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ORTHOGONAL DESIGNS WITH ZERO DIAGONAL

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1. Introduction.

Definition 1. An *orthogonal design* of order n and type $(s_1, \ldots, s_l)$ (s_i positive integers) on the commuting variables $x_1, \ldots, x_l$ is an $n \times n$ matrix X with entries chosen from $\{0, \pm x_1, \ldots, \pm x_l\}$ such that

$$XX^t = \left(\sum_{i=1}^{l} s_i x_i^2\right) I_n.$$

Alternatively, each row of X contains s_i entries of the form $\pm x_i$ and the rows are formally orthogonal. In [**2**] it was established that $X^tX = XX^t$, and further, that $l \leqq \rho(n)$ (Randon's function) where for $n = 2^a b$, b odd, $a = 4c + d$, $0 \leqq d \leqq 3$, $\rho(n) = 8c + 2^d$. More on orthogonal designs is contained in [**2**; **3**; **4**; **5**; **9**; **10**].

Definition 2. A *weighing matrix* of weight k and order n is an $n \times n$ matrix A with entries chosen from $\{0, 1, -1\}$ such that $AA^t = kI_n$. (A weighing matrix is usually denoted as $W(n, k)$.)

The relationship between weighing matrices and orthogonal designs is illustrated by

THEOREM 1. [**2**]. *There is an orthogonal design X of order n and type $(s_1, s_2, \ldots, s_l)$ if and only if there are matrices $A_1, \ldots, A_l$ such that*

(i) *A_i is a $W(n, s_i)$, $1 \leqq i \leqq l$,*

(ii) *$A_iA_j^t + A_jA_i^t = 0$, $1 \leqq i \neq j \leqq l$, and*

(iii) *$A_i * A_j = 0$* (Hadamard product) *for $i \neq j$.*

For various n and k, weighing matrices have been extensively studied. In particular, a $W(n, n)$ is a Hadamard matrix and there is considerable interest in such matrices. (A recent comprehensive survey is given in [**8**].)

The principal thrust of this article is motivated by work that has been done on weighing matrices of order n and weight $n - 1$. If $n \equiv 0 \pmod 4$ and A is a $W(n, n-1)$ for which $A = -A^t$ then $H = I + A$ is a Hadamard matrix (of skew-type). If $n \equiv 2 \pmod 4$ and A is a $W(n, n-1)$ having zero diagonal and satisfying $A = A^t$, then A is called a symmetric conference matrix. In [**8**] both of these types of matrices are extensively used to construct other Hadamard matrices. These matrices are also discussed in [**1**] and the following elegant result is proved there. Here, it is restated in a form appropriate for further use with orthogonal designs.

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THEOREM 2. [**1**] (Delsarte-Goethals-Seidel) *Let A be a $W(n, n-1)$ with the rows reordered so that the diagonal consists of zeroes.*

(i) *If $n \equiv 2 \pmod 4$ then multiplication by -1 of rows (or columns) of A as necessary yields a matrix $\bar{A}$ with $\bar{A} = \bar{A}^t$.*

(ii) *If $n \equiv 0 \pmod 4$ then multiplication by -1 of rows (or columns) of A as necessary yields a matrix $\bar{A}$ with $\bar{A} = -\bar{A}^t$.*

This result may be generalized to orthogonal designs as follows.

THEOREM 3. *Let X be an orthogonal design of order n and type $(s_1, \ldots, s_l)$ with $\sum_{i=1}^{l} s_i = n - 1$.*

(i) *If $n \equiv 2 \pmod 4$ there is an orthogonal design $\bar{X}$ of order n and type $(s_1, \ldots, s_l)$ where $\bar{X}$ has 0-diagonal and $\bar{X} = \bar{X}^t$.*

(ii) *If $n \equiv 0 \pmod 4$ there is an orthogonal design $\bar{X}$ of order n and type $(s_1, \ldots, s_l)$ where $\bar{X}$ has 0-diagonal and $\bar{X} = -\bar{X}^t$.*

