BULL. AUSTRAL. MATH. SOC. MOS 4745
VOL. 1 (1969), 397-401.

# The approximation problem for compact operators

**S. R. Caradus**

The following sufficient condition is obtained for the uniform approximability of compact operators on a reflexive Banach space by operators of finite rank: if $S$ is the unit ball of $X$ and $\phi : X^* \to C(S)$ is the imbedding $\phi(x^*)x = x^*(x)$ then we require $\phi(X^*)$ to be complemented in $C(S)$ .

Let $X$ denote a Banach space, $B(X)$ the space of continuous endomorphisms of $X$ , $C$ the set of compact operators in $B(X)$ and $F$ the set of finite dimensional operators in $B(X)$ . By the *approximation problem*, we shall mean: is $C$ equal to the uniform closure of $F$ ? Only sufficient conditions on $X$ are known for an affirmative solution to this problem. Most important among these is the *condition d'approximation* due to Grothendieck ([2] p. 165): every operator in $B(X)$ can be uniformly approximated on compact subsets by operators in $F$ . It is, of course, not known whether this condition holds for all $X$ . However the conjecture that it is true for all $X$ is known ([2] pp. 170-175) to be equivalent to a large number of other conjectures, some apparently more tractable.

At least we know that the approximation problem has an affirmative solution for a large number of the common Banach spaces. This can be deduced from the following ideas.

(1) A Banach space $Y$ is called a $P_\lambda$ *space*, if, given any Banach space $\tilde{Y} \supseteq Y$ , there exists a projection of norm not exceeding $\lambda$ from $\tilde{Y}$ to $Y$ .

(2) In [3], Lindenstrauss introduced the concept of an $N_\lambda$ *space*: a Banach space $Y$ is an $N_\lambda$ space if there exists a set $\{Y_\tau\}$ of finite

Received 25 July 1969.



https://doi.org/10.1017/S0004972700042313 Published online by Cambridge University Press

dimensional subspaces of $Y$ , directed by inclusion, such that $\cup Y_\tau$ is dense in $Y$ and such that each $Y_\tau$ is a $P_\lambda$ space.

(3) The concept of the $\lambda$ *projection approximation property* *($\lambda$-P.A.P.)* was also introduced in [3]: a Banach space $Y$ has the $\lambda$-*P.A.P.* if there exists a set $\{Y_\tau\}$ of finite dimensional subspaces of $Y$ , directed by inclusion, such that $\cup Y_\tau$ is dense in $Y$ and such that, for each $\tau$ , there exists a projection of $Y$ onto $Y_\tau$ with norm not exceeding $\lambda$ .

It is evident that every $N_\lambda$ space has the $\lambda$-*P.A.P.* Moreover, it is not difficult to deduce that the $\lambda$-*P.A.P.* implies the *condition d'approximation.* For suppose $K$ is compact and $\varepsilon > 0$ . Then we can find $x_1, x_2, \ldots, x_n \in K$ such that for every $x \in K$ there exists $x_i$ with $\|x-x_i\| < \varepsilon$ . Then we can choose $y_1, y_2, \ldots, y_n \in \cup Y_\tau$ such that $\|y_i-x_i\| < \varepsilon$ . Thus there exists $\tau_0$ such that $y_1, y_2, \ldots, y_n \in Y_{\tau_0}$ and a projection $P_0$ of $Y$ onto $Y_{\tau_0}$ such that $\|P_0\| \leq \lambda$ . If, given $x \in K$ , $\|x-x_i\| < \varepsilon$ then we can write

$$\text{(i)} \quad \begin{aligned} \|Tx-TP_0x\| &\leq \|Tx-Tx_i\| + \|Tx_i-Ty_i\| + \|TP_0y_i-TP_0x_i\| + \|TP_0x_i-TP_0x\| \\ &\leq 2\|T\|(1+\lambda)\varepsilon . \end{aligned}$$

Hence the result follows.

Now the separable Banach spaces with a basis, the $L_p(\mu)$ spaces ($1 \leq p \leq \infty$ , $\mu$ arbitrary) and $C(\Omega)$ ($\Omega$ any topological space) are known to be spaces with the $\lambda$-*P.A.P.* for some $\lambda$ (see [3] pp. 25, 29). Hence for these spaces the approximation problem has an affirmative solution.

In that which follows, another sufficient condition is obtained for the case where $X$ is reflexive. For any reflexive Banach space $X$ , let $S$ denote the unit ball with the weak topology and let $\phi : X^* \to C(S)$ denote the isometric imbedding $\phi(x^*)x = x^*(x)$ .

**THEOREM.** *If* $\phi(X^*)$ *has a closed complement in* $C(S)$ *, then the approximation problem in* $X$ *has an affirmative solution.*

https://doi.org/10.1017/S0004972700042313 Published online by Cambridge University Press

Proof. Let $P_o$ denote a projection $C(S) \to \phi(X^*)$ . By a result of Lindenstrauss ([3] p. 29), we know that $C(S)$ is a $N_\lambda$ space for each $\lambda > 1$ . Moreover ([3] p. 22), every finite dimensional subspace of a $N_\lambda$ space is contained in a finite dimensional $P_{\lambda'}$ space for any $\lambda' > \lambda$ . We shall fix $\lambda$ and $\lambda'$ , $\lambda' > \lambda > 1$ .

