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CONJUGACY CLASS SIZES OF CERTAIN DIRECT PRODUCTS

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Abstract

We consider finite groups in which, for all primes p, the p-part of the length of any conjugacy class is trivial or fixed. We obtain a full description in the case in which for each prime divisor p of the order of the group there exists a noncentral conjugacy class of p-power size.

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1. Introduction

Let G be a finite group, and $g \in G$; we denote by x^G the conjugacy class of x, that is, $x^G = \{g^{-1}xg \mid g \in G\}$. Then $|x^G| = |G : C_G(x)|$ is the conjugacy class size (sometimes called the index) of x. We let

$$\mathrm{cl}(G) = \{|x^G| \mid x \in G\}.$$

It is of some interest to investigate the relationship between the structure of a finite group G and the arithmetical properties of $\mathrm{cl}(G)$. The earliest and best-known instance of such a relationship is Burnside's p^α-lemma: a group which admits a conjugacy class whose order is a nontrivial prime-power cannot be simple. More recently, there have appeared a number of papers addressing the question of the recognition of certain classes of groups (for example nilpotent or soluble) by the set of conjugacy class sizes, and particular effort has been devoted to direct products. Before mentioning some of the known results, it is convenient to set a specific notation. If π_1, π_2 are nonempty sets of positive integers, their product is defined to be the set

$$\pi_1 \times \pi_2 = \{n_1 n_2 \mid n_1 \in \pi_1, n_2 \in \pi_2\}.$$

Thus, given finite groups H and G, one has $\mathrm{cl}(H \times G) = \mathrm{cl}(H) \times \mathrm{cl}(G)$. In particular, the set of class sizes of a finite nilpotent group is the product of sets all of whose elements are powers of the same prime. The converse of this fact seems indeed rather far from being true: in [7] A. R. Camina and R. D. Camina produce examples

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of nonnilpotent groups with the same class sizes of a nilpotent group (the smallest example being a nonnilpotent group G with $\mathrm{cl}(G) = \{1, 2, 4\} \times \{1, 5\}$). However, in [4], Camina proves that if $\mathrm{cl}(G) = \{1, p^a\} \times \{1, q^b\}$, for primes p and q, then G is nilpotent: a result that has later been generalized by Beltrán and Felipe [2, 3], by showing that the same conclusion holds when n and m are coprime and $\mathrm{cl}(G) = \{1, n\} \times \{1, m\}$. Moreover, in [5], A. R. Camina and R. D. Camina prove that if $\mathrm{cl}(G) = \{1, p_1\} \times \{1, p_2\} \times \cdots \times \{1, p_k\}$, with distinct primes $p_1, p_2, \ldots, p_k$, then G is nilpotent (further results in this direction may be found in [6]). The aim of this note is to extend this kind of result. As a special case, we shall obtain the following theorem.

Theorem 1.1. *Let $p_1^{m_1}, p_2^{m_2}, \ldots, p_k^{m_k}$ be powers of distinct primes, and let G be a group with*

$$\mathrm{cl}(G) = \{1, p_1^{m_1}\} \times \{1, p_2^{m_2}\} \times \cdots \times \{1, p_k^{m_k}\}.$$

Then G is nilpotent.

Let us now explain the more general result that we in fact establish.

For $k \geq 1$, and given nontrivial powers $p_1^{m_1}, \ldots, p_k^{m_k}$ of distinct primes, we denote by $\mathcal{D}(p_1^{m_1}, \ldots, p_k^{m_k})$ the class of all finite groups G such that

$$\mathrm{cl}(G) \subseteq \{1, p_1^{m_1}\} \times \cdots \times \{1, p_k^{m_k}\},$$

and we say that a group G is a $\mathcal{D}$-group if G belongs to one of such classes. The object of this paper is a strict subclass of $\mathcal{D}$ (but containing, for instance, all nilpotent $\mathcal{D}$-groups). Namely, we denote by $\overline{\mathcal{D}}(p_1^{m_1}, p_2^{m_2}, \ldots, p_k^{m_k})$ the class of those groups G such that

$$\{p_1^{m_1}, \ldots, p_k^{m_k}\} \subseteq \mathrm{cl}(G) \subseteq \{1, p_1^{m_1}\} \times \cdots \times \{1, p_k^{m_k}\}.$$

We then say that a group G is a $\overline{\mathcal{D}}$-group if it belongs to one of these last classes. Our aim is to give a complete description of all $\overline{\mathcal{D}}$-groups. Now, the basic examples of $\overline{\mathcal{D}}$-groups are the following:

(I) p-groups with a unique nontrivial conjugacy class size;
(II) semidirect products $G = PQ$, where p, q are distinct primes, P is a normal abelian p-subgroup of G, Q is an abelian Sylow q-subgroup, $C_Q(P) = Z(G)$, and $Q/Z(G)$ is a cyclic group acting fixed point freely on P.

Ito [10] proved that if $\mathrm{cl}(G) = \{1, m\}$, then $m = p^\alpha$ for some prime p and $G = A \times P$, with A abelian and P a group of type (I). Groups of type (I) have been subject to a rather intensive study; in particular, Ishikawa [9] proved that they have nilpotency class at most 3. Relevant extensions of Ishikawa's work may be found in a series of papers of Mann; in particular, we shall make use of results in [12]. The proof that a group G such that $\mathrm{cl}(G) = \{1, p^\alpha, q^\beta\}$, with p, q distinct primes, is a direct product $G = A \times H$ with A abelian and H a group of type (II) essentially goes back to Baer [1].

We call groups of type (I) and type (II), described above, *basic* $\overline{\mathcal{D}}$-groups. Our main result (Theorem 3.10), is then the following theorem.

Theorem 1.2. *Any $\overline{\mathcal{D}}$-group is the direct product of basic $\overline{\mathcal{D}}$-groups of pairwise coprime order.*

We have not attempted to generalize the Beltrán–Felipe theorem, although it does not seem implausible that our approach may prove useful also in that direction. All groups considered in this paper are finite. For a group G, $Z(G)$ and $F(G)$ will denote, respectively, the centre and the Fitting subgroup of G.

2. Preliminaries

Let us start by recalling a standard and elementary fact, which we will use without any further reference.

