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PROPERTIES OF A CLASS OF (0,1)-MATRICES COVERING A GIVEN MATRIX

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1. Introduction. We wish to consider the class of (0,1)-matrices with prescribed row and column sums. Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be vectors with nonnegative integral entries and $r_1 + r_2 + \ldots + r_m = s_1 + s_2 + \ldots + s_n$. We define the class $\mathscr{U}(R, S)$ to be the set of $m \times n$ (0, 1)-matrices with i^{th} row sum r_i and j^{th} column sum s_j for $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$.

Gale and Ryser independently found simple necessary and sufficient conditions for $\mathscr{U}(R, S)$ to be nonempty [**9**, **14**]. From R, we form an $m \times n$ (0, 1)-matrix $\bar{A}$ as follows. The i^{th} row sum of $\bar{A}$ is r_i and the 1's are as far to the left as possible. Let $\bar{s}_j$ be the j^{th} column sum of $\bar{A}$. We define the sequence $(\bar{s}_i)$ to be the *conjugate* of the sequence (r_i). Define S to be *monotone* if $s_1 \geqq s_2 \geqq \ldots \geqq s_n$. The same definition applies to R.

THEOREM 1.1 (Gale, Ryser). *There exists a matrix $A \in \mathscr{U}(R, S)$ if and only if*

$$(1.1) \qquad \sum_{i=1}^{t} \bar{s}_i \geqq \sum_{i=1}^{t} s_i \quad (1 \leqq t \leqq n),$$

where the sequence $(\bar{s}_i)$ is the conjugate of the sequence (r_i) and S is monotone.

The class $\mathscr{U}(R, S)$ has been studied extensively and most of the basic results can be found in [**17**]. Brualdi has recently written an excellent survey article [**3**]. Some of the work in this paper can be found in [**1**].

We can now describe the matrices that will be studied in this paper. Let $A = (a_{ij})$ and $B = (b_{ij})$. Then we say $A \geqq B$ or A *covers* B if and only if $a_{ij} \geqq b_{ij}$ for all pairs i, j. Let P be an $m \times n$ (0, 1)-matrix with column sums at most 1. Define

$$(1.2) \qquad \mathscr{U}_P(R, S) = \{A \in \mathscr{U}(R, S) | \quad A \geqq P\}.$$

Thus $\mathscr{U}_P(R, S)$ consists of the matrices in $\mathscr{U}(R, S)$ which cover P. We aim to generalize some results in $\mathscr{U}(R, S)$ to $\mathscr{U}_P(R, S)$. Note that for $P = 0$, the zero matrix, the two coincide. In Section 2, we prove a

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generalization of Theorem 1.1 for $\mathscr{U}_P(R, S)$. It also generalizes a result of Fulkerson on matrices with zero trace [**7**].

In Section 3, we state a theorem in $\mathscr{U}_P(R, S)$ that involves interchanges and triangle interchanges. These consist of small changes in a matrix which preserve row and column sums. There are a number of interesting corollaries to our result. We obtain that given $A, B \in \mathscr{U}_P(R, S)$, one can get from A to B by a series of interchanges and triangle interchanges. This reduces to Ryser's Interchange Theorem when $P = 0$ [**17**]. Let $R' = (r_1', r_2', \ldots, r_m')$ with $r_i' \leqq r_i$ and $S' = (s_1', s_2', \ldots, s_n')$ with $s_i' = s_i - k$ for some given k. We show there exists an $A \in \mathscr{U}_P(R, S)$ and a $B \in \mathscr{U}_P(R', S')$ with $A \geqq B$ if and only if $\mathscr{U}_P(R, S)$ and $\mathscr{U}_P(R', S')$ are nonempty. Setting $P = 0$ and letting $s_i' = s_i - k$ or $s_i - k - 1$, we obtain the same result which generalizes a result of [**4**].

Digraphs with specified indegree and outdegree sequences correspond to $\mathscr{U}_I(R, S)$ where I is the identity matrix. We may obtain an interchange theorem in this setting which can be specialized to undirected graphs and tournaments and prove results due to Fulkerson *et al* [**6**] and Ryser [**1**]. Our other result translates into a theorem of Kundu [**12**].

In Section 4 we prove the result stated in Section 3. Section 5 determines minimal and maximal possible columns in $\mathscr{U}_P(R, S)$. This provides a generalization of Ford and Fulkerson's (0, 1)-matrix rule [**5**].

2. Existence theorem. Consider a class $\mathscr{U}(R, S)$. Define a matrix P to be *acceptable* if it is an $m \times n$ (0, 1)-matrix with column sums at most 1. In addition, we require that no row (column) sum of P is greater than the corresponding row (column) sum in $\mathscr{U}(R, S)$. We will derive necessary and sufficient conditions under which $\mathscr{U}_P(R, S)$ is nonempty. Our results have appeared in [**1**].

From P and R, we define an $m \times n$ (0, 1)-matrix A^* as follows. Let A^* have i^{th} row sum r_i $(1 \leqq i \leqq m)$ and let A^* have 1's wherever P has 1's with the remaining 1's as far to the left as possible. Define s_j^* to be the j^{th} column sum A^*. Let the sequence (s_i^*) be the *P-required conjugate* of the sequence (r_i).

THEOREM 2.1. *Let P be an acceptable matrix. There exists a matrix $A \in \mathscr{U}(R, S)$ with $A \geqq P$ (i.e., $A \in \mathscr{U}_P(R, S)$) if and only if*

$$(2.1) \qquad \sum_{i=1}^{t} s_i^* \geqq \sum_{i=1}^{t} s_i \quad (1 \leqq t \leqq n),$$

where the sequence (s_i^*) is the P-required conjugate of the sequence (r_i) and S is monotone.

Proof. Assume there is a matrix $A \in \mathscr{U}_P(R, S)$. Then (2.1) holds since A can be obtained from A^* by shifting 1's to the right.