Proof. If necessary, reorder the rows (or columns) so that the orthogonal design X has 0-diagonal. In this form if x_1 (say) occurs in position (i, j), $i \leqq j$, then position (j, i) contains $\pm x_1$. For, suppose not and assume, without loss of generality, that position (j, i) contains $\pm x_2$. Consider the various incidences between the ith and jth rows. Count all occurrences of $\begin{pmatrix} \pm x_1 \\ \pm x_1 \end{pmatrix}$ and assume there are t_1 of these; similarly assume there are a total of t_2 occurrences of $\begin{pmatrix} \pm x_1 \\ \pm x_2 \end{pmatrix}$ and $\begin{pmatrix} \pm x_2 \\ \pm x_1 \end{pmatrix}$, and a total of t_3 occurrences of $\begin{pmatrix} \pm x_1 \\ \pm x_k \end{pmatrix}$ and $\begin{pmatrix} \pm x_k \\ \pm x_1 \end{pmatrix}$, $k \neq 1, 2$. Since rows i and j are orthogonal it follows that each of t_1, t_2 and t_3 must be even. Observe that these incidences account for all but one of the x_1's in rows i and j, namely that occurring as $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$. Thus, $2t_1 + t_2 + t_3 = 2s_1 - 1$ and this is a contradiction.

Now suppose $n \equiv 2 \pmod 4$ and multiply rows and columns of the orthogonal design by -1, as necessary, so that each variable in the first row and column appears with coefficient $= +1$. Call the resulting matrix $\bar{X}$ and replace every variable in it by $+1$ to obtain the $W(n, n-1)$

$$\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & * & \\ \vdots & * & \ddots & \\ 1 & & & 0 \end{bmatrix}.$$

By Theorem 2(i), multiplication of appropriate rows and columns of this matrix by -1 will make it symmetric. However, as the first row and column are already symmetric it follows that the entire matrix is symmetric. Hence, $\bar{X} = \bar{X}^t$ as was to be shown.

For $n \equiv 0 \pmod 4$, multiply the rows and columns of X so that each variable in the first row appears with coefficient $= +1$, and each variable in the first column appears with coefficient $= -1$. Call the resulting matrix $\bar{X}$ and set each variable $= +1$. The argument above, with Theorem 2(ii), implies $\bar{X} = -\bar{X}^t$.

2. Applications.

THEOREM 4. *Let $n \equiv 0 \pmod 4$. There is an orthogonal design of order n and type $(s_1, \ldots, s_l)$ with $\sum_{i=1}^{l} s_i = n - 1$ if and only if there is an orthogonal design of order n and type $(1, s_1, s_2, \ldots, s_l)$ with $1 + \sum_{i=1}^{l} s_i = n$.*

Proof. The sufficiency is evident. To establish the necessity, one observes that in view of Theorem 3(ii), if there is an orthogonal design of the type described then there is one, $\bar{X}$, where $\bar{X} = -\bar{X}^t$ on the variables $x_1, x_2, \ldots, x_l$. It is then easily verified that $Y = yI + \bar{X}$ is an orthogonal design of type $(1, s_1, s_2, \ldots, s_l)$.

In order to give the next application of Theorem 3 recall the following theorem.

THEOREM 5. [**2**; **5**]. *For $n = 4t$, t odd, a necessary condition that there exist an orthogonal design of type*

(i) *(a, b) in order n is that b/a be a sum of fewer than four rational squares.*
(ii) *(a, a, b) in order n is that b/a be a sum of one or two rational squares.*
(iii) *(a, a, a, b) in order n is that b/a be a rational square.*

It was conjectured in [**5**] that these conditions were also sufficient for existence. This conjecture is defeated by

THEOREM 6. *There exists no orthogonal design of type*

(i) $(1, 1, 1, 16)$ *in order* 20;
(ii) $(1, 1, 17)$ *in order* 20;
(iii) $(7, 12)$ *in order* 20.

Proof. By Theorem 4 these designs exist if and only if the designs $(1, 1, 1, 1, 16)$, $(1, 1, 1, 17)$ and $(1, 7, 12)$ exist in order 20. The first is impossible since $\rho(20) = 4$. The second is impossible by Theorem 5 (iii) and the fact that 17 is not a rational square. The third is impossible since it would imply the existence of an orthogonal design of type $(1, 7)$ in order 20 which is not possible by

Theorem 5 (i) and the fact that 7 is not the sum of fewer than four rational squares.

Observe that these designs were not prohibited by Theorem 5 since $16 = 4^2$, $17 = 1^2 + 4^2$ and $12/7 = (8/7)^2 + (4/7)^2 + (2/7)^2$.

Remarks. (1) For reasons analogous to those given in Theorem 6, there exist no orthogonal designs in order 20 of the following types: (1, 5, 5, 8), (1, 1, 8, 9), (1, 6, 12), (2, 7, 10), (5, 6, 8).

(2) In [**9**] it was separately conjectured that there exists an orthogonal design of type $(1, k)$ in order $n = 4t$, t odd, for every $k < n$ such that k is a sum of three rational squares.