Lemma 2.1. *Suppose that the prime p does not divide any conjugacy class size of the group G. Then G has a central Sylow p-subgroup (possibly trivial).*

We now introduce a weaker condition than $\overline{\mathcal{D}}$, which refers to a single prime. It is hoped that, in its generality, the main result of this section (Theorem 2.3), might be useful in further investigation. Given a group G and a prime p, we denote by $\mathrm{cl}(G)_p$ the set of all p-parts of the conjugacy class sizes of G, and by $e_p(G)$ the number of elements of $\mathrm{cl}(G)_p$ which are distinct from 1, that is,

$$e_p(G) = |\{|x^G|_p \mid x \in G, p \text{ divides } |x^G|\}| = |\mathrm{cl}(G)_p| - 1.$$

Thus, $e_p(G) = 0$ if and only if G has central Sylow p-subgroup. The following is a simple but very useful remark, whose proof is rather easy.

Lemma 2.2. *Let p be a prime, and let N be a normal subgroup of the group G.*

(i) *If $(p, |N|) = 1$, then $\mathrm{cl}(G/N)_p \subseteq \mathrm{cl}(G)_p$; in particular, $e_p(G/N) \leq e_p(G)$.*
(ii) *If $(p, |G/N|) = 1$, then $\mathrm{cl}(N)_p \subseteq \mathrm{cl}(G)_p$: in particular, $e_p(N) \leq e_p(G)$.*

Our interest is in groups G such that $e_p(G) = 1$. If this is the case, we denote by $p^{\alpha(p)}$ the only nontrivial power of p that may occur as the p-part of a conjugacy class size of G. Thus, $e_p(G) \leq 1$ if and only if $\mathrm{cl}(G) \subseteq \Delta \times \{1, p^{\alpha(p)}\}$, where Δ is a set of p'-numbers. Observe also that every group G in which the Sylow p-subgroups have prime order p satisfies $e_p(G) = 1$, and there are plenty of groups with this condition that are not p-soluble. However, for p-soluble groups, the condition $e_p(G) = 1$ has strong consequences.

Theorem 2.3. *Let G be a p-soluble group with $e_p(G) = 1$. Then G has p-length 1. If, furthermore, the Sylow p-subgroups of G are not abelian, G has a normal p-complement.*

Proof. Let G and p satisfy the hypotheses of the theorem. Denote by $p^a = p^{\alpha(p)}$ the only nonidentity power of p that divides some conjugacy class size of G, and let $p^b = |G|_p$. We first show that G has p-length one. By Lemma 2.2 we may well assume $O_{p'}(G) = 1$. Let $A = O_p(G)$; then $A = F(G)$ and, in particular, $C_G(A) = Z(A)$. We want

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to show that A is a Sylow p-subgroup of G. If $G = A$ there is nothing to prove. Thus, assume G has p'-elements.

Let x be a p'-element of G, and let P_0 be a Sylow p-subgroup of $C_G(x)$. First, observe that P_0 centralizes $C_G(x) \cap A$. In fact let $y \in P_0$; then y commutes with the p'-element x, and so $C_G(xy) = C_G(x) \cap C_G(y) \le C_G(x)$. Thus p does not divide $|C_G(x) : C_G(xy)|$, which means that y centralizes a Sylow p-subgroup of $C_G(x)$, in particular, y centralizes $C_G(x) \cap A$. Now we have the following result.

(i) $P_0 \le Z(A)$.

Suppose, in contradiction, that there exists $g \in P_0 \setminus Z(A)$. Let $X = C_G(x) \cap A = P_0 \cap A$; then, as $x \notin Z(A) = C_G(A)$, $X < A$; in particular, $Y = N_A(X) > X$. Clearly, X and Y are $C_G(x)$-invariant: let $S/X = C_{Y/X}(g)$. As g is a p-element, and Y/X a p-group, we have $S/X \neq 1$ (that is, $S > X$). Now, $[\langle g\rangle, \langle x\rangle, S] = 1$ (as g centralizes x), and $[S, \langle g\rangle, \langle x\rangle] \le [X, \langle x\rangle] = 1$. Consequently, by the three subgroups lemma, $[S, \langle x\rangle, \langle g\rangle] = 1$. Thus, $[S, \langle x\rangle] \le C_G(g) \cap A$. Now, we know that g centralizes a Sylow p-subgroup of $C_G(x)$; since g does not centralize A, we deduce that $C_G(g) \cap A = X$. Thus, $[S, \langle x\rangle] \le X$, and so $[S, \langle x\rangle, \langle x\rangle] = 1$. Since x is a p'-element, this forces $[S, \langle x\rangle] = 1$, and we have the contradiction $S \le C_G(x) \cap A = X$. Hence $P_0 \setminus Z(A) = \emptyset$, and (i) is proved.

Let P be a Sylow p-subgroup of G, containing P_0. Observe that, assuming such x exists, point (i) implies that $p^a = |x^G|_p = |P : P_0| \ge |P/Z(A)|$.

We also have the following result.

(ii) $A = Z(A)$.

Suppose, in contradiction, that there exists $u \in A \setminus Z(A)$. Then $Z(A) < C_A(u) < A$, and so $1 < |G : C_G(u)|_p \le |P : C_P(u)| \le |P : Z(A)|$. By the remark just made above, this is a contradiction. Hence, we have proved (ii). Again, notice that, as $A \not\le C_G(x)$, this implies the following result.

(iii) $p^a = |P : P_0| > |P/A|$.

Now, let P be a Sylow p-subgroup of G, and suppose, in contradiction, that $P > A$. Let $L/A = O_{p'}(G/A)$, and let H be a complement of A in L (that is, a Hall p'-subgroup of L). Now, it follows from (i) that if $g \in P \setminus A$, then $C_G(g)$ is a p-subgroup. Thus, P/A acts as a group of fixed point free automorphisms of L/A. In particular, since $C_G(G/A) \le L/A$, P/A is either cyclic or a quaternion group. In any case, $LP/L = O_p(G/L)$, and so (as, by Lemma 2.2, we may suppose $O^{p'}(G) = G$) $G = LP$. Then G/A is a Frobenius group with kernel L/A and complement P/A. In particular, we have $P = N_G(P)$, and so P has $h = |H|$ distinct conjugates in G, any two of which intersect in A.

Observe that, if $y \in A$, then $C_G(y) \ge A$, and so $|y^G|_p < |P : A|$. Hence, by (iii), y centralizes some Sylow p-subgroup of G, that is, some P^h with $h \in H$. We thus get

$$A = \bigcup_{h \in H} C_A(P^h). \tag{2.1}$$

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Also, if $g \in P \setminus A$, then g does not centralize A, and $C_G(g) \leq P$, so, as $g \in C_G(g)$,

$$|P/C_A(P)| \geq |P : C_A(g)| > |P : C_P(g)| = p^a. \tag{2.2}$$

Denoting by t the number of distinct orbits of the action by conjugation of H on A, from (2.1) and (2.2), we get

$$t \leq |C_A(P)| < |P|/p^a. \tag{2.3}$$

On the other hand, for every $1 \neq h \in H$, we have that $C_A(h)$ is a Sylow p-subgroup of $C_G(h)$, and so $|C_A(h)| = |P|/p^a$. Then Burnside's counting lemma yields

$$t|H| = \sum_{h \in H} |C_A(h)| = |A| + (|H| - 1)|P|/p^a,$$

therefore

$$(t - |P|/p^a)|H| = |A| - |P|/p^a > 0,$$

and hence, finally, $t > |P|/p^a$, which contradicts (2.3). This completes the proof that $P = A$ and, consequently, that G has p-length 1.