Assume (2.1) holds and that P is acceptable. Following the spirit of Ryser's proof of the Gale-Ryser Theorem [**17**], we give an algorithm for constructing an $A \in \mathscr{U}_P(R, S)$. We define a 1 in a $m \times n$ (0, 1)-matrix to be *free* if it is not in the same position as a 1 of P.

Algorithm.

Step 1. $B \leftarrow A^*$; $j \leftarrow n$.

Step 2. If B has at least s_j 1's in column j then go to Step 5.

Step 3. Among those rows of B which do not have a 1 in column j and among columns $1, 2, \ldots, j - 1$ we select a free 1 in B such that the following holds. Let C be the matrix obtained from B by shifting the selected 1 to the right into column j. Let C have j^{th} column sum c_j. We require

$$(2.2) \quad \sum_{i=1}^{t} c_i \geqq \sum_{i=1}^{t} s_i \quad (1 \leqq t \leqq n).$$

Step 4. $B \leftarrow C$; go to Step 2.

Step 5. $j \leftarrow j - 1$; if $j > 1$ then go to Step 2. Otherwise halt and output B.

The process of shifting 1's is clearly finite, thus the algorithm will terminate as long as Step 3 can always be performed. We will verify this shortly. We claim that the output is a matrix $A \in \mathscr{U}_P(R, S)$. We note that at every stage, B has row sums given by R since A^* does and 1's are never shifted between rows. The column sums of A are given by S unless, at some Step 2, column j has more than s_j 1's. Consider B in such a case where b_i is the i^{th} column sum of B. We have $b_n = s_n$, $b_{n-1} = s_{n-1}, \ldots, b_{j+1} = s_{j+1}$ and $b_j > s_j$. Counting all the 1's we have

$$b_1 + b_2 + \ldots + b_n = s_1 + s_2 + \ldots + s_n$$

and thus

$$(2.3) \quad \sum_{i=1}^{j-1} b_i < \sum_{i=1}^{j-1} s_i.$$

This contradicts (2.2) (or (2.1) in the case $B = A^*$). Thus $A \in \mathscr{U}(R, S)$. We note that $A^* \geqq P$ and only free 1's are shifted in the algorithm, thus $B \geqq P$. We deduce that $A \geqq P$, establishing our claim.

We must now verify that Step 3 can always be performed. Let B be a matrix at some stage in the algorithm and let B have i^{th} column sum b_i. Then by (2.2) (or (2.1) if $B = A^*$), we obtain

$$(2.4) \quad \sum_{i=1}^{t} b_i \geqq \sum_{i=1}^{t} s_i \quad (1 \leqq t \leqq n).$$

We claim that among the free 1's under consideration in Step 3, the one in column p, where p is as large as possible, will work. At the end of this

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proof we consider the case that no such free 1 exists. Consider C to be the matrix obtained by shifting that 1 and let C have i^{th} column sum c_i. Since column i of C is the same as column i of B for $1 \leqq i \leqq p$ and $j \leqq i \leqq n$, we obtain

$$(2.5) \qquad \sum_{i=1}^{t} c_i = \sum_{i=1}^{t} b_i \geqq \sum_{i=1}^{t} s_i \quad (1 \leqq t < p; j \leqq t \leqq n).$$

We have $b_j < s_j$ by Step 2. We claim $b_q \leqq s_q$ for $p < q < j$. Now S is monotone, so $s_j \leqq s_q$ for $p < q < j$. There is at most one 1 of P in column q since P is acceptable. Free 1's in column q can occur only in the b_j rows where column j of B has 1's. Thus $b_q \leqq b_j + 1$ for $p < q < j$. Combining, we obtain

$$b_q \leqq b_j + 1 \leqq s_j \leqq s_q$$

as claimed. Considering the effect of shifting the 1, we obtain

$$(2.6) \qquad \sum_{i=1}^{t} c_i \geqq \sum_{i=1}^{t} s_i \quad (p \leqq t < j).$$

Combining this with (2.5) yields (2.2) as desired.

We can now dispose of the remaining case that there are no free 1's in the given rows and columns. We have $b_n = s_n, b_{n-1} = s_{n-1}, \ldots, b_{j+1} = s_{j+1}$, and $b_j < s_j$ since we are at Step 3. By our above arguments

$$b_{j-1} \leqq s_{j-1}, b_{j-2} \leqq s_{j-2}, \ldots, b_1 \leqq s_1.$$

This contradicts that

$$b_1 + b_2 + \ldots + b_n = s_1 + s_2 + \ldots + s_n$$

and so we may conclude that there is some free 1 which can be selected in Step 3.

COROLLARY 2.2. Gale-Ryser Theorem (Theorem 1.1).

Proof. Simply take $P = 0$ and note that $\bar{s}_i = s_i^*$.

A quick examination of the algorithm reveals that any matrix $A \in \mathscr{U}_P(R, S)$ can be generated in this way. We may restate Theorem 2.1 in terms of finding a matrix $A \in \mathscr{U}(R, S)$ with $A + P \leqq J$ where J is the matrix of 1's. We say that A *avoids* P. Define P to be *avoidable* if it is an $m \times n$ (0, 1)-matrix with column sums at most 1 and if the i^{th} row sum (j^{th} column sum) does not exceed $n - r_i$ $(m - s_j)$ for $1 \leqq i \leqq m$ $(1 \leqq j \leqq n)$. From P and R, we define an $m \times n$ (0, 1)-matrix A^{**} as follows. Let A^{**} have i^{th} row sum r_i $(1 \leqq i \leqq m)$ and let A^{**} have 0's wherever P has 1's with the 1's of A^{**} as far to the left as possible. Define s_j^{**} to be the j^{th} column sum of A^{**}. Let the sequence (s_i^{**}) be the *P-restricted conjugate* of the sequence (r_i).

