Can. J. Math., Vol. XXVI, No. 4, 1974, pp. 978–984

SOME COUNTEREXAMPLES IN LINK THEORY

DALE ROLFSEN

This note is concerned specifically with links of two (disjoint) n-spheres in an $(n + 2)$-manifold M, i.e. embeddings $L:S^n + S^n \to M$. The links L_0 and L_1 are *isotopic* if they are the ends of a continuous family $L_t:S^n + S^n \to M$ $(0 \leqq t \leqq 1)$ of links. They are *ambient isotopic* or *equivalent* if there is a continuous family of self-homeomorphisms $h_t:M \to M$ $(0 \leqq t \leqq 1)$ such that h_0 = identity and $h_1 \circ L_0 = L_1$. Ambient isotopic links are isotopic, but not conversely. For example, an isotopy can tie and untie little knots (as in Figure 3) in the components of a link, thus changing the original link into one which is inequivalent to the original. It is natural to propose the following conjecture, which would provide the best possible relation between the two types of isotopy.

CONJECTURE 1. *Suppose L_0 and L_1 are n-dimensional links in M^{n+2} which are isotopic. If the first component of L_0 is equivalent (as a knot) to the first component of L_1, and likewise for the second, then L_0 and L_1 are ambient isotopic.*

Closely related to this is

CONJECTURE 2. *If the n-dimensional link L is isotopic to a trivial link in M^{n+2} then L is splittable.*

It was shown in [**5**] that

THEOREM. *Conjectures* 1 *and* 2 *are true in the PL category for the classical case* $n = 1$, $M = R^3$ *or* S^3.

The examples presented below show that this Theorem cannot be very much generalized.

A *trivial* link of n-spheres in M^{n+2} is one whose components bound disjoint locally flat $(n + 1)$-disks in M. A link in M^{n+2} is *splittable* if its components lie in disjoint $(n + 2)$-disks in M. In the PL category it is understood that the spaces, links, and isotopies (as maps $(S^n + S^n) \times I \to M$ or $M \times I \to M$) are all required to be piecewise linear. It should be noted that the converse of Conjecture 1 is obviously true, that (at least in the PL category) the converse of Conjecture 2 always holds, and that Conjecture 1 implies Conjecture 2 for PL links (see [**5**]). Both conjectures are true in the smooth (C^∞) category, since C^∞ isotopy implies ambient isotopy.

Example 1, *showing that Conjectures* 1 *and* 2 *are false in the topological category.* This is a "folk" example, but does not seem to appear anywhere in print. A

Received March 6, 1973.

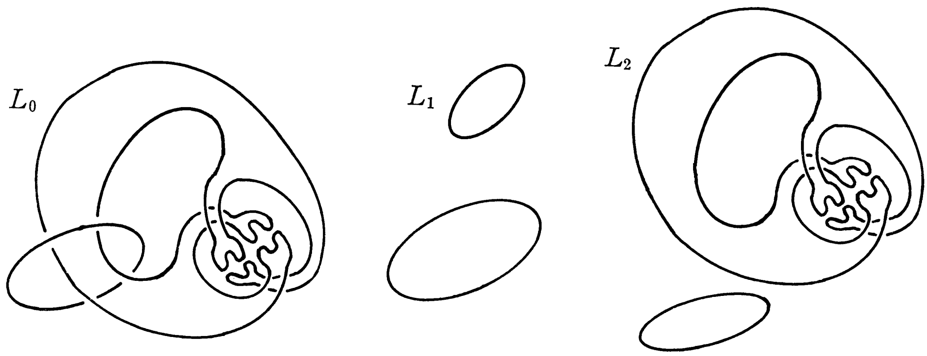

FIGURE 1

2-sphere may be embedded in R^3 as Alexander's horned sphere [**1**] in such a way that the equator runs through all the wild points. L_0 consists of this embedding of the equator, together with a curve that runs around the horns (Figure 1). Now, sliding the equator on the horned sphere to, say, the arctic circle provides an isotopy from L_0 to the trivial link L_1. Conjecture 2 would imply that L_0 is splittable. It can be shown that this is not the case; in fact the round component of L_0 is not even contractible in the complement of the horned component.