This conjecture is also defeated. There is no orthogonal design of type (1, 42) in order 44 since such a design, in view of Theorem 3(ii), would imply the existence of a design of type (1, 1, 42) in order 44—a contradiction to Theorem 5(ii).

For analogous reasons there exists no orthogonal design of type (1, 66) in order 68, nor of type (1, 114) in order 116.

The appendix lists those designs in order 20 whose existence is still in doubt. For the construction of the known designs in order 20 the reader is referred to [**5**] and [**10**].

Further consequences of Theorem 4 are included in the next two corollaries. The first answers questions (c), (d) and (g) of [**2**].

COROLLARY 1. *If* $n \neq 1, 2, 4, 8$ *then there is a* $\rho(n)$*-tuple* $(s_1, \ldots, s_{\rho(n)})$ *with* $s_i > 0$ *and* $\sum_{i=1}^{\rho(n)} s_i \leqq n$ *which is not the type of an orthogonal design of order* n.

Proof. First observe that if $n = 1, 2, 4, 8$ then every $\rho(n)$-tuple (as described in the corollary) is the type of an orthogonal design of order n. This is ultimately a consequence of the classical 1, 2, 4 and 8 squares identity.

Now if n is odd, $n > 1$ then $\rho(n) = 1$ and by [**2**] there is no $W(n, 2)$.

For $n = 2t$, t odd and $t > 1$, $\rho(n) = 2$. A necessary condition for the existence of an orthogonal design of type (a, b) in order n is that b/a be a rational square [**2**]. As $1 + 2 < n$ and 2 is not a rational square, there is no orthogonal design of type (1, 2) in order n.

For $n = 4t$, t odd, $t > 1$, $\rho(n) = 4$. Now $1 + 1 + 1 + 2 < n$ and by Theorem 5 (iii) there is no orthogonal design of type (1, 1, 1, 2) in order n.

For $n = 8t$, t odd, $t > 1$, $\rho(n) = 8$. By a theorem of D. Shapiro [**7**], there is no orthogonal design of type (1, 1, 1, 1, 1, 1, 1, 2) in order n.

Finally, let $n = 2^a t$, t odd, $t \geqq 1$ and $a \geqq 4$. In this case note that $\rho(n) < n$ and consider the $\rho(n)$-tuple $(1, 1, \ldots, 1, n - \rho(n))$. There is no orthogonal design of this type, for otherwise, by Theorem 4, there would exist an orthogonal design in order n on $\rho(n) + 1$ variables. This is a contradiction, for an orthogonal design in order n can never involve more than $\rho(n)$ variables [**2**].

COROLLARY 2. *If* $n = 16$, *then every* 4-*tuple* (s_1, s_2, s_3, s_4) *where* $0 < s_1 \leqq s_2 \leqq s_3 \leqq s_4$, $\sum_{i=1}^{4} s_i \leqq 16$ *is the type of an orthogonal design in order* 16.

Proof. In [**2**] it was shown that all three-variable designs in order 16 exist, and the only four variable design in doubt was (1, 5, 5, 5). A design of this type is now obtained from a design of type (5, 5, 5) with the aid of Theorem 4.

Remark. The fact that all four-variable designs exist in order 16 leads to many of the five variable designs in order 16 that were missing in [**2**]. The current status of orthogonal designs in order 16 is included in the appendix.

Furthermore, the methods outlined here defeat some conjectures made by J. S. Wallis in [**9**] about the existence of orthogonal designs in order $8t$, t odd.

It has been proved by D. Shapiro that if $n = 8t$, t odd, a necessary condition for the existence of an orthogonal design in order n of type

(i) (a, a, a, a, a, b) is that b/a be a sum of fewer than four rational squares.
(ii) (a, a, a, a, a, a, b) is that b/a be a sum of one or two rational squares.
(iii) (a, a, a, a, a, a, a, b) is that b/a be a rational square.

The conjectures referred to amount to the statement that these necessary conditions are also sufficient.

Theorem 4 and Shapiro's Theorem may be invoked to establish that there are no orthogonal designs of type (1, 1, 1, 1, 1, 66), (1, 1, 1, 1, 1, 1, 65) or (1, 1, 1, 1, 1, 1, 1, 64) in order 72; these provide appropriate counter-examples to the conjectures.