Now assume that a Sylow p-subgroup P of G is not abelian. As before, we may suppose $O_{p'}(G) = 1$, and thus aim at proving $P = G$. If, in contradiction, G contains a nontrivial p'-element x, the same argument as in the proof of point (i) above shows that, since $P = O_p(G)$,

$$p^a = |P : C_P(x)| \geq |P/Z(P)|.$$

On the other hand, there exists $y \in P \setminus Z(P)$, and hence

$$p^a = |y^G|_p = |P : C_P(y)| < |P : Z(P)|.$$

This contradiction proves the second claim of the theorem. □

In the case in which the Sylow p-subgroups are not abelian a little more can easily be established.

Lemma 2.4. *Let G be a p-soluble group with $e_p(G) = 1$ and let P be a Sylow p-subgroup of G. Assume that P is not abelian. Then $Z(P) \leq Z(G)$.*

Proof. Let G and p satisfy the hypotheses of the lemma. Then, by Theorem 2.3, G has a normal p-complement N. Thus, $G = NP$ and $C_P(N) = O_p(G)$. Let $a \in P \setminus Z(P)$, then $C_G(a) = C_N(a)C_P(a)$, and so $|P : C_P(a)| = |G : C_G(a)|_p = p^{\alpha(p)}$.

Let $b \in N$, let Y be a Sylow p-subgroup of $C_G(b)$, and let $x \in N$ such that $Y \leq P^x$. If $Y = P^x$ then, in particular, b centralizes $Z(P)^x$. Otherwise, $|P^x : Y| = p^{\alpha(p)}$. Take $y \in Y$; then $C_G(by) = C_G(b) \cap C_G(y)$. Thus, by centralizes some conjugate of Y in $C_G(b) = C_N(b)Y$, say Y^c with $c \in C_N(b)$. Hence, $[NY, y] = [NY^c, y] \leq N[Y^c, y] = N$. This holds for every $y \in Y$ and so $Y \simeq YN/N$ is abelian. Let $a \in Y \setminus Z(P^x)$. Then $C_{P^x}(a) \geq YZ(P^x)$ and, by the initial observation,

$$p^{\alpha(p)} = |P^x : C_{P^x}(a)| \leq |P^x : YZ(P^x)| \leq |P^x : Y| = p^{\alpha(p)}.$$

Hence $Y = C_{P^x}(a)$ and so $Z(P^x) \leq Y$. We have thus shown that every element of N centralizes a N-conjugate of $Z(P)$. It then follows that $Z(P)$ centralizes N, and, consequently $Z(P) \leq Z(G)$. □

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We end this section with two lemmas from the literature. The first is a straightforward application of results of Dolfi and Lucido, and of Camina; we remark that it does not assume p-solubility.

Lemma 2.5. *Let p be a prime, G a group, and let $p^a = \max(\mathrm{cl}(G)_p)$. Let $x \in G$ be such that p^a divides $|x^G|$. Assume that x is a q-element for some prime q. Then we have the following results.*

(i) (Dolfi and Lucido [8]) *If $q \neq p$, $C_G(x)$ is p-soluble with abelian Sylow p-subgroups; moreover $O^q(C_G(x))$ has a normal p-complement.*

(ii) (Camina [4]) *If $q = p$, a Sylow p-subgroup of $C_G(x)$ is a direct factor of $C_G(x)$.*

Proof. Let G and x satisfy the hypotheses of the lemma, and write $H = C_G(x)$. Let y be a any q'-element of H. Then, since $(|x|, |y|) = 1$ and $xy = yx$, we have $C_G(xy) = H \cap C_G(y) = C_H(y)$. Thus, by the assumption on x, p does not divide $|H : C_G(xy)| = |H : C_H(y)|$. Hence, we conclude that every q'-element of H centralizes some Sylow p-subgroup of H. Point (i) now follows from [8, Theorem 5], and point (ii) from [4, Theorem 1]. □

The second result we quote, due to Mann, dealing with 2-groups, is strictly related to the theory of p-groups with only two class sizes.

Lemma 2.6 (Mann [12, Theorem 7]). *Let G be a 2-group, and let $x \in G \setminus Z(G)$ be such that the conjugacy class of x has minimal size for noncentral classes of G. Then $\langle x\rangle^G$ is abelian.*

3. $\overline{\mathcal{D}}$-groups

Clearly, the class $\mathcal{D}$ defined in the introduction is just the class of all groups G such that $e_p(G) = 1$ for every prime $p \in \pi(G/Z(G))$. Observe that the following corollary is an immediate consequence of Theorem 2.3.

Corollary 3.1. *Let G be a soluble $\mathcal{D}$-group, and assume that G does not have any abelian Sylow subgroups. Then G is nilpotent.*

Another simple but useful fact is the following lemma, whose proof is immediate.

Lemma 3.2. *Let G be a $\mathcal{D}$-group, and let $x, y \in G$. If $C_G(y) \leq C_G(x)$, then $|G : C_G(x)|$ and $|C_G(x) : C_G(y)|$ are coprime.*

It is not difficult to show that $\overline{\mathcal{D}}$ is a proper (and indeed, much stricter) subclass of $\mathcal{D}$. For instance, $\mathrm{cl}(A_5) = \{1, 12, 15, 20\}$ and so A_5 belongs to $\mathcal{D}$ but not to $\overline{\mathcal{D}}$ (indeed, we shall show in due course that every $\overline{\mathcal{D}}$-group is soluble). We will not pursue here the study of the more general class $\mathcal{D}$, though it is not unlikely that much could be said about it too, at least in the soluble context. However, as we shall see from the proofs, the condition that G contains classes of prime power size is essential to make our arguments effective. An indication of how decisive this extra condition might be is Burnside's lemma, a recent development of which, due to A. R. Camina and R. D. Camina, will be important in our proofs.

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Lemma 3.3 [5, Proposition 1]. *Let u be an element of the group G such that $|u^G|$ is a power of the prime p. Then $\langle u^G\rangle' \leq O_p(G)$.*

We now establish that any $\overline{\mathcal{D}}$-group is soluble. In fact we prove slightly more.