A counterexample to Conjecture 1 can be obtained by re-introducing the horns on the first component, via an isotopy, in such a way that the other component is free of the horns, forming a link L_2. Now L_0 and L_2 are isotopic, their respective components are equivalent knots, yet they are not ambient isotopic (since L_2 is splittable). A similar argument could be made using the infinite "knitted" link of Milnor [**4**], which has only one wild point.

Example 2, *showing that Conjecture* 1 *is false* (*even in the PL category*) *for* $n = 1$ *if* M *is an arbitrary* 3-*manifold.* In particular, consider $M = S^2 \times S^1$. This may be visualized as $S^2 \times [0, 1]$ with the identification of every $x \times 0$ with $x \times 1$ (Figure 2). If n and s denote the north and south poles of S^2, then L_1

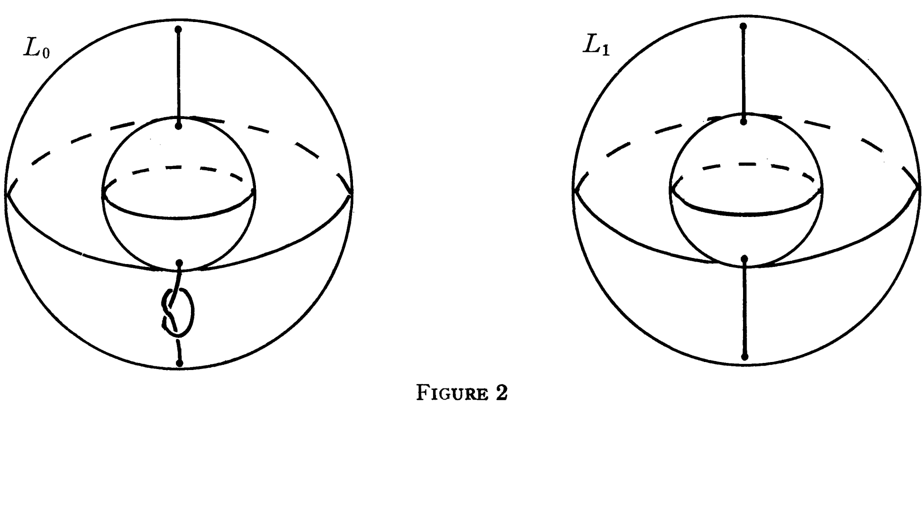

FIGURE 2

is the link whose image is $n \times [0, 1]$ and $s \times [0, 1]$ (with identifications). L_0 is the same, except that a small overhand knot is tied in $s \times [0, 1]$.

The sequence of details of Figure 3 illustrates an isotopy between L_0 and L_1 (which can be done piecewise-linearly). Furthermore, if the northern component of L_0 is removed, the overhand knot can be undone via an *ambient* isotopy by slipping a loop around the inner sphere. Thus, the corresponding components of L_0 and L_1 are equivalent as knots, so all the hypotheses of Conjecture 1 are fulfilled.

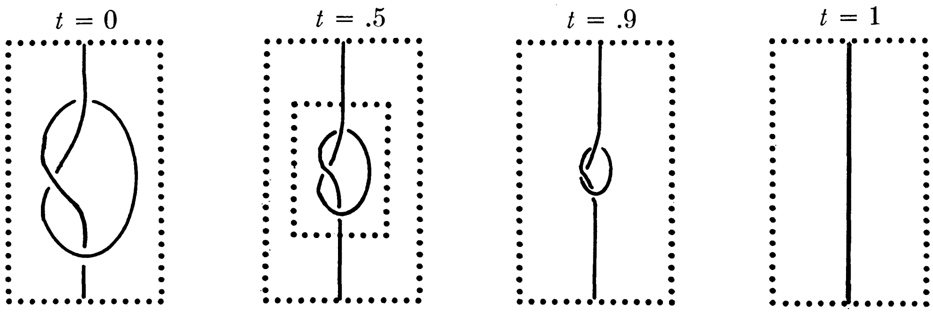

FIGURE 3

However, L_0 and L_1 are not ambient isotopic in M; in fact their complements are not even homeomorphic. An easy computation of the fundamental groups gives presentations:

$$\pi_1(M - L_0) = (x, y, z; xyx = yxy, xz = zx)$$
$$\pi_1(M - L_1) = (a, b; ab = ba) \cong Z \times Z.$$

These groups are non-isomorphic; the second is abelian, whereas the first is not.