As a final application, the main theorem may be invoked, with a rather tedious argument, to defeat yet another conjecture concerning orthogonal designs. Recall

THEOREM 7 [**2**]. *For* $n = 2t$, t *odd, necessary conditions that there be an orthogonal design in order* n *of type* (a, b) *include*
(1) $a + b < n$,
(2) a, b *are each a sum of almost two squares, and*
(3) ab *is a square.*

It has been conjectured [**9**] that these three conditions are also sufficient for the existence of orthogonal designs on two variables in the stated orders. While this conjecture was verified in [**2**] for orders $n = 2, 6, 10$, and in [**2**] and [**6**] for $n = 14$, it is not valid for $n = 18$.

THEOREM 8. *There is no orthogonal design of type* (1, 16) *in order* 18.

Proof. In the form given, the proof cannot be directly extended to establish the non-existence of other designs of type $(1, n - 2)$ in order $n = 2t$, t odd, $n - 2$ a square. However, in the hope that some interested reader may take up the challenge, the proof begins in this general setting. (The authors suspect that such designs may exist if and only if $(t - 1)/2$ is odd.)

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1. Let X be an orthogonal design of type $(1, n - 2)$ in order $n = 2t$, t odd, and write $X = x_1 I + x_2 A$. (This assumption of existence eventually leads to a contradiction when $n = 18$.) Orthogonality of rows implies that A is skew-symmetric. Hence, without loss of generality, assume that the 2×2 diagonal blocks of A contain only zeros. Further, assume that the remaining entries of its first row are $+1$: by orthogonality, the corresponding entries of the second row are $(t - 1)$ each of ± 1.

2. Let $H = \underset{t}{\oplus} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; then $Y = HX$ has zero diagonal. Hence the proof of Theorem 3 implies that multiplication of appropriate rows of Y by -1 will yield a symmetric matrix. Thus there exists a diagonal matrix P with diagonal entries ± 1 such that PY is symmetric.

The symmetry of the entries x_1 in Y implies that each of the t 2×2 diagonal blocks of P is either I_2 or $-I_2$. If the first row of Y is invariant under P, skew-symmetry of A and symmetry of x_2 in PY imply that P has $(t + 1)/2$ blocks I_2 and $(t - 1)/2$ blocks $-I_2$. Further, without loss of generality, it may be assumed that $X = x_1 I + x_2 A$ where

$$A = \left[\begin{array}{c|c:c}
\begin{matrix}0&0\\0&0\end{matrix} & \overbrace{\begin{matrix}+&+\\+&+\end{matrix}\ \Big|\ \cdots\ \Big|\ \begin{matrix}+&+\\+&+\end{matrix}}^{\frac{t-1}{2}\text{ blocks}} & \overbrace{\begin{matrix}+&+\\-&-\end{matrix}\ \Big|\ \cdots\ \Big|\ \begin{matrix}+&+\\-&-\end{matrix}}^{\frac{t-1}{2}\text{ blocks}} \\ \hline
 & \begin{matrix}0&0\\0&0\end{matrix}\ \ddots & \\ \hdashline
 & & \ddots\ \begin{matrix}0&0\\0&0\end{matrix}
\end{array}\right]$$

and $Q = PH = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \underset{(t-1)/2}{\oplus} \left\{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\right\} \underset{(t-1)/2}{\oplus} \left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}.$

Here $A = -A^t$, and for $B = QA$, $B = B^t$.

3. Now $QBQ = QQAQ = AQ = -A^tQ = -(QA)^t = B^t = -B$. Partition B and Q into 2×2 blocks, denoted B_{ij} and Q_{ij} respectively $1 \leqq i, j \leqq t$, and

observe that $Q_{ii}B_{ij}Q_{jj} = -B_{ij}$. It is easy to verify that if $Q_{ii} = Q_{jj}$, B_{ij} has the form $v = \begin{bmatrix} a & b \\ -b & -a \end{bmatrix}$, and if $Q_{ii} = -Q_{jj}$, B_{ij} has the form $u = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$. Hence, for B written as

$$B = \left[\begin{array}{cc|cc|c|cc|cc|c|cc}
0 & 0 & + & + & \cdots & + & + & + & + & \cdots & + & + \\
0 & 0 & + & + & & + & + & - & - & & - & - \\
\hline
+ & + & 0 & 0 & & & & & & & & \\
+ & + & 0 & 0 & & & & & & & & \\
\cdot & & & & \ddots & & & & & & & \\
\cdot & & & & W_1 & & & & W_2 & & & \\
\cdot & & & & & & & & & & & \\
+ & + & & & & & & & & & & \\
+ & + & & & & & & & & & & \\
\hdashline
+ & - & & & & & & & & & & \\
+ & - & & & & & & & & & & \\
\cdot & & & & & & & & & & & \\
\cdot & & & & W_2^T & & & & W_3 & & & \\
\cdot & & & & & & & & & & & \\
+ & - & & & & & & & & & & \\
+ & - & & & & & & & & & &
\end{array}\right],$$

sections W_1 and W_3 have only v-type blocks, and W_2 has only u-type blocks.