Theorem 3.4. *Let G be a group with $e_2(G) = 1$ and $2^{\alpha(2)} \in \mathrm{cl}(G)$. Then G is soluble.*

Proof. Let G be a counterexample of minimal order, and write $2^\alpha = 2^{\alpha(2)}$.

Let $u \in G$ be such that $|u^G| = 2^\alpha$, and set $C = C_G(u)$. Let P be a Sylow 2-subgroup of G, and $Y = P \cap C$, so that $|P : Y| = |G : C| = 2^\alpha$ and $G = CP$. Since C is maximal among proper centralizers, we may suppose that u is a q-element for some prime q.

(1) $q \neq 2$.

In fact, if u is a 2-element, it follows by Lemma 2.5(ii) that Y is a direct factor of C. Then $u \in Y = C_P(u)$; thus $Z(P) \leq Y$, and consequently $Z(P) \leq Z(G)$. Let $g \in P$, and suppose $|g^P| < 2^\alpha$; then g centralizes some Sylow 2-subgroup of G, that is, $g \in Z(P)$. It follows that u is a noncentral element of P with minimal conjugacy class size; by Lemma 2.6, $\langle u\rangle^P = \langle u\rangle^G$ is abelian. Hence $\langle u\rangle^G \leq Y$, and $C = C_G(\langle u\rangle^G)$ is normal in G, which is thus soluble.

Now, as $q \neq 2$, point (i) of Lemma 2.5 applies, yielding the following result.

(2) Y is abelian and $O^q(C)$ has a normal 2-complement.

The next step is to show the following result.

(3) $Y \trianglelefteq P$ or $C_P(Y) = Y$.

Write $R = \bigcap_{x\in P} Y^x$. Then $Y \trianglelefteq P$ if and only if $Y = R$. Now, every element of G may be written as $g = ha$ with $h \in P$, $a \in C$, hence, as $R \trianglelefteq P$, $R^g = R^a$, and so $R^G = \langle R^g \mid g \in G\rangle = R^C \leq Y^C \leq C$. Observe that $R^C \cap P$ is a normal subgroup of P contained in $C \cap P = Y$; thus $R^C \cap P = R$. Let $y \in Y$ be such that $C_P(y) > Y$. Then $|y^G|_2 \leq |P : C_P(y)| < |P : Y| = 2^\alpha$, and so $C_G(y)$ contains a Sylow 2-subgroup of G, that is, $y \in Z(P^c)$ for some $c \in C$. Also, as $C_G(y) \geq Y$, we may take P^c to contain Y, and hence $Y^c \leq P^c \cap C = Y$. Then $c \in N_C(Y)$, and we obtain

$$y \in Y \cap Z(P^c) = Y \cap Z(P)^c = (Y \cap Z(P))^c \leq R^c \cap Y \leq R.$$

Since Y is abelian, this proves claim (3).

(4) $Y \trianglelefteq P$.

Suppose $Y \ntrianglelefteq P$, and, as before, let R be the largest normal subgroup of P contained in Y. Then, as $C_P(Y) = Y$, we have $Z(P) \leq R$. Now, $R^G = R^C \leq Y^C$, and so, by Lemma 2.5(i), R^G has a normal 2-complement M. Then $M \trianglelefteq G$, and $R^G = M(R^G \cap P) = M(R^G \cap Y) = MR$. Now, Y is abelian and not normal in P, so $Y < C_P(R) \trianglelefteq P$. Also, $[R^G, C_P(R)] \leq M$.

Suppose $C_P(R) < P$, and let $D = C_G(R^G/M)$. Then $C_P(R) \leq D$ and $[D \cap P, R] \leq M \cap P = 1$; therefore $D \cap P = C_P(R) < P$. In particular, 2 divides $|G/D|$. Now, by [8,

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Theorem 5], $[O^q(C), Y] \le O_{p'}(C)$. Hence, $[O^q(C), R^G] \le O_{p'}(C) \cap R^G \le M$, that is, $O^q(C) \le D$. Therefore, G/D is a q, p-group. In fact $G/D = (CD/D)(PD/D)$, where CD/D is a q-group.

Let $a \in R$; then $|a^P| \le |P/C_P(R)| < |P : Y| = p^\alpha$. Thus, $a \in Z(P^x)$ for some $x \in C$ and, *a fortiori*, $a \in Z(P)^G$. Therefore, $R \le Z(P)^G \cap P \le R$, and so $R^G = Z(P)^G$.

Now, consider the faithful action of G/D on the abelian 2-group $V = R^G/M$. As CD/D is a q-group, there is a q-element $y \in C$ such that $1 \ne yD \in Z(CD/D)$. Suppose that $C_V(yD) \ge Z(P)M/M$. Then, for every $x \in C$,

$$[yD, Z(P^x)] = [(yD)^x, Z(P)] = [yD, Z(P)] \le M,$$

and so yD centralizes $Z(P)^x M/M$; as this holds for every $x \in C$, y centralizes $Z(P)^G M/M = R^G/M$, which contradicts $yD \ne 1$. Thus $C_V(y)$ does not contain any conjugate of $Z(P)M/M$. In particular, as R^G/M is a normal 2-subgroup of G/M, $|y^G|_2 \ne 1$, and so $|y^G|_2 = 2^\alpha$.

Let S be a Sylow 2-subgroup of $C_G(y)$, which, by possibly replacing y with a conjugate, we may assume to be contained in P. By Lemma 2.5, S is abelian, and by what we proved above, $Z(P) \nleq S$. Then, if $b \in S$, $C_P(s) \ge SZ(P) > S$. Hence $|s^G|_2 \le |P : C_P(s)| < |P : S| = 2^\alpha$, and therefore s centralizes some Sylow 2-subgroup of G, that is, $s \in Z(P)^G \cap P = R$. Thus $S \le R$, and so $|P : R| \le |P : S| = 2^\alpha = |P : Y|$. This implies the contradiction $R = Y$.

We are left with the case $C_P(R) = P$. Then $R = Z(P)$, and $R^G = MZ(P)$. Let $g \in P$, with $|g^P| < 2^\alpha = |P : Y|$. We have the fact that g centralizes some Sylow 2-subgroup of G, and so there exists $x \in C$ such that

$$g \in Z(P^x) \cap P \le MZ(P) \cap P = Z(P).$$

Thus, 2^α is the minimal size among the noncentral conjugacy classes of P. Now, if $y \in Y \setminus Z(P)$, then, since Y is abelian, $Y \le C_P(y) < P$; that is, $1 < |y^P| \le |P : Y| = 2^\alpha$. Hence y is a noncentral element of P with minimal class size. By Lemma 2.6, we have the fact that $\langle y\rangle^P$ is abelian; thus $\langle y\rangle^P \le C_P(y) = Y$, and so $\langle y\rangle^P \le R = Z(P)$, which is a contradiction. This concludes the proof of claim (4).