COROLLARY 2.3. *Let P be an avoidable matrix. There exists a matrix $A \in \mathcal{U}(R, S)$ with $A + P \leqq J$ if and only if*

$$(2.8) \quad \sum_{i=1}^{t} s_i^{**} \geqq \sum_{i=1}^{t} s_i \quad (1 \leqq t \leqq n),$$

where the sequence (s_i^{**}) is the P-restricted conjugate of the sequence (r_i) and S is monotone.

Proof. Let P have i^{th} row sum r_i'' and j^{th} column sum s_j''. Define $R' = (r_1', r_2', \dots, r_m')$ and $S' = (s_1', s_2', \dots, s_n')$ as follows

$$(2.9) \quad \begin{aligned} r_i' &= r_i + r_i'' \quad (1 \leqq i \leqq m), \\ s_j' &= s_j + s_j'' \quad (1 \leqq j \leqq n). \end{aligned}$$

There exists an $A \in \mathcal{U}(R, S)$ with $A + P \leqq J$ if and only if there exists a $B \in \mathcal{U}_P(R', S')$ with $B \geqq P$. If such an A exists, take $B = A + P$. If such a B exists, take $A = B - P$. By Theorem 2.1, such a B exists if and only if

$$(2.10) \quad \sum_{i=1}^{t} s_i^{*} \geqq \sum_{i=1}^{t} s_i' \quad (1 \leqq t \leqq n),$$

where the sequence (s_i^*) is the P-required conjugate of the sequence (r_i'). We note that $s_i^* = s_i^{**} + s_i''$ and $s_i' = s_i + s_i''$ for $1 \leqq i \leqq n$. Thus (2.10) holds if and only if (2.8) holds and the result is proven.

We can use Corollary 2.3 to prove a result of Fulkerson on matrices with zero trace. Simply set $P = I$, the identity matrix.

COROLLARY 2.4 (Fulkerson [**7**]). *Consider $\mathcal{U}(R, S)$ with $m = n$. There exists a matrix $A \in \mathcal{U}(R, S)$ with* $\operatorname{tr}(A) = 0$ *if and only if*

$$(2.11) \quad \sum_{i=1}^{t} s_i^{**} \geqq \sum_{i=1}^{t} s_i \quad (1 \leqq t \leqq n),$$

where the sequence (s_i^{**}) is the I-restricted conjugate of the sequence (r_i) and S is monotone.

There are a number of applications of Theorem 2.1 and Corollary 2.3. One could determine the minimum and maximum value of the trace and the maximum term rank for matrices in $\mathcal{U}(R, S)$ [**1**]. The formulas obtained are different but approximately as easy as those of Ryser [**15**, **16**]. The case of P being a permutation matrix is the most interesting. Ryser proved the following.

THEOREM 2.5 (Ryser [**17**]). *Consider $\mathcal{U}(R, S)$ with $m = n$. If there exists a matrix $A \in \mathcal{U}(R, S)$ which covers a permutation matrix, then there* *exists a matrix in* $\mathscr{U}(R, S)$ *which covers*

$$(2.12) \quad M = \begin{bmatrix} 0 & & & & 1 \\ & & & \cdot & \\ & & \cdot & & \\ & \cdot & & & \\ & 1 & & & 0 \\ 1 & & & & \end{bmatrix}$$

A companion result can be proven using Theorem 2.1 [**1**].

COROLLARY 2.6. *Consider* $\mathscr{U}(R, S)$ *with* $m = n$. *If there is a matrix* $A \in \mathscr{U}(R, S)$ *with* $A \geqq I$, *then for any permutation matrix* P *of order* n, *there exists a matrix in* $\mathscr{U}(R, S)$ *which covers* P.

Thus M and I are, respectively, the easiest and hardest permutations to cover.

3. A theorem on interchanges and triangle interchanges. Consider a matrix $A \in \mathscr{U}(R, S)$ with a submatrix $B \in \mathscr{U}(R', S')$. Then replacing B by any $B' \in \mathscr{U}(R', S')$ results in a new matrix $A' \in \mathscr{U}(R, S)$. Row and column sums are unaffected. The simplest possibility is with $R' = S' = (1, 1)$ and the two matrices in $\mathscr{U}(R', S')$ are

$$(3.1) \quad \text{i)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{ii)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Replacing a submatrix i) by ii) or vice versa is called an *interchange.*

We will also use a more complicated version in $\mathscr{U}_P(R, S)$. Consider the following two matrices.

$$(3.2) \quad \text{iii)} \begin{bmatrix} 0 & 1 & c \\ 1 & b & 0 \\ a & 0 & 1 \end{bmatrix}, \quad \text{iv)} \begin{bmatrix} 1 & 0 & c \\ 0 & b & 1 \\ a & 1 & 0 \end{bmatrix}.$$

Replacing a submatrix iii) by iv) or vice versa leaves the row and column sums unchanged. We restrict ourselves to the case $a = b = c = 1$ and that these are 1's of P. Thus iii) and iv) are both triangles as defined by Ryser [**19**]. A *triangle interchange* is such a replacement or any version of (3.2) obtained by applying the same row permutation to both iii) and iv).

Our main result in this section unifies some previous results as well as providing new ones. The theorem is messy to state but it has simple corollaries. We will use some proof techniques introduced by Brualdi and Ross [**4**]. Define $m(A, B)$ to be the number of -1's in $A - B$. We note that if $m(A, B) = 0$ then $A \geqq B$, which is the object of some of our corollaries.