Example 3, *showing that Conjectures* 1 *and* 2 *are false for* $n > 1$ *in either the PL or topological categories, even if* $M = R^{n+2}$. The main task is to construct, for $n > 1$, a *PL* isotopy converting a trivial link of two n-spheres in R^{n+2} into an unsplittable link. The following generalization of the process illustrated in Figure 3 is the key to this construction.

Isotopy by disk replacement. Suppose $L_0: S^n + S^n \to M^{n+2}$ is a link and A is an n-disk in one component of $S^n + S^n$. Further suppose that B is an $(n + 2)$-disk in M whose intersection with the image of L_0 is exactly the set $L_0(A)$. Then if $L_1: S^n + S^n \to M$ is another link satisfying (i) $L_0 = L_1$ off A and (ii) $L_1(A) \subset B$, it may be shown that L_0 and L_1 are isotopic, using

LEMMA. (*PL* or topological category) *Suppose* $f_i: D^n \to D^{n+2}$ $(i = 0, 1)$ *are embeddings of an n-disk in an* $(n + 2)$*-disk such that* $f_i(\partial D^n) \subset \partial D^{n+2}$ *and* $f_0|\partial D^n = f_1|\partial D^n$. *Then they can be connected by a continuous family of embeddings* $f_t: D^n \to D^{n+2}$ $(0 \leqq t \leqq 1)$ *such that* $f_t|\partial D^n = f_0|\partial D^n$ *for all t.*

The Lemma may be proved [**5**] by a version of a well-known technique of Alexander, moving each f_i via an isotopy to the cone on $f_i|\partial D^n$.

The construction. In R_+^3, the closed upper half of 3-space, arrange the following objects as in Figure 4: arcs Σ_1 and Σ_2 with endpoints in $R^2 = \partial R_+^3$, a 3-disk C with $C \cap R^2 = a$ 2-disk in ∂C, and a 3-disk $H \cong D^2 \times D^1$ with $H \cap C$ and $H \cap R^2$ 2-disks corresponding to the ends $D^2 \times 1$ and $D^2 \times (-1)$. Let $P = \Sigma_1 \cap C$ and $Q = \overline{\Sigma_1 - P}$.

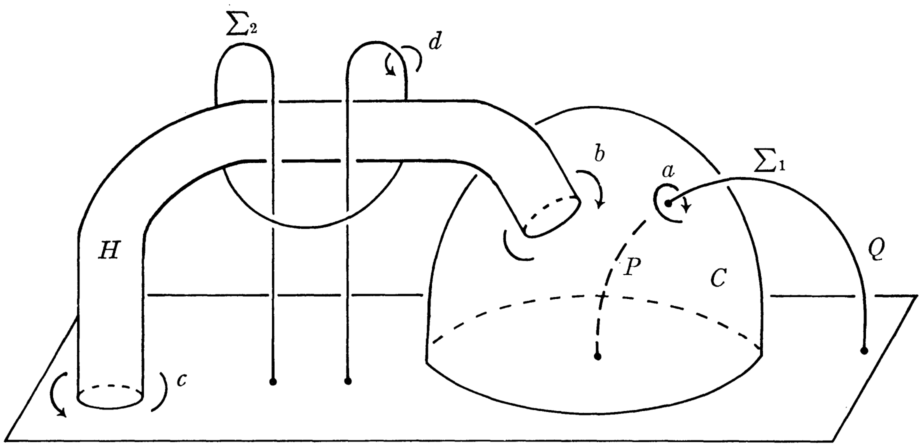

FIGURE 4

Fix an integer $n > 1$ and spin R_+^3 by S^{n-1} around R^2 (see, for example, [**6**]). The resulting total space is homeomorphic to R^{n+2}. The objects obtained by spinning Σ_1, Σ_2, etc. will be denoted by the same symbols, so Figure 4 may be viewed as a cross-section of the spun objects, cut by the half-hyperplane R_+^3 in R^{n+2}. Thus after spinning:

Σ_1 and Σ_2 are n-spheres which form a trivial link in R^{n+2}, P and Q are n-disks forming hemispheres of Σ_1, C and H are $(n + 2)$-disks with $H \cong D^2 \times D^n$ an n-handle attached to C along $D^2 \times S^{n-1}$.