As $Y \trianglelefteq P$, we have $Y^G = Y^C \le C = C_G(u)$, and so, in particular, $Y^G \le \bigcap_{x\in P} C^x = C_G(\langle u\rangle^G)$. Let $M = O^2(Y^G)$. We know that M is a normal $2'$-subgroup of G. We next prove the following result.

(5) $[Y^G, G] \le M$, and $Y \le Z(P)$.

Again let $D = C_G(Y^G/M)$; as before, we have $O^q(C) \le D$, so G/D is a $\{q, 2\}$-group and thus it is soluble. Let $g \in D$, and let Q be a Sylow 2-subgroup of $C_G(g)$. Then, as $D \trianglelefteq G$, $Q \cap D$ is a Sylow 2-subgroup of $C_D(g)$. Thus, if Q is a Sylow 2-subgroup of G, $Q \cap D$ is a Sylow 2-subgroup of D as well. Otherwise, $|Q| = |P|/2^\alpha = |Y| = |Y^G/M|$. Now, g centralizes the normal 2-section Y^G/M, and so $QM \ge Y^G$, yielding, by order reasons, $Q \le Y^G = MY$; thus, Q is a conjugate of Y and is already contained in D.

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This shows that $e_2(D) \leq 1$. If $e_2(D) = 0$, D is a central Sylow 2-subgroup, and so it is soluble; since G/D is also soluble, we obtain a contradiction with the choice of G. Thus, $e_2(D) = 1$; this, in particular, means that the Sylow 2-subgroups of D have order $2^b > |Y|$. Now, $u \in D$, and Y is a Sylow 2-subgroup of $C_D(u)$. Thus $|u^D| = 2^b/|Y|$, and D satisfies the hypotheses of the theorem. If $D < G$, D is soluble, and therefore G is also soluble. Thus, $D = G$. Consequently, $[P, Y] \leq P \cap [P, Y^G] \leq P \cap M = 1$, and point (5) follows.

(6) $Y \leq Z(G)$.

With the same notation as before, let $L = N_G(Y)$. By (5) and the Frattini argument, we have $G = ML$ and $L = C_G(Y)$. Observe that $u \in L$, and, as $M \leq C$, $C = M(C \cap L) = MC_L(u)$; hence

$$|u^L| = |L : C_L(u)| = |G : C_G(u)| = 2^\alpha.$$

Let $g \in L$, then $[Y, g] = 1$, so there exists a Sylow 2-subgroup Q of $C_G(g)$ with $Y \leq Q$. Then, however, by (5), $[Y, Q] \leq Q \cap M = 1$ and so $Q \leq L$. This shows that $\mathrm{cl}(L)_2 = \mathrm{cl}(G)_2$, and therefore that L satisfies the hypotheses of the theorem. If $L < G$, it is then soluble by the choice of G; consequently G/M is soluble and therefore, since M is a group of odd order, G is soluble, contrary to its choice. Thus $L = G$.

(7) Let g be a 2-element of $G \setminus Y$; then $C_G(g)$ contains a unique Sylow 2-subgroup of G as a direct factor.

In fact, let g be a 2-element of $G \setminus Y$ and let h be a $2'$-element in $C_G(g)$ (possibly $h = 1$). Then, by (6), $C_G(hg) = C_G(h) \cap C_G(g) \geq Y\langle g\rangle > Y$. Thus, $|(hg)^G|_2 < |P|/|Y| = 2^\alpha$, and therefore hg centralizes a Sylow 2-subgroup Q of G, with $Q \leq C_G(g)$. It follows that every $2'$-element of $C_G(g)$ centralizes a Sylow 2-subgroup of $C_G(g)$. By [4, Theorem 1], $C_G(g)$ has a unique Sylow 2-subgroup Q which is a direct factor; by what we observed before, Q is a Sylow 2-subgroup of G.

(8) $Y = O_2(G)$.

Indeed, for every $x \in G$, $Z(P^x)$ centralizes $O_2(G)$; but it follows from (7) that every 2-element of G is contained in the centre of some Sylow 2-subgroup, and so $P^G \leq C_G(O_2(G))$. If $O_2(G) > Y$ it then follows from (7) that $O_2(G)$ is contained in a unique Sylow 2-subgroup, and this means that $P \trianglelefteq G$, and consequently that G is soluble. Therefore, by choice of G, we have $O_2(G) = Y$.

(9) Conclusion.

Let $A = \langle u^G\rangle$. Then, since u is a q-element, A/A' is an abelian q-group. Now, by Lemma 3.3, $A' \leq O_2(G)$. Thus, by point (8), $A' \leq Y$, and since (by point (6)) Y is central in G, we deduce that $A' = 1$ and so that A is a normal abelian q-subgroup of G. Consider the action of P on A, observing that, since $Y = C_P(u)$, Y is the kernel of this action. Suppose that, for some $g \in P \setminus Y$, $C_A(g) \neq 1$, and let $1 \neq x \in C_A(g)$. Then, by point (7), g is centralized by a unique Sylow 2-subgroup of G, which clearly contains $Z(P)$, and which centralizes x as well. Thus, $C_A(g) \leq C_A(Z(P))$. From this it follows that P/Y acts as a group of fixed point free automorphisms on $[A, Z(P)]$. Since $Z(P) > Y$, $[A, Z(P)]$ is not trivial. Therefore, P/Y is either cyclic or a quaternion group. Now, that P cannot be a quaternion group follows from point (7), which implies, in particular, that every 2-element of G is central in some Sylow 2-subgroup. Hence P/Y is cyclic, and so G/Y is soluble, yielding the final contradiction. □

Corollary 3.5. *Every $\overline{\mathcal{D}}$-group is soluble.*

Examples. (1) Theorem 3.4 does not hold for primes $p \geq 5$, even with respect to p-solubility. In fact, for $p \geq 5$, let $H = S_p$ be the full symmetric group on p points, choose a prime $q \neq p$, and let M be the natural permutational module for $GF(q)[H]$. Then consider the semidirect product $G = MH$: one has $\mathrm{cl}(G)_p = \{1, p\}$, $p \in \mathrm{cl}(G)$, but G is not p-nilpotent. (I believe that $e_3(G) = 1$ and $3^{\alpha(3)} \in \mathrm{cl}(G)$ together imply 3-solubility, but do not have a complete proof of this.)