THEOREM 3.1. *Let* $\mathscr{U}_P(R, S)$ *be nonempty. Let* $R' = (r_1', r_2', \ldots, r_m')$ *with* $r_i' \leqq r_i$ *and let* $S' = (s_1', s_2', \ldots, s_n')$ *with* $s_i' = s_i - k$ *or* $s_i - k - 1$ *for some given* k. *Assume* $\mathscr{U}_P(R', S')$ *is nonempty. Then there is an* $A \in \mathscr{U}_P(R, S)$ *and a* $B \in \mathscr{U}_P(R', S')$ *with* $m(A, B)$ *no larger than the number of columns* i *with* $s_i' = s_i - k$ *and no larger than the number of columns* i *with* $s_i' = s_i - k - 1$ *and containing a* 1 *of* P. *For an arbitrary pair* $A \in \mathscr{U}_P(R, S)$, $B \in \mathscr{U}_P(R', S')$ *with* $m(A, B)$ *larger than this bound, there is an interchange or a triangle interchange resulting in a pair* $A' \in \mathscr{U}_P(R, S)$, $B' \in \mathscr{U}_P(R', S')$ *with* $m(A', B') < m(A, B)$.

We postpone the proof to Section 4 and proceed to consider the corollaries. Our first result appears in [**1**].

COROLLARY 3.2. *Given a pair* $A, B \in \mathscr{U}_P(R, S)$, *one can get from* A *to* B *by a series of interchanges and triangle interchanges without leaving* $\mathscr{U}_P(R, S)$.

Proof. Let $R' = R$ and $S' = S$ with $k = 0$ in Theorem 3.1. Thus there are no columns with $s_i' = s_i - 1$. Then any pair $A, B \in \mathscr{U}_P(R, S)$ can be changed by a series of interchanges or triangle interchanges, without leaving $\mathscr{U}_P(R, S)$, to a pair A', B' with $m(A', B') = 0$ i.e., $A' = B'$. This proves the result.

We note that the interchanges and triangle interchanges can be chosen so that $m(A, B)$ is reduced at each step. Since $m(A, B) = 1$ is impossible when $A, B \in \mathscr{U}_P(R, S)$, we deduce that $m(A, B) - 1$ is an upper bound on the number of operations required to obtain B from A. As the next corollary shows, only interchanges are required in $\mathscr{U}(R, S)$. Exact lower bounds on the number of interchanges required in this case were given by Walkup [**20**].

COROLLARY 3.3 ([**17**]). Interchange theorem. *Given a pair of matrices* $A, B \in \mathscr{U}(R, S)$, *one can get from* A *to* B *by a series of interchanges.*

Proof. Set $P = 0$ in Corollary 3.2. We note that triangle interchanges require 1's of P and there are none.

COROLLARY 3.4. *Let* $R' = (r_1', r_2', \ldots, r_m')$ *with* $r_i' \leqq r_i$ *and* $S' = (s_1', s_2', \ldots, s_n')$ *with* $s_i' = s_i - k$. *Then there exists an* $A \in \mathscr{U}_P(R, S)$ *and a* $B \in \mathscr{U}_P(R', S')$ *with* $A \geqq B$ *if and only if* $\mathscr{U}_P(R, S)$ *and* $\mathscr{U}_P(R', S')$ *are nonempty.*

Proof. One direction is obvious. For the other direction, we note that the number of columns i with $s_i' = s_i - k - 1$ is zero. Thus by Theorem 3.1, there is a pair of matrices $A \in \mathscr{U}_P(R, S)$, $B \in \mathscr{U}_P(R', S')$ with $m(A, B) = 0$, i.e., $A \geqq B$.

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Let $\mathbf{k} = (k, k, \ldots, k)$ be the vector with n k's. Then a version of Birkhoff's Theorem, stated in (0, 1)-matrix terms, says that any matrix in $\mathcal{U}(\mathbf{k}, \mathbf{k})$ is the sum of k disjoint permutation matrices [**17**]. If we take $P = 0$ and $S' = S - \mathbf{k}$, then Corollary 3.4 can answer the question: does there exist a matrix in $\mathcal{U}(R, S)$ which covers k disjoint permutation matrices? If we take P to be a permutation matrix and $S' = S - \mathbf{k}$, then Corollary 3.4 can answer the question: does there exist a matrix in $\mathcal{U}_P(R, S)$ which covers k disjoint permutation matrices all disjoint from P? Thus we answer in the whole class the question considered by Fulkerson for a specific (0, 1)-matrix [**8**].

We may obtain a generalization of a result of Brualdi and Ross [**4**], which appears in [**1**]. The following result is akin to several results in graph theory concerning the existence of subgraphs with vertex degrees k or $k + 1$ [**10, 11, 12**]. In fact the proof of Brualdi and Ross was derived from a paper of Lovasz on 1-factors [**13**].

COROLLARY 3.5. *Let $R' = (r_1', r_2', \ldots, r_m')$ with $r_i' \leqq r_i$ and let $S' = (s_1', s_2', \ldots, s_n')$ with $s_i' = s_i - k$ or $s_i - k - 1$. Then there exists an $A \in \mathcal{U}(R, S)$ and a $B \in \mathcal{U}(R', S')$ with $A \geqq B$ if and only if $\mathcal{U}(R, S)$ and $\mathcal{U}(R', S')$ are nonempty.*

Proof. Set $P = 0$. Then the number of columns i with 1's of P is zero. Thus by Theorem 3.1, there is a pair of matrices $A \in \mathcal{U}(R, S)$ $B \in \mathcal{U}(R', S')$ with $m(A, B) = 0$, i.e., $A \geqq B$.