Now suppose $T$ is a compact operator in $B(X)$ . Then by a theorem of Lacey ([1] p. 85), given $\varepsilon > 0$ , there exists a closed subspace $N_\varepsilon$ of $X$ with finite codimension such that $\|T|N_\varepsilon\| < \varepsilon$ . Let $N_\varepsilon^\perp$ denote the functionals in $X^*$ which vanish on $N_\varepsilon$ . Then $N_\varepsilon^\perp$ is a finite dimensional subspace of $X^*$ and $\phi\left(N_\varepsilon^\perp\right)$ is finite dimensional in $C(S)$ . By the remark about $N_\lambda$ spaces, there exists a finite dimensional subspace $X_\varepsilon$ in $C(S)$ with $X_\varepsilon \supseteq \phi\left(N_\varepsilon\right)$ and a projection $P_\varepsilon : C(S) \to X_\varepsilon$ with $\|P_\varepsilon\| < \lambda'$ . Consider the product $P_oP_\varepsilon$ . Evidently this is a finite dimensional operator whose range $R$ satisfies $\phi\left(N_\varepsilon^\perp\right) \subseteq R \subseteq \phi(X^*)$ . Hence we can define $\tilde{P}_\varepsilon = \phi^{-1}P_oP_\varepsilon\phi$ which will be a finite dimensional operator in $X^*$ .

Now $\phi\left(N_\varepsilon^\perp\right) \subseteq X_\varepsilon$ so that

$$\begin{aligned} N_\varepsilon^\perp \subseteq \phi^{-1}(X_\varepsilon) = \phi^{-1}R(P_\varepsilon) &= \left\{x^* \in X^* : \phi x^* \in R(P_\varepsilon)\right\} \\ &= \{x^* \in X^* : P_\varepsilon\phi x^* = \phi x^*\} \\ &\subseteq \{x^* \in X^* : P_oP_\varepsilon\phi x^* = \phi x^*\} \\ &= \{x^* \in X^* : \tilde{P}_\varepsilon x^* = x^*\} \\ &= N\left(I-\tilde{P}_\varepsilon\right) \\ &= R\left(I-\tilde{P}_\varepsilon{}^*\right)^\perp \end{aligned}$$

where we are identifying $\tilde{P}_\varepsilon{}^*$ with the corresponding operator in $B(X)$ .

Hence $N_\varepsilon \supseteq R(I-\tilde{P}_\varepsilon{}^*)$ .

We can now write

https://doi.org/10.1017/S0004972700042313 Published online by Cambridge University Press

$$\|T(I-\tilde{P}_\varepsilon{}^*)\| \le \sup_{x\neq 0} \frac{\|T(I-\tilde{P}_\varepsilon{}^*)x\|}{\|(I-\tilde{P}_\varepsilon{}^*)x\|} \sup_{x\neq 0} \frac{\|(I-\tilde{P}_\varepsilon{}^*)x\|}{\|x\|}$$
$$\le \sup_{0\neq y\in N_\varepsilon} \frac{\|Ty\|}{\|y\|} \cdot \|I-\tilde{P}_\varepsilon{}^*\|$$
$$\le \varepsilon\left(1+\|\tilde{P}_\varepsilon\|\right)$$
$$\le \varepsilon\left(1+\|P_0\|\lambda'\right) .$$

Since $\lambda'$ and $\|P_0\|$ are fixed and $\varepsilon$ arbitrary, and since $\tilde{P}_\varepsilon{}^*$ has finite dimensional range, the proof is complete.

REMARK. Examination of the above proof reveals that the hypotheses can be weakened considerably. Suppose we define the following property *(P)* : a Banach space $X$ will be said to have property *(P)* if there exists $M > 0$ such that, given any finite dimensional subspace $F$ of $X$, there exists a finite dimensional projection $P$ in $B(X)$ such that $\|P\| \le M$ and $R(P) \supseteq F$. An examination of the proof of Lemma 3.1 of [3] shows that any space with the $\lambda$-*P.A.P.* also has property *(P)* ; moreover, using simple calculations similar to (i), we can deduce that property *(P)* implies the *condition d'approximation* of Grothendieck. Hence we can state

COROLLARY. *Let* $X$ *be a reflexive Banach space such that there exists a Banach space* $Y$ *with property (P) and a linear homeomorphism* $\phi : X^* \to Y$ *with* $\phi(X^*)$ *complemented. Then the approximation problem for* $X$ *has an affirmative solution.*

## References

[1] Seymour Goldberg, *Unbounded linear operators* (McGraw-Hill, New York, 1966).

[2] Alexandre Grothendieck, "Produits tensoriels topologiques et espaces nucléaires", *Mem. Amer. Math. Soc.* 16 (1955).

https://doi.org/10.1017/S0004972700042313 Published online by Cambridge University Press

[3] Joram Lindenstrauss, "Extension of compact operators", *Mem. Amer. Math. Soc.* 48 (1964).

Department of Mathematics, IAS,
Australian National University,
Canberra, ACT,

and

Queen's University at Kingston,
Ontario, Canada.