(2) For $p = 2$, none of the assumptions on 2-parts of conjugacy class sizes in Theorem 3.4 may be dropped. We have already observed that $e_2(A_5) = 1$ (although, I have not been able to decide whether, in the case of nonabelian Sylow 2-subgroups, the hypothesis that $2^{\alpha(2)} \in \mathrm{cl}(G)$ is necessary to get the conclusion of Theorem 3.4). On the other hand, let p be an odd prime and let $H = \mathrm{PSL}(2, 7)$. Then H has a 2-transitive action on a set of eight points (for example the set of points of the projective line $\mathbb{P}(1, 7)$); let M be the permutational $GF(p)[H]$-module with respect to this action and $G = MH$ the resulting semidirect product. While G is not soluble, one checks that $\mathrm{cl}(G)_2 = \{2, 4, 8\}$ and $8 \in \mathrm{cl}(G)$. Observe that 8 is the largest 2-part of the conjugacy class sizes, so this group also provides an example that [7, Theorem 1] does not hold without the assumption of the group being an A-group. Finally, let K be the direct product of three copies of S_3, and G as before; then $\mathrm{cl}(K \times G)_2 = \{2^i \mid 0 \leq i \leq 6\}$ and $K \times G$ admits a conjugacy class of size 2^i for each $0 \leq i \leq 6$.

We now turn to the soluble case, and start by looking at a single prime. We recall that a group G is an A-group if all Sylow subgroups of G are abelian. Let us remind ourselves of an elementary and well-known fact about A-groups (see for example [11, Section 7, Exercise 3]).

Lemma 3.6. *Let G be an A-group. Then $G' \cap Z(G) = 1$.*

The following statement could be proved without restricting ourselves to A-groups, but in this form it is simpler to prove, and is enough for our applications.

Lemma 3.7. *Let G be a soluble A-group in $\mathcal{D}$, and p a prime divisor of the order of G such that $p^{\alpha(p)} \in \mathrm{cl}(G)$. Let P be a Sylow p-subgroup of G. Then one of the following properties is true.*

(1) *$P \leq F(G)$.*
(2) *$O_p(G) \leq Z(G)$, and $P/O_p(G)$ is cyclic of order $p^{\alpha(p)}$.*

Proof. Let G be as in the statement; write $A = O_{p'}(G)$, and let P be a Sylow p-subgroup of G. By Theorem 2.3, $AP \trianglelefteq G$, whence $C_P(A) = F(G) \cap P = O_p(G)$, and $G = AN_G(P)$. By assumption, there exists an element $u \in G$, of p'-order, such that $|G : C_G(u)| = p^{\alpha(p)}$. Observe that $A \leq C_G(u)$, and so $u \in C_G(A)$.

Set $C = C_G(u)$, and let Y be a Sylow p-subgroup of C; by possibly replacing u with a conjugate we may well assume $Y \leq P$ (and consequently $|P : Y| = p^{\alpha(p)}$). Observe also that $AY = A(P \cap C) = AP \cap C$ and so, being P abelian, $AY \trianglelefteq APC = G$. Furthermore, $u \in C_G(AY) \trianglelefteq G$. Since AY is an A-group we have, by Lemma 3.6,

$$(AY) \cap C_G(AY) = (AY)' \cap Z(AY) = 1.$$

Let $a \in (AY)'$; then a and u belong to trivially intersecting normal subgroups, and hence

$$C_G(au) = C_G(a) \cap C_G(u) \leq C.$$

By Lemma 3.2, p does not divide $|C : C_G(au)|$, which means that a centralizes a Sylow p-subgroup of C. Now, since $AY \trianglelefteq G$, all Sylow p-subgroups of C are contained in AY and so are conjugates of Y by elements in $[A, Y]$. However, $[A, Y] \leq (AY)'$, and so we have that every element of $[A, Y]A$ centralizes some Y^x with $x \in [A, Y)$. This forces $[A, Y]$ to be centralized by Y; thus $[A, Y, Y] = 1$, and so, by coprime action, $[A, Y] = 1$. Hence $Y \leq C_P(A) = O_p(G)$.

If $O_p(G) = P$, then $P \leq F(G)$ and we are in case (1) of the statement.

Thus assume from now on that $C_P(A) = O_p(G) < P$. Then there exists an element $a \in A$ which is not centralized by any conjugate of P, and so

$$Y \leq O_p(G) \leq C_G(a) \cap P < P,$$

but also

$$p^{\alpha(p)} = |P : P \cap C_G(a)| \leq |P : Y| = p^{\alpha(p)}.$$

Hence $P \cap C_G(x) = Y$, showing that $Y = C_P(A) = O_p(G)$.

This, in particular, implies that u centralizes $F(G) \leq AY$, and consequently $u \in F(G) \cap A$, that is, $u \in Z(A)$. Thus P/Y is an abelian p-group acting faithfully on $Z(A)$. Suppose that there exists $b \in Z(A)$ such that $C_P(b) > Y$. Then b must centralize some Sylow p-subgroup of G, that is, some conjugate P^x with $x \in A$. Since $b \in Z(A)$, this implies that b centralizes P. Hence P/Y acts faithfully on $[Z(A), P]$ as a group of fixed point free automorphisms and therefore it is a cyclic p-group.

It remains to show that $Y \leq Z(G)$.

Observe that, C being maximal among the centralizers of noncentral elements of G, we may suppose that u is a q-element for some prime $q \neq p$. Let $b \in C$ be a q'-element. Then, as we have already argued, b centralizes a Sylow p-subgroup of C. However, $Y = O_p(G)$ is the only such Sylow subgroup, and so $b \in C_G(Y)$. Set $R = C_G(Y)$; it follows that

$$R \geq APO^q(C) \geq O^q(C).$$

Hence, G/R is a q-group.

We will show that $R = G$. Now, since G/R is an abelian q-group, it has a regular orbit on the p-group Y; that is, there exists $y \in Y$ such that $C_G(y) = R$. In particular, $|G/R| = q^{\alpha(q)}$. By Lemma 3.6, $R' \cap Y \leq R' \cap Z(R) = 1$, and so, by the same argument used in the first part of the proof, every element of R' centralizes a Sylow q-subgroup of R. In particular, as $u \in Z(A) \trianglelefteq R$ is a q-element, we conclude that $R' \leq C_G(u) = C$.