We can apply some of our results to directed graphs (digraphs). For suitable definitions, see [**2**]. Associated with any (0, 1)-matrix $A = (a_{ij})$ of order n is a digraph D on n vertices where there is an edge from v_i to v_j ($v_i \rightarrow v_j$) if and only if $a_{ij} = 1$. We only consider digraphs with no multiple edges (edges from v_i to v_j and from v_j to v_i can occur) and no loops. The *indegree* (*outdegree*) of a vertex is the number of edges directed into (out from) a vertex. Thus the class of all digraphs with a given indegree sequence S and outdegree sequence R corresponds directly to $\mathcal{U}_I(R, S)$. For any $B \in \mathcal{U}_I(R, S)$ we take the digraph associated with $B - I$. Corollary 2.4, due to Fulkerson, gives necessary and sufficient conditions for the existence of a digraph with given indegree and outdegree sequences. Corollary 3.4, with $P = I$, yields the following result due to Kundu.

COROLLARY 3.6 (Kundu [**12**]). *There exists a digraph with indegree sequence S and outdegree sequence R containing a k-factor (subgraph including all vertices with all degrees k) if and only if there exists a digraph with indegree sequence S and outdegree sequence R and a digraph with indegree sequence $S - \mathbf{k}$ and outdegree sequence $R - \mathbf{k}$.*

We obtain the following interesting result using Corollary 3.2 with $P = I$.

COROLLARY 3.7. *Consider two digraphs with the same indegree sequences and the same outdegree sequences. Then one can be obtained from the other by a series of transformations of Type I and Type II, as given in Figure* 1, *in which certain edges are "switched".*

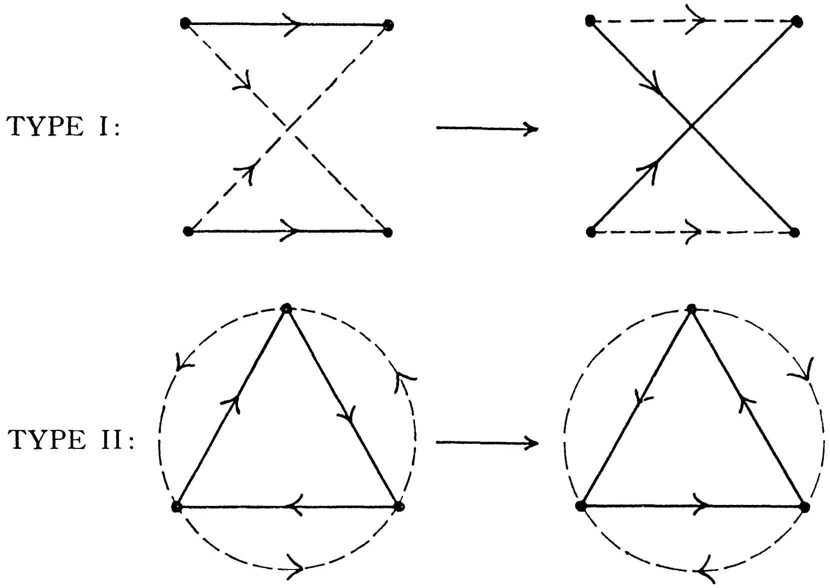

FIGURE 1

In Figure 1, *the dotted edges are edges not in the digraph.*

Proof. Apply Corollary 3.2 to $\mathscr{U}_I(R, S)$. The interchange translates into Type I transformations and the triangle interchange translates into Type II transformations.

We can apply Corollary 3.7 to undirected graphs and tournaments. An undirected graph (or simply graph) corresponds to a digraph where there is an edge from v_i to v_j whenever there is an edge from v_j to v_i and the two together are called the edge joining v_i and v_j The indegree and outdegree sequences are equal and are called the *degree sequence*. Thus a graph with a given degree sequence R corresponds to a matrix $A \in \mathscr{U}_I(R, R)$ with $A^T = A$, where A^T denotes the transpose of A. The matrix $A - I$ is called the adjacency matrix. Consider another graph $B \in \mathscr{U}_I(R, R)$ with $B^T = B$ and $m(A, B) > 0$. Transformations of Type II are not possible in A. Thus a transformation of Type I in A is possible, using Theorem 3.1, which reduces $m(A, B)$. Using the symmetry $A^T = A$, we deduce that there is another interchange which, combined with the first one, corresponds to the transformation in Figure 2 and reduces $m(A, B)$.

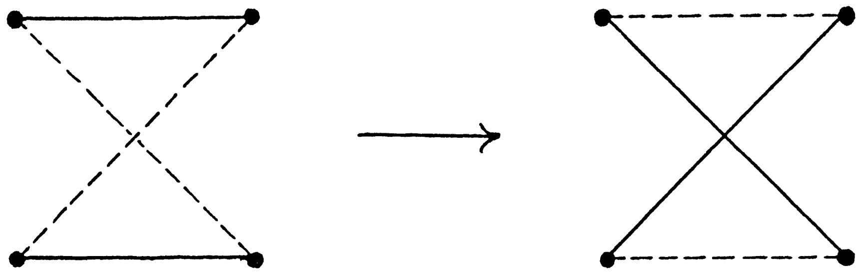

FIGURE 2

In Figure 2, dotted edges are not in the graph. The above argument proves a result of [**6**].

COROLLARY 3.8 (Fulkerson, Hoffman and McAndrew [**6**]). *Given two graphs with the same degree sequence, one can get from one to the other by "switching" edges as in Fig.* 2.

A tournament is a digraph in which any pair of vertices are joined by exactly one directed edge. We refer to the outdegree sequence as the *score sequence.* A tournament on n vertices with score sequence R is an $A \in \mathscr{U}_I(R, \mathbf{n} - R)$ with $A + A^T = J + I$. Consider another tournament B with the same score sequence. An interchange of Type I in Figure 1 in A which reduces $m(A, B)$ has a corresponding one, using $A + A^T = J - I$, which also reduces $m(A, B)$. Together they correspond to reversing the edges of a four cycle $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \rightarrow v_1$. If there is an edge $v_2 \rightarrow v_4$, then one could first reverse the edges of the three cycle $v_2 \rightarrow v_4 \rightarrow v_1 \rightarrow v_2$ and then reverse the three cycle $v_4 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4$, which corresponds to two transformations of Type II. The same works in the remaining case that there is an edge $v_4 \rightarrow v_2$. Corollary 3.7 was proven by Ryser for tournaments [**18**]. We have proven a slightly stronger result.