Now $C \cup H$ is an $S^n \times D^2$ and may be made into a disk by removing a 2-handle. Instead of the obvious method (removing a disk bounded by b shown in Figure 4) a trick of Hudson and Sumners [**3**] may be used. Choose an embedding $F:S^1 \to \partial C - (H \cup \Sigma_1)$ which represents $b^2ab^{-1}a^{-1}$ as in Figure 5. Since F is nullhomotopic in $C - P$ it may be extended to an embedded 2-disk $D \subset C - P$ (by general position if $n > 2$ and by a construction of [**3**] for $n = 2$). Remove an open tubular neighborhood $N(D)$ from $C \cup H$ to obtain

$$B = C \cup H - N(D).$$

https://doi.org/10.4153/CJM-1974-091-6 Published online by Cambridge University Press

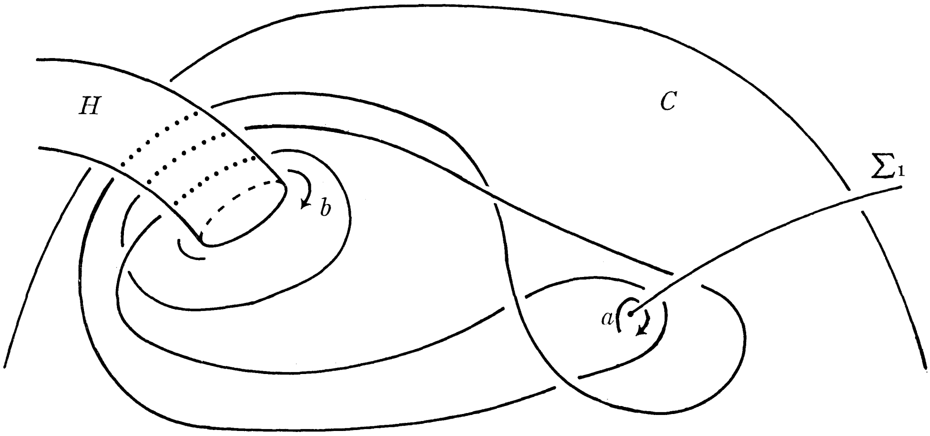

FIGURE 5

It is easily seen that F is equivalent in $\partial(C \cup H)$ to a loop going once around b, so $\partial B \cong S^{n+1}$ and B is an $(n + 2)$-disk.

The inclusion of a cross-section into its spun object induces an isomorphism of fundamental groups (provided the object intersects R^2 when $n = 2$) so the following presentations may be read from Figure 4 by traditional methods (E denotes the interior of $C \cup H$).

$$(1) \qquad \pi_1(R^{n+2} - (\Sigma_1 \cup E)) = (a, b) = Z * Z$$

$$(2) \qquad \pi_1(R^{n+2} - (\Sigma_2 \cup E)) = (b, c, d; \bar{r})$$

$$(3) \qquad \pi_1(R^{n+2} - (\Sigma_1 \cup \Sigma_2 \cup E)) = (a, b, c, d; \bar{r})$$

The generators a, b, c, d are represented by loops in $R_+{}^3$ as in Figure 4 and $\bar{r}$ is the relation $[b, [d, b^{-1}c]] = b^{-1}c$ (see [**6**] for a calculation). The symbol $[x, y]$ denotes the commutator $x^{-1}y^{-1}xy$.

After the surgery of $C \cup H$ along D, one computes

$$(4) \qquad \pi_1(\partial B - \partial Q) = (a, b; b^2a = ab)$$

$$(5) \qquad \pi_1(R^{n+2} - (\Sigma_1 \cup \text{int } B)) = (a, b; b^2a = ab)$$

$$(6) \qquad \pi_1(R^{n+2} - (\Sigma_1 \cup \Sigma_2 \cup \text{int } B)) = (a, b, c, d; b^2a = ab, \bar{r}).$$

PROPOSITION. *There are no two disjoint $(n + 2)$-disks in R^{n+2} such that one contains $\Sigma_1 \cup B$ and the other contains Σ_2.*

Proof. If $\Sigma_1 \cup B$ and Σ_2 could be so separated, the group (6) of their complement would split into the free product of the subgroup generated by d, and another subgroup. However, each generator of (6) represents a nontrivial element, and none of a, b, c are in the subgroup generated by d. (This may be seen by mapping (6) into the group of permutations of the symbols 1, 2, 3, 4 via $a \to (12)$, $b \to (123)$, $c \to (134)$, $d \to (24)$ in the notation of cycles.)

https://doi.org/10.4153/CJM-1974-091-6 Published online by Cambridge University Press

The fact that $b^{-1}c \neq 1$ and the nontrivial relation $\bar{r}$ holds in (6) between b, c and d thus contradicts the conclusion that d generates a free factor.