Let $N = N_G(P)$. Then $R = A(N \cap R) = AC_N(Y)$ (observe that $P \leq N \cap R$). We have

$$[P, N \cap R] \leq R' \cap P \leq C \cap P = Y$$

and, consequently, $[P, N \cap R, N \cap R] \leq [Y, R] = 1$. Therefore, as P is abelian, $N \cap R = C_N(P) = C_A(P)P$, yielding

$$R = A(N \cap R) = AP \quad \text{and} \quad |G/AP| = q^{\alpha(q)}.$$

Now let $g \in P$ such that $P = Y\langle g\rangle$ (remember that P/Y is cyclic), and let $ax \in C_G(b)$, with $a \in A$, $x \in N$; then $1 = [ax, b] = [a, b]^x[x, b]$, and so $[a, b]^x = [b, x] \in A \cap P = 1$: in particular, $a \in C_G(g)$. Thus a centralizes $Y\langle g\rangle = P$, and so $a \in N$. Therefore $C_G(b) \leq N$. Write $S = C_N(P/Y)$; then $P = YC_P(S)$, and we may take b as above, with $[B, S] = 1$. Let $y \in Y$ with $C_G(y) = C_G(Y) = R$; then yb generates P modulo Y and so, as we have seen for b, $C_G(yb) \leq N$. Now, as yb generates P modulo Y, $C_G(yb) = C_N(yb) \leq S \leq C_G(b)$. Hence, $C_G(yb) = C_G(b) \cap C_G(b) = R \cap N = C_G(P)$. As $C_G(P) \leq R = CG(y)$, we then have that q does not divide $|R : C_G(P)| = |A : C_A(P)|$. However, this is a contradiction, because $u \in O_q(A)$ does not centralize P, and this proves $R = C_G(Y) = G$. □

To proceed, we have now to impose the full $\overline{\mathcal{D}}$ property.

Lemma 3.8. *Let G be a soluble A-group in $\overline{\mathcal{D}}$, and p a prime divisor of $|G|$ such that G has a normal Sylow p-subgroup P. Then we have the following results.*

(i) $|P/(P \cap Z(G))| = p^{\alpha(p)}$.

(ii) *$G/C_G(P)$ is a q-group for some prime $q \neq p$.*

Proof. Since P is a normal Sylow subgroup of G, it admits a complement H; and, as P is abelian, by coprime action we have $P = (P \cap Z(G)) \times [P, H]$. Thus, $G = (P \cap Z(G)) \times [P, H]H$, and we may therefore assume $P \cap Z(G) = 1$, that is, $C_P(H) = 1$.

Now, there exists $x \in G$ such that $|G : C_G(x)| = p^{\alpha(p)}$, and since P is abelian we may assume that x is a q-element for some prime $q \neq p$. By replacing x with a conjugate, we may also suppose $x \in Z(H)$. Write $C = C_G(x) \cap P = C_P(x)$. Then $|P : C| = p^{\alpha(p)}$ and $C \trianglelefteq G$.

(i) By the standard argument, every q'-element of H centralizes the unique Sylow p-subgroup of $C_G(x)$ which is C. Hence $O^q(H) \leq C_G(C)$. Let Q be a Sylow q-subgroup of H; then, as $x \in Q$ does not centralize P, Q is not contained in $F(G)$, and thus, by Lemma 3.7, $|Q/O_q(G)| = q^{\alpha(q)}$. Now, $H = C_H(C)Q$ and so $|H/C_H(C)| = |Q/C_Q(C)|$. However, $x \in C_Q(C) \setminus O_q(G)$, hence $|C_Q(C)| > |O_q(G)|$, therefore

$$|G : C_G(C)| = |H : C_H(C)| < |Q : O_q(G)| = p^{\alpha(p)},$$

and so $|G : C_G(C)|$ is a power of q strictly smaller than $|G : C_G(C)|$. It then follows that $G = C_G(g)$ for every $g \in C$, and we conclude that $C \leq Z(G) = 1$, proving (i).

(ii) We have $|G : C_G(x)| = |P| = p^{\alpha(p)}$ and $C_G(x) \cap P = 1$, and hence $C_G(x) = H$. Let $F(G) = P \times N$, where N is the p'-component of $F(G)$. Then $N = F(G) \cap H$. As before, let Q be a Sylow q-subgroup of H with $x \in Q$. By Lemma 3.7, $O_p(G) \leq Z(G)$, and thus $O_q(G)\langle x\rangle \leq Z(H)$. In particular, $O_q(G)\langle x\rangle$ centralizes N. Since, by Lemma 3.7, $|Q/O_q(G)| = q^{\alpha(q)}$, we have $|G/C_G(N)|_q < q^{\alpha(q)}$; thus q does not divide $|G : C_G(y)|$ for every $y \in N$. Now, let $b \in G$ be such that $|G : C_G(b)| = q^{\alpha(q)}$. Then $C_G(b) \geq NO_p(G)$ and so $b \in F(G)$. Let $b = ac$ with $a \in P$ and $c \in N$. Then $C_G(c) \geq C_G(b)$, and, on the other hand, we know that c centralizes some Sylow q-subgroup of G. Hence $c \in Z(G)$. We may thus assume $b \in P$. Finally, let $g \in C_G(b)$; then $C_P(g) \neq 1$, and so

$$|G : C_G(g)|_p = |P : C_P(g)| < |P| = p^{\alpha(p)},$$

yielding $P \leq C_G(g)$, which proves point (ii). □

Lemma 3.9. *Let G be a group in $\overline{\mathfrak{D}}(p_1^{m_1}, p_2^{m_2}, \ldots, p_k^{m_k})$. Let $p = p_1$, P a Sylow p-subgroup of G, and $A = O_{p'}(G)$. Suppose that P is not abelian. Then we have the following results.*

(i) *A is a group in $\overline{\mathfrak{D}}(p_2^{m_2}, \ldots, p_k^{m_k})$.*
(ii) *$Z(A)$ is centralized by P.*

Proof. (i) Write $\alpha = \alpha(p) = m_1$. Let $a \in A$, and $m = |a^A| = |A : C_A(a)|$. Then, since by Theorem 2.3 $A \trianglelefteq G = AP$, we clearly have $|a^G| = mp^i$, where $i \in \{0, p^\alpha\}$. Thus $\mathrm{cl}(A) \subseteq \{1, p_2^{m_2}\} \times \cdots \times \{1, p_k^{m_k}\}$.

Let now $i \in \{2, \ldots, k\}$. There exists $r \in G$ such that $|g^G| = p_i^{m_i}$. Write $g = bx$, with $b \in A$, x a p-element, and $[a, x] = 1$. As $C_G(x)$ contains $C_G(g)$ which has p'-index in G, x is central in some Sylow p-subgroup of G. By Lemma 2.4, $x \in Z(G)$. Hence $C_G(g) = C_G(b)$, and from this it follows that $|b| = p^{m_i}$. This holds for every prime p_i with $i \in \{2, \ldots, k\}$, and thus (i) is proved.