4. The remainder of the argument assumes $n = 18$, and much of the detail is omitted.

The u-type blocks are denoted

$$A^+ = \begin{bmatrix} + & + \\ + & + \end{bmatrix},\ B^+ = \begin{bmatrix} - & - \\ - & - \end{bmatrix},\ C^- = \begin{bmatrix} + & - \\ - & + \end{bmatrix},\ D^- = \begin{bmatrix} - & + \\ + & - \end{bmatrix}$$

(where the sign on the name denotes the sign of the inner product of its rows) and those of v-type are denoted

$$A^- = \begin{bmatrix} + & + \\ - & - \end{bmatrix},\ B^- = \begin{bmatrix} - & - \\ + & + \end{bmatrix},\ C^+ = \begin{bmatrix} + & - \\ + & - \end{bmatrix},\ D^+ = \begin{bmatrix} - & + \\ - & + \end{bmatrix}.$$

Orthogonality of each of the first two rows of B with the rows in any 2×18 block row of B, after the first, implies that

$$\begin{aligned}
&\text{number of blocks } A^- \text{ in } W_1\ (W_3) \\
= {}&\text{number of blocks } B^- \text{ in } W_1\ (W_3) \\
= {}&\text{number of blocks } A^+ \text{ in } W_2\ (W_2{}^t) \\
= {}&\text{number of blocks } B^+ \text{ in } W_2\ (W_2{}^t).
\end{aligned}$$

Hence symmetry may be used to show that W_1 has one of the forms:

$$\begin{matrix} 0 & A^- & B^- & D^+ \\ C^+ & 0 & A^- & B^- \\ D^+ & C^+ & 0 & C^+ \\ B^- & D^+ & A^- & 0 \end{matrix} \quad \text{or} \quad \begin{matrix} 0 & B^- & A^- & D^+ \\ D^+ & 0 & A^- & B^- \\ C^+ & C^+ & 0 & C^+ \\ B^- & D^+ & A^- & 0 \end{matrix},$$

or one of these with A, D interchanged with B, C respectively, or one of these with simultaneous row and column interchanges.

5. The remainder of the argument is the same for each case, and only the first version of W_1 is considered. Remark 4, and the necessity that W_2 and W_2^T have the same format imply that the corresponding W_2 has the form

$$\begin{matrix} A^+ & B^+ & Z & Z \\ B^+ & Z & Z & A^+ \\ Z & Z & Z & Z \\ Z & A^+ & Z & B^+ \end{matrix}$$

(where each Z is either C^- or D^-), or this form with column interchanges. Block rows 2 to 5 may be written

$$\begin{matrix} A^+ & 0 & A^- & B^- & D^+ & A^+ & B^+ & Z_1 & Z \\ A^+ & C^+ & 0 & A^- & B^- & B^+ & Z & Z_2 & A^+ \\ A^+ & D^+ & C^+ & 0 & C^+ & Z & Z & Z & Z \\ A^+ & B^- & D^+ & A^- & 0 & Z & A^+ & Z_3 & B^+ \end{matrix}$$

Orthogonality of blocks row 2 and 3 imply $Z_1 = Z_2$, of block rows 2 and 5 imply $Z_1 = Z_3$, of block rows 3 and 5 imply $Z_2 \neq Z_3$ — a contradiction.

3. Appendix.

Order 20. The existence of the following designs in order 20 is still in doubt. The reader should also consult the appendix in [**5**]. Many of these designs have recently been shown not to exist by methods quite different from those in this paper. See [**10**].