The unsplittable link. Since B is an $(n + 2)$-disk it has a natural radial structure, which can be used to embed the cone on ∂P in B. Form a link L whose image is $Q \cup \text{cone } (\partial P) \cup \Sigma_2$. By the Lemma, L is isotopic to the trivial link with image $\Sigma_1 \cup \Sigma_2$. However, since B collapses to cone (∂P), the Proposition implies that L is unsplittable. This example has the flaw that L if not locally flat, since (4) implies that $\partial P = \partial Q$ is knotted in ∂B.

A locally flat example may be constructed by a trick. Add a point ∞ to R^{n+2}; then ∂B bounds a ball on either side and there is a "reflection" homeomorphism $r:R^{n+2} + \infty \rightarrow R^{n+2} + \infty$, fixed on ∂B and interchanging the sides. Now let L' be a link whose image is $Q \cup rQ \cup \Sigma_2$. Note that the inclusion homomorphism

$$\pi_1(\partial B - \partial Q) \rightarrow \pi_1(B - rQ)$$

is equivalent to the inclusion map of (4) into (5), hence an isomorphism. It follows by Van Kampen's theorem that $\pi_1(R^{n+2} - L')$ has the same presentation as (6) and an argument as in the Proposition shows that L' is not splittable. As before L' is isotopically trivial and, by construction, locally flat.

Finally, a counterexample to Conjecture 1 may be obtained from L (or L') as follows. Let $L_0 = L$. A small knot may be tied in Σ_1 via an isotopy (using the Lemma) so that it has the same knot type as the first component of L_0, but is still splittable from Σ_2. Call this link L_1. Then L_0 and L_1 satisfy the hypotheses of Conjecture 1, but are not ambient isotopic (since L_1 splits while L_0 does not).

Comments and open questions. (1) Is the *PL* version of Conjecture 2 true in any M^3? More generally, for what class of 3-manifolds other than R^3 and S^3 are the conjectures valid? It can be shown that Conjectures 1 and 2 are true ($n = 1$, *PL*) in any M, provided the links all have unknotted components (i.e. each component bounds a disk in M). That is, if all components are unknotted we have: isotopic $\Leftrightarrow$ ambient isotopic, and splittable $\Leftrightarrow$ trivial.

(2) All *PL* knots are isotopic to one another. Is this true of wild knots? A knot in 3-space which is not isotopic to a tame knot would have to be so wild as to fail to pierce a disk at each of its points. The "Bing sling" [**2**] is such a candidate.

(3) If L_0 and L_1 are *PL* links connected by a topological isotopy, are they *PL* isotopic?

(4) Is there an n-dimensional link ($n > 1$) which satisfies the conditions of the link of Example 3 (unsplittable, yet isotopically trivial) and the further condition that both of its components are unknotted?

(5) Example 3 would be empty if, in fact, *PL* links in higher dimensions were isotopically trivial. It will be shown in a subsequent paper ("Localized Alexander invariants and isotopy of links", to appear in Ann. of Math.) that, unlike the case of knots, there are infinitely many *PL* isotopy classes of links in every dimension.

References

1. J. W. Alexander, *An example of a simply connected surface bounding a region which is not simply connected*, Proc. Nat. Acad. Sci. U.S.A. *10* (1924), 8–10.
2. R. H. Bing, *A simple closed curve that pierces no disk*, J. Math. Pures Appl. *35* (1956), 337–43.
3. J. F. P. Hudson and D. W. L. Sumners, *Knotted ball pairs in unknotted sphere pairs*, J. London Math. Soc. *41* (1966), 712–22.
4. J. Milnor, *Isotopy of links, in algebraic geometry and topology*, A symposium in honor of S. Lefschetz, Princeton, 1957, 280–306.
5. D. Rolfsen, *Isotopy of links in codimension two*, J. Indian Math. Soc. (to appear).
6. E. C. Zeeman, *Linking spheres*, Abh. Math. Sem. Univ. Hamburg *24* (1960), 149–52.

University of British Columbia,
Vancouver, British Columbia