(ii) It is enough to show that P centralizes every q-element $a \in Z(A)$, for any prime q. If $q \notin \{p_2, \ldots, p_k\}$, then $a \in Z(G)$ and there is nothing to show. Thus, let $q = p_i$ for some $i = 2, \ldots k$. By (i) there exists $b \in A$ such that $|b^G| = p_i^{m_i}$. Hence $a \in C_G(b)$ and, up to conjugation, $P \leq C_G(b)$. Then, for every $x \in P$, bx centralizes a Sylow p_i-subgroup of $C_G(b)$; in particular, as $a \in O_{p_i}(G)$, x centralizes a. Thus $[P, a] = 1$, and (ii) follows. □

We are now in a position to prove our main result.

Theorem 3.10. *Let G be a $\overline{\mathfrak{D}}$-group. Then G is a direct product,*

$$G = A \times G_1 \times G_2 \times \cdots \times G_t,$$

where A is abelian, each G_i is a basic group of type $\overline{\mathfrak{D}}$ (that is, groups of type (I) and (II) described in the introduction), and $(|G_i|, |G_j|) = 1$ for $i \neq j$.

Proof. Let G belong to $\overline{\mathfrak{D}}(p_1^{m_1}, \ldots, p_k^{m_k})$, for distinct primes $p_1, \ldots, p_k$. This, in particular, implies that $\pi(G/Z(G)) = \{p_1, \ldots, p_k\}$. If $k = 1$, the group G is, up to a central factor, a p-group, and so it must be a basic $\overline{\mathcal{D}}$-group of type (I). We then proceed by induction on k. Thus, suppose $k \geq 2$.

(A) Assume that, for some $i \in \{1, \dots, k\}$, the Sylow p_i-subgroups of G are not abelian. We may as well suppose that $i = 1$, and write P for a fixed Sylow p_1-subgroup of G. By Theorem 2.3, G has a normal p_1-complement N, and by Lemma 3.9, N is a group belonging to $\overline{\mathcal{D}}(p_2^{m_2}, \dots, p_k^{m_k})$. By inductive assumption, N is a direct product $N = A \times G_2 \times \cdots \times G_k$, with $A \leq Z(N)$ and $G_2, \dots, G_n$ basic $\overline{\mathcal{D}}$-groups of coprime orders. Then each G_i is normalized by P. Also, by Lemma 3.9, P centralizes A, and so $A \leq Z(G)$.

Let $i \in \{2, \dots, k\}$. Suppose that G_i is a q-group for some prime q. Then N (and consequently G) has a unique nonabelian Sylow q-subgroup $Q \geq G_i$. By Theorem 2.3, P commutes with Q, and thus with G_i.

Suppose now that G_i is a basic $\overline{\mathcal{D}}$-group of type (II). Then, say, $G_i = TQ$, T is a normal abelian t-subgroup, Q is an abelian Sylow q-subgroup (t, q are distinct primes belonging to $\{p_2, \dots, p_k\}$), $C_Q(T) = Z(G_i)$, and $Q/Z(G_i)$ is a cyclic group acting fixed point freely on T. It is then clear that $|T| = t^{\alpha(t)}$, and $|Q/C_Q(T)| = q^{\alpha(q)}$. Let g be an element of G such that $|g^G| = q^{\alpha(q)}$. It is easy to see that g must have a nontrivial t-component $a \in T$ and that $|a^G| = q^{\alpha(q)}$. Up to conjugation, we may suppose that a is centralized by P. Thus, for every $y \in P$, $C_T(y) \neq 1$, and so $|y^G|_t < |T| = t^{\alpha(t)}$, forcing y to centralize T. Hence $[T, P] = 1$. Thus $[T, P, G_i] = 1$, $[G_i, T, P] = 1$, and so, by the three subgroup lemma, $[G_i, P] \leq C_{G_i}(T) = TZ(G_i)$. It follows, by also applying Lemma 3.9, that $[G_i, P, P] \leq [G_i, TZ(G_i)] = 1$, whence, by coprime action, $[G_i, P] = 1$.

We have thus shown that P centralizes N. Hence $G = N \times P$, and we are done.

(B) We now assume that G is an A-group. Clearly, G is not nilpotent; hence, by Lemma 3.7, there exists a prime $p = p_i$ such that G has a normal (noncentral) Sylow p-subgroup P. Let H be a p'-complement of P in G; then, since P is abelian, $P = [P, H] \times C_P(H)$. Thus, $C_P(H) = C_P(G)$ is a central direct factor of G, and so we may well assume $C_P(G) = 1$. Then, by Lemma 3.8, $|P| = p^{\alpha(p)} = p_i^{m_i}$ and $H/C_H(P) \simeq G/C_G(P)$ is a q-group for some prime $q = p_j \neq p$. Let Q be a Sylow q-subgroup of H. Then clearly, $Q \nleq F(G)$, and so, by Lemma 3.7, $Q/(Q \cap Z(G))$ is cyclic of order $q^{\alpha(q)}$. It follows that $H/C_H(P)$ is cyclic of order $q^{\alpha(q)}$, and this implies, in particular, that $H/C_H(P)$ acts as a group of fixed point free automorphisms of P. Moreover, $C_H(P) = O_q(G) \times N$, where N is a (normal) q'-Hall subgroup of $C_H(P)$. Therefore, $G = NG_1$, where $N \trianglelefteq G$, $N \cap G_1 = 1$, and $G_1 = PQ$ is a basic $\overline{\mathcal{D}}$-group of type (II). Now, the same remark as used in case (A) guarantees the existence of an element $u \in Q$ such that $|u^G| = p^{\alpha(p)}$. Thus, in particular, $[N, u] = 1$. Since u does not belong to $C_G(P) = Q \cap Z(G)$, and $Q/C_Q(P)$ is cyclic, it follows that $[N, Q] = 1$. Thus N centralizes G_1, and therefore $G = G_1 \times N$. Since $(|G_1|, |N|) = 1$, the conclusion now follows by inductive assumption. □

Observe that the statement of Theorem 3.10 provides also, in principle, a way of describing, given prime powers $p_1^{m_1}, p_2^{m_2}, \dots, p_k^{m_k}$, the possible class size patterns of

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groups in the class $\overline{\mathcal{D}}(p_1^{m_1}, p_2^{m_2}, \ldots, p_k^{m_k})$. In particular, nonnilpotent direct factors in the decomposition of G provided by the theorem correspond to pairs of distinct primes p_i, p_j such that the product $p_i p_j$ does not divide any conjugacy class size. If no such pair exists, the group G is nilpotent, and this proves Theorem 1.1.

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