COROLLARY 3.9. *Given two tournaments with the same score sequence, one can get from one to the other by transformations of Type II in Figure* 1.

4. Proof of the theorem.

Proof of Theorem 3.1. We combine the techniques used to prove Corollary 3.2 and Corollary 3.5 in [**1**]. To conserve space, we will take the transposed version of Theorem 3.1 in the proof that follows. We now have

$$(4.1) \quad \begin{aligned} r_i' &= r_i - k \quad \text{or} \quad r_i - k - 1, \\ s_i' &\leqq s_i, \end{aligned}$$

and P has row sums at most 1.

Take $A \in \mathscr{U}_P(R, S)$ and $B \in \mathscr{U}_P(R', S')$ with $m(A, B) > 0$. Let $A = (a_{ij})$, $B = (b_{ij})$, and $P = (p_{ij})$. Assume A has no interchange or triangle interchange that would result in a matrix $A' \in \mathscr{U}_P(R, S)$ with $m(A', B) < m(A, B)$. Assume the same for B. We consider the implications of these assumptions in the following two cases.

Case I. Let A and B have corresponding 2×2 submatrices of the form

$$(4.2) \qquad \begin{bmatrix} 0 & 1 \\ 1 & a_{ij} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & b_{ij} \end{bmatrix}.$$

If $a_{ij} = 0$, then an interchange in A will result in a matrix A' with $m(A', B) < m(A, B)$ and $A' \in \mathscr{U}_P(R, S)$. If $b_{ij} = 1$ and $p_{ij} = 0$, then an interchange in B will result in a matrix $B' \in \mathscr{U}_P(R', S')$ with $m(A, B') < m(A, B)$. Thus under our assumptions, either $a_{ij} = 1$ and $b_{ij} = 0$ or $a_{ij} = b_{ij} = p_{ij} = 1$.

Case II. Let A and B have corresponding 2×2 submatrices of the form

$$(4.3) \qquad \begin{bmatrix} 1 & 0 \\ 0 & a_{ij} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & b_{ij} \end{bmatrix}.$$

By similar arguments, we conclude that either $a_{ij} = 0$ and $b_{ij} = 1$ or $a_{ij} = b_{ij} = p_{ij} = 1$.

We now apply these arguments and try to reduce the number of -1's in $A - B$. Let row p of $A - B$ have the most number of nonzero entries.

Case 1. Row p has t -1's and $t + k + 1$ 1's.

Since $A - B$ has positive column sums ($s_i' \leqq s_i$), there must be a 1 in some row q in the same column as a -1 of row p. Using Case I, we deduce that row q has 1's in the columns containing the 1's of row p with one exception where $a_{qi} = b_{qi} = p_{qi} = 1$. If they were all 1's, then row q would have $t + k + 2$ 1's and, by (4.1), at least $t + 1$ -1's which contradicts the choice of row p. We also note that P has at most one 1 per row. Thus row q has $t + k + 1$ 1's and so has t -1's. Using Case II, the -1's of row q lie in columns with a -1 in row p and in a column with a 1 of P in row p. We may write out rows p and q (after a column permutation and deleting columns of 0's) as follows. The symbol **0** refers to a 1 of P which is a 0 in $A - B$.

$$(4.4) \qquad \begin{array}{lccccccccc} \text{row } p: & \mathbf{0} & -1 & -1 & \cdots & -1 & 1 & 1 & \cdots & 1 \\ \text{row } q: & -1 & 1 & -1 & \cdots & -1 & \mathbf{0} & 1 & \cdots & 1. \end{array}$$

We know that $A - B$ has nonnegative column sums. Since any row with a 1 in the same column as a -1 of row p must take the form of row q, we deduce that $t = 1$.

There must be some row r with a 1 in the same column as the -1 in row q. The remaining entries of row r occur as shown forced by the same arguments that determined row q (we ignore columns of 0's).

$$(4.5)\qquad \begin{array}{lrrrrcr} \text{row } p: & \mathbf{0} & -1 & 1 & 1 & \cdots & 1 \\ \text{row } q: & -1 & 1 & \mathbf{0} & 1 & \cdots & 1 \\ \text{row } r: & 1 & \mathbf{0} & -1 & 1 & \cdots & 1. \end{array}$$

We note that a triangle interchange in A or B in rows p, q, r and the first three columns in (4.5) will reduce $m(A, B)$ by 3. This is a contradiction and so Case 1 does not occur.

Cases 2, 3, and 4 consider the possibility that row p has t -1's and $t + k$ 1's. We deduce that row q, chosen as before, has either $t + k$ or $t + k + 1$ 1's. Now $t + k + 1$ 1's forces at least t -1's which is a contradiction. Thus row q has $t + k$ 1's. We consider the three variations of this possibility.

Case 2. Row p has t -1's and $t + k$ 1's. Row q, chosen as before, has $t - 1$ -1's and $t + k$ 1's. There is a 0 in row q in the same column as a -1 in row p.

