuum-components.

To show that Z is rigid, take any autohomeomorphism $h: Z \longrightarrow Z$.

First, it follows from the above consideration that $h(p_n) = p_n$ for each $n \geq 2$. So all we need to show is that $\{p_n \mid n\}$ is dense in Z, because that immediately implies that $h = 1_Z$. Take any basic open set $\pi_n^{-1}(U)$ of Z, where U is an open set of Z_n. Since $Z_1 \setminus \{y_k \mid k \leq n\}$ is dense in Z_n and $\{y_k \mid k > n\}$ is dense in Z_1, there is a $k > n$ so that $y_k \in U$. Then we have that $\pi_n^{-1}(U) = \pi_k^{-1}(\pi^k_n)^{-1}(U) \supseteq \pi_k^{-1}(y_k) \ni p_{k+1}$.

Therefore Z is rigid and is the desired continuum.

As a corollary to Theorem 1, we have:

THEOREM 2. *There is a rigid infinite-dimensional continuum Z with many embeddings. (cf. the Remark below)*

PROOF. $S = \mathbb{Q}$ satisfies the condition of Theorem 1. Take any two points x, y of Z. Consider Z as a subspace of $\mathbb{Q}$. There is a subspace A of Z such that $y \in A \approx \mathbb{Q}$. Since $\mathbb{Q}$ is homogeneous, there is a homeomorphism $h: \mathbb{Q} \longrightarrow A$ such that $h(x) = y$. Then $h|Z$ is an embedding which sends x to y.

REMARK. The referee points out that the Menger Universal Curve M satisfies the hypotheses of Theorem 1. Hence the proof of Theorem 2 with $S = M$ gives a rigid curve (*i.e.* 1-dimensional continuum) Z with many embeddings.

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