4-variables:

(1, 1, 2, 16)	(1, 2, 2, 9)	(1, 4, 5, 5)
(1, 1, 5, 8)	(1, 2, 6, 11)	(1, 5, 5, 9)
(1, 1, 5, 13)	(1, 2, 8, 9)	(2, 2, 5, 5)
(1, 1, 8, 10)	(1, 3, 6, 8)	(2, 3, 7, 8)
(1, 2, 2, 8)	(1, 4, 4, 9)	(3, 3, 6, 6)

3-variables:

(1, 2, 10)	(1, 5, 13)	(2, 8, 9)
(1, 2, 16)	(1, 6, 13)	(3, 3, 12)
(1, 3, 9)	(1, 8, 10)	(3, 4, 10)
(1, 3, 10)	(2, 3, 8)	(3, 4, 11)
(1, 3, 11)	(2, 3, 13)	(3, 6, 8)
(1, 3, 14)	(2, 5, 6)	(3, 7, 8)
(1, 3, 16)	(2, 5, 7)	(3, 7, 10)
(1, 4, 6)	(2, 6, 11)	(5, 5, 9)
(1, 4, 13)	(2, 7, 8)	(5, 6, 7)
(1, 5, 8)	(2, 7, 11)	

2-variables:

(3, 16) (6, 13) (7, 10)

Order 16. The reader should also consult [**2**]. All one, two, three and four variable designs exist in order 16. Recall also that $\rho(16) = 9$.

9-variables: There are 45 9-tuples each of which might be the type of orthogonal designs in order 16. In [**2**] it is shown that each of (1, 1, 1, 1, 1, 1, 1, 1, 1) and (1, 1, 2, 2, 2, 2, 2, 2, 2) is the type of an orthogonal design in order 16.

In view of Theorem 4 the following 11 9-tuples do not correspond to the type of an orthogonal design thus leaving 32 still in doubt.

(1, 1, 1, 1, 1, 1, 1, 1, 7)	(1, 1, 1, 1, 1, 1, 3, 3, 3)
(1, 1, 1, 1, 1, 1, 1, 2, 6)	(1, 1, 1, 1, 1, 2, 2, 2, 4)
(1, 1, 1, 1, 1, 1, 1, 3, 5)	(1, 1, 1, 1, 1, 2, 2, 3, 3)
(1, 1, 1, 1, 1, 1, 1, 4, 4)	(1, 1, 1, 1, 2, 2, 2, 2, 3)
(1, 1, 1, 1, 1, 1, 2, 2, 5)	(1, 1, 1, 2, 2, 2, 2, 2, 2)
(1, 1, 1, 1, 1, 1, 2, 3, 4)	

8-variables: There are 67 8-tuples each of which might be the type of an orthogonal design in order 16. Eleven of these are given in [**2**] and by the method of this paper so also are the following two, leaving 54 still in doubt.

(1, 1, 1, 1, 2, 3, 3, 4) (1, 1, 1, 1, 3, 3, 3, 3)

7-variables: There are 94 possible 7-tuples to consider. In [**2**] it is shown that at least 37 are the type of an orthogonal design. Using the new 8-variable designs above the following 10 new designs are obtained leaving 47 still in doubt.

(1, 1, 1, 1, 2, 3, 3)	(1, 1, 1, 1, 3, 4, 5)
(1, 1, 1, 1, 2, 3, 4)	(1, 1, 1, 2, 3, 3, 5)
(1, 1, 1, 1, 2, 3, 7)	(1, 1, 1, 2, 3, 4, 4)
(1, 1, 1, 1, 2, 4, 6)	(1, 1, 2, 2, 3, 3, 4)
(1, 1, 1, 1, 3, 3, 6)	(1, 1, 2, 3, 3, 3, 3)

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6-variables: Of the 125 6-tuples that could be the type of an orthogonal design all are, except possibly for the following 20 which are still undecided.

(1, 1, 1, 1, 1, 8)	(1, 1, 1, 2, 2, 8)
(1, 1, 1, 1, 1, 9)	(1, 1, 1, 2, 2, 9)
(1, 1, 1, 1, 1, 10)	(1, 1, 1, 2, 5, 5)
(1, 1, 1, 1, 1, 11)	(1, 1, 1, 4, 4, 4)
(1, 1, 1, 1, 2, 8)	(1, 1, 2, 2, 2, 7)
(1, 1, 1, 1, 2, 9)	(1, 1, 2, 2, 3, 6)
(1, 1, 1, 1, 3, 8)	(1, 1, 2, 2, 4, 5)
(1, 1, 1, 1, 4, 7)	(1, 1, 2, 2, 5, 5)
(1, 1, 1, 1, 5, 5)	(1, 2, 2, 2, 3, 5)
(1, 1, 1, 1, 5, 6)	(2, 2, 2, 3, 3, 3)

5-variables: Of the 149 5-tuples that could be the type of an orthogonal design all are, except possibly for the following 3 which are still undecided.

(1, 1, 1, 1, 11) (1, 1, 2, 2, 9) (1, 2, 2, 5, 5)

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