There is freedom in choosing a column for the 0 for there is at most one such 0 otherwise there couldn't be $t - 1$ -1's. Using arguments as in Case 1, we may write the entries of rows p and q as follows (ignoring columns of 0's)

$$(4.6)\qquad \begin{array}{lrrrrcrrrcr} \text{row } p: & \mathbf{0} & -1 & -1 & -1 & \cdots & -1 & 1 & 1 & \cdots & 1 \\ \text{row } q: & -1 & 1 & 0 & -1 & \cdots & -1 & \mathbf{0} & 1 & \cdots & 1. \end{array}$$

We need $t \geqq 2$ for this case to occur. In order for the column sums in $A - B$ to be nonnegative, we deduce that $t = 2$ considering the possibilities for any additional row with a 1 in the same column as a -1 of row p (Case 2, 3, or 4). Consider a row r with a 1 in the same column as the -1 in row q. We obtain the following entries for row r (ignoring columns of 0's)

$$(4.7)\qquad \begin{array}{lrrrrrcr} \text{row } p: & \mathbf{0} & -1 & -1 & 1 & 1 & \cdots & 1 \\ \text{row } q: & -1 & 1 & 0 & \mathbf{0} & 1 & \cdots & 1 \\ \text{row } r: & 1 & \mathbf{0} & 0 & -1 & 1 & \cdots & 1. \end{array}$$

Other than the first 1 in row r, the remaining entries are forced as follows. There are $k + 2$ 1's in row r and at most one -1, so the -1 is forced by (4.1). Comparing row q and row r in the first and second column in (4.7), we deduce that the position in row r with a **0** is either a **0** or a 1. If it were a 1 then comparing rows p and r, we would have $k + 3$ 1's in row r which is impossible. Thus the **0** of row r is forced. Comparing rows p and r, we deduce that the only place for the -1 in row r is as shown. The remaining entries are now forced. We note that the first, second, and fourth columns of rows p, q, and r yield a triangle interchange in A or B which reduces $m(A, B)$ by 3. This contradiction ensures that Case 2 does not occur.

Case 3. Row p has t -1's and $t+k$ 1's. Row q, chosen as before, has t -1's and $t+k$ 1's.

This is similar to Case 1. We immediately obtain that the entries of rows p and q can be written as follows (ignoring columns of 0's)

$$(4.8)\quad \begin{array}{lcccccccc} \text{row } p: & \mathbf{0} & -1 & -1 & \cdots & -1 & 1 & 1 & \cdots & 1 \\ \text{row } q: & -1 & 1 & -1 & \cdots & -1 & \mathbf{0} & 1 & \cdots & 1. \end{array}$$

With Case 2 eliminated, we deduce that $t = 1$ in order that $A - B$ have nonnegative column sums. Consider a row r with a 1 in the same column as the -1 of row q. If that row also has a -1 then we can get a contradiction using the arguments of Case 1 with $k+1$ replaced by k. In the remaining case that row r has no -1, we obtain

$$(4.9)\quad \begin{array}{lcccccc} \text{row } p: & \mathbf{0} & -1 & 1 & 1 & \cdots & 1 \\ \text{row } q: & -1 & 1 & \mathbf{0} & 1 & \cdots & 1 \\ \text{row } r: & 1 & \mathbf{0} & 0 & 1 & \cdots & 1. \end{array}$$

This follows by comparing rows q and r and noting that row r has k 1's and one 1 of P in the $k+1$ columns where row q has 1's. The 0 in row r is now forced. Comparing rows p and r in the second and third columns of (4.9), we see that the $\mathbf{0}$ is forced. Consider the first three columns of (4.9). The 0 indicates that either both A and B have 0's in that position or both A and B have 1's. In the former case, a triangle interchange in A will reduce $m(A, B)$ by 2. In the latter case, a triangle interchange in B will reduce $m(A, B)$ by 2. Both possibilities are a contradiction and we conclude that Case 3 does not occur.

We conclude the proof with Case 4 which is the only remaining possibility.

Case 4. Row p has t -1's and $t+k$ 1's. Row q, chosen as before, has $t-1$ -1's and $t+k$ 1's. There are no 0's in the same column as a -1 of row p.

Using similar arguments, we may write the entries of rows p and q as follows (ignoring columns of 0's)

$$(4.10)\quad \begin{array}{lcccccccc} \text{row } p: & -1 & -1 & \cdots & -1 & 1 & 1 & \cdots & 1 \\ \text{row } q: & 1 & -1 & \cdots & -1 & \mathbf{0} & 1 & \cdots & 1. \end{array}$$

In order that the column sums be nonnegative, we deduce that $t = 1$. Thus (4.10) may be written

$$(4.11)\quad \begin{array}{lccccc} \text{row } p: & -1 & 1 & 1 & \cdots & 1 \\ \text{row } q: & 1 & \mathbf{0} & 1 & \cdots & 1. \end{array}$$

Thus each -1 in $A - B$ occurs once per row in a row with row sum k

and that row can be paired off with a row with row sum $k + 1$ and having a 1 of P. This yields the bound in Theorem 3.1, after taking transposes, and completes the proof.

5. Results on possible columns. We define a column α of m entries to be a *possible* k^{th} column in $\mathscr{U}_P(R, S)$ if there is a matrix in $\mathscr{U}_P(R, S)$ with α as its k^{th} column. Certainly α must have 1's where column k of P has 1's and in rows i where $r_i = n$. Let P' be the matrix obtained from P by deleting the k^{th} column. Define $S = (s_1', s_2', \ldots, s_{n-1}')$ by $s_i' = s_i$ for $i < k$ and $s_i' = s_i + 1$ for $i \geqq k$. Let $R' = (r_1', r_2', \ldots, r_m')$ with $R' = R - \alpha^T$. Then α is a possible k^{th} column if and only if $\mathscr{U}_{P'}(R', S')$ is nonempty and α has 1's wherever the k^{th} column of P has 1's.

It is useful to define t_i to be the column of the free 1 in row i of A^* which is furthest to the right and zero if there is no free 1. We permute the rows so that $t_1 \geqq t_2 \geqq \ldots \geqq t_m$ in the results that follow and we call R *t-monotone* with this ordering. We wish to form a partial order on possible columns. We define a 1 to be *moveable* if it is not a 1 in the k^{th} column of P and it is not in a row i for which $r_i = n$. Then for two columns α, β, we define $\alpha \leqq \beta$ if the i^{th} moveable 1 of α (from the top) is in row j and the i^{th} moveable 1 of β is in row k with $j \leqq k$ $(t_j \geqq t_k)$. The following result on minimal columns in this ordering occurs in [**1**].

