BULL. AUSTRAL. MATH. SOC. 22A05
VOL. 49 (1994) [519–521]

A REMARK ON EMBEDDING TOPOLOGICAL GROUPS INTO PRODUCTS

VLADIMIR PESTOV

Let $\mathcal{P}$ be a class of topological groups such that every topological group is isomorphic to a topological subgroup of the direct product (with Tychonoff topology) of a subfamily of $\mathcal{P}$. Then every Tychonoff space is homeomorphic to a subspace of a group from $\mathcal{P}$.

QUESTION. *Let $\mathcal{P}$ be a certain class of topological groups. Is every topological group isomorphic to a topological subgroup of the direct product (with the Tychonoff topology) of a subfamily of $\mathcal{P}$?*

This question was discussed for various classes of topological groups $\mathcal{P}$ by Arhangel'skiĭ [**1, 2**]. In particular, it was answered in the negative independently by the author [**8**] and Guran [**4**] in the case where $\mathcal{P}$ was the class of all topological groups with unity of type G_δ, and also by Guran [**5**] in the case where $\mathcal{P}$ was the class of all topological groups whose underlying topological spaces were sequential.

The following result throws new light on all possible questions of this kind.

THEOREM. *Let $\mathcal{P}$ be a class of topological groups. Suppose that continuous homomorphisms to the groups from $\mathcal{P}$ separate points in every Hausdorff topological group G. Then every Tychonoff space is homeomorphic to a subspace of a group from $\mathcal{P}$.*

PROOF: Let X be an arbitrary Tychonoff space. One can assume without loss in generality that X is compact. It can be embedded into a Tychonoff space Y such that any ordered n-tuple of pairwise distinct elements of Y can be sent to any other such n-tuple by means of an autohomeomorphism of Y [**7**]. The full autohomeomorphism group, $\operatorname{Aut} Y$, of Y acts on the free topological group, $F(Y)$, on Y [**6**] if one extends autohomeomorphisms of Y to automorphisms of $F(Y)$. This action is continuous if

Received 25th August, 1993
This work was done during a visit of the author to the Analysis Research Group of the University of Wollongong in January 1993, and the author is thankful to Professor Sidney Morris for his kind hospitality. A Small Research Grant V212/451/RGNT/594/153 from the Internal Grants Committee of the Victoria University of Wellington is acknowledged.

the group $\operatorname{Aut} Y$ is endowed with the discrete topology. Form the semidirect product $G = \operatorname{Aut} Y \ltimes F(Y)$.

It follows from the assumption of the Theorem that here exist an $H \in \mathcal{P}$ and a continuous $f: G \to H$ such that $f(Y)$ is non-trivial. The restriction of f to Y is one-to-one. (Indeed, otherwise there exist pairwise distinct $x, y, z \in Y$ such that $f(x) = f(y) \neq f(z)$. There is an $h \in \operatorname{Aut} Y$ with $h^{-1}xh = y$, $h^{-1}yh = z$, and $h^{-1}zh = x$. One has $f(y) = f(h)^{-1}f(x)f(h) = f(h)^{-1}f(y)f(h) = f(z)$, a contradiction.) Therefore, $f|_X$ is a homeomorphism by virtue of the compactness of X. ∎

Corollary. *Let $\mathcal{P}$ be a class of topological groups. Suppose that every Hausdorff topological group G is isomorphic to a topological subgroup of the direct product (with the Tychonoff topology) of a subfamily of $\mathcal{P}$. Then every Tychonoff space is homeomorphic to a subspace of a group from $\mathcal{P}$.* ∎

In particular, the two cases mentioned above receive simple answers in the negative.

We hope that our remark can be put in the context of generating varieties of topological groups [**3**]. In particular, we suggest the following.

Conjecture. *Let $\mathcal{P}$ be a class of topological groups. The following are equivalent:*

(i) *every Hausdorff topological group G is isomorphic to a topological subgroup of the direct product (with the Tychonoff topology) of a subfamily of $\mathcal{P}$;*

(ii) *every Hausdorff topological group G is isomorphic to a topological subgroup of a group from $\mathcal{P}$.*

References

[1] A.V. Arhangel'skiĭ, 'Relations among invariants of topological groups and their subspaces', *Russian Math. Surveys* **35** (1980), 1–23.

[2] A.V. Arhangel'skiĭ, 'Classes of topological groups', *Russian Math. Surveys* **36** (1981), 151–174.

[3] M.S. Brooks, S.A. Morris and H.B. Thompson, 'Generating varieties of topological groups', *Proc. Edinbourgh Math. Soc.* **18** (1973), 191–197.

[4] I.I. Guran, 'Topology of an infinite symmetric group and condensations', (in Russian), *Comment. Math. Univ. Carolin.* **22** (1981), 311–316.

[5] I.I. Guran, A private communication, 1992.

[6] A.A. Markov, 'Three papers on topological groups', *Amer. Math. Soc. Transl.* **30** (1950).

[7] M.G. Megrelishvili, 'On the imbedding of topological spaces into spaces with strong properties of homogenuity', (in Russian), *Bull. Acad. Sci. Georgian SSR* **121** (1986), 257–260.

[8] V.G. Pestov, 'On embeddings and condensations of topological groups', *Math. Notes* **31** (1982), 228–230.

Department of Mathematics
Victoria University of Wellington
Wellington
New Zealand
e-mail: vladimir.pestov@vuw.ac.nz