THEOREM 5.1. *Let α be a column of m entries and s_k* 1's *with* 1's *wherever column k of P has* 1's *and* 1's *in rows i for which $r_i = n$. The remaining* 1's *are as high as possible. Then α is the minimal possible k^{th} column for $\mathscr{U}_P(R, S)$, with R being t-monotone, if $\mathscr{U}_P(R, S)$ is nonempty.*

Proof. Let $B \in \mathscr{U}_P(R, S)$ and let β be its k^{th} column. Let $R'' = (r_1'', r_2'', \ldots, r_m'')$ with $R'' = R - \beta^T$. Let the sequence $(\bar{s}_i'')$ be the P'-required conjugate of the sequence (r_i''). Using Theorem 2.1, we have

$$(5.1) \qquad \sum_{i=1}^{t} \bar{s}_i'' \geqq \sum_{i=1}^{t} s_i' \quad (1 \leqq t \leqq n - 1).$$

Using the definition of α, we consider the matrix A^* associated with each sequence $(\bar{s}_i')$ and $(\bar{s}_i'')$ and deduce

$$(5.2) \qquad \sum_{i=1}^{t} \bar{s}_i' \geqq \sum_{i=1}^{t} \bar{s}_i'' \quad (1 \leqq t \leqq n - 1).$$

Combining (5.1) and (5.2), we obtain that $\mathscr{U}_{P'}(R', S')$ is nonempty, using Theorem 2.1. Hence α is a possible k^{th} column. It is certainly minimal.

This result provides us with the analogue of the (0, 1)-matrix rule of Ford and Fulkerson [**5**]. We can generate a matrix in $\mathscr{U}_P(R, S)$, if one exists, as follows. For column k we choose α as described and then repeat on the smaller class $\mathscr{U}_{P'}(R', S')$. Since α is so easy to determine, it would be a useful algorithmic tool. It could be applied, as was the (0, 1)-matrix rule in [**1**], to determining whether there is a matrix in $\mathscr{U}_P(R, S)$ with a submatrix in $\mathscr{U}_{P'}(R', S')$ in certain rows and columns.

We can also find the maximal column in this ordering. Let

$$(5.3) \qquad d_j = \min \left\{ \sum_{i=1}^{t} s_i^* - \sum_{i=1}^{t} s_i' \middle| \quad j \leqq t \leqq n \right\},$$

where the sequence (s_i^*) is the P-required conjugate of the sequence (r_i). Define a column ω of m entries and s_k 1's as follows. It has 1's wherever column k of P has 1's and in rows i for which $r_i = n$. The remaining 1's are placed as far down in the column as possible subject to the following condition: There are no more than $d_j - \delta$ 1's in rows i for which $t_i \leqq j$ (apart from the 1's of P in ω) where $\delta = 1$ if $j \geqq k$ and column k of P has a 1 and $\delta = 0$ otherwise.

THEOREM 5.2. *Let ω be the column of m entries and s_k 1's as defined above. Then ω, if it exists, is the maximal possible k^{th} column in $\mathscr{U}_P(R, S)$ where R is t-monotone. If ω does not exist then $\mathscr{U}_P(R, S)$ is empty.*

Proof. Let β be a possible k^{th} column and let β have c_j 1's in rows i for which $t_i \leqq j$ (apart from 1's of P in β). Let $R'' = (r_1'', r_2'', \ldots, r_m'')$ with $R'' = R - \beta^T$. Let the sequence $(\bar{s}_i'')$ be the P-required conjugate of the sequence (r_i'') and have it arise from $\bar{A}$ (A^* in the notation used before Theorem 2.1). Let the sequence (s_i^*) be the P-required conjugate of the sequence (r_i) and have it arise from A^*. We note that $\bar{A}$ can be obtained from A^* by deleting 1's. A 1 of P in column k must be deleted, reducing s_k^* by 1. A 1 of β (not a 1 of P) in row i causes a 1 in row i and column t_i to be deleted, reducing s_t^* by 1 where $t = t_i$. Let c be the number of 1's in P in column k. Thus $c = 0$ or 1. We obtain

$$(5.4) \qquad \sum_{i=1}^{t} \bar{s}_i'' = \begin{cases} \sum_{i=1}^{t} s_i^* - c_t & 1 \leqq t < k \\ \sum_{i=1}^{t} s_i^* - c_t - c & k \leqq t \leqq n. \end{cases}$$

Thus

$$(5.5) \qquad \sum_{i=1}^{t} \bar{s}_i'' \geqq \sum_{i=1}^{t} s_i' \quad (1 \leqq t \leqq n),$$

only if $c_j \leqq d_j - \delta$, where δ is as defined in the theorem, using (5.3). This is true for any possible k^{th} column β and so we deduce that ω is the maximal possible k^{th} column.

If ω did not exist, then no such β could exist because ω could be obtained from any possible k^{th} column by shifting 1's down. Thus $\mathscr{U}_P(R, S)$ is empty. This completes the proof.

Let β be a column of m entries and s_k 1's with 1's where the k^{th} column of P has 1's and in rows i for which $r_i = n$. By Theorem 5.2, if $\beta \leqq \omega$, then β is a possible k^{th} column. Thus all possible columns can be determined. Our arguments do not apply to rows (unless P also has row sums at most 1) since we have used Theorem 2.1 heavily.

We note that we could use these results to generate all possible matrices in $\mathcal{U}_P(R, S)$ by generating all possible k^{th} columns and then repeating this on what is left, $\mathcal{U}_{P'}(R', S')$. Ignoring any row and column permutations which would give isomorphisms, this process would generate each matrix in $\mathcal{U}_P(R, S)$ exactly once.

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