CERTAIN HOMOMORPHISMS OF A COMPACT SEMIGROUP ONTO A THREAD

R. P. HUNTER and L. W. ANDERSON [1]

(Received 7 October 1965)

Let S be a compact semigroup and f a continuous homomorphism of S onto the (compact) semigroup T. What can be said concerning the relations among S, f, and T? It is to one special aspect of this problem which we shall address ourselves. In particular, our primary considerations will be directed toward the case in which T is a standard thread. A *standard thread* is a compact semigroup which is topologically an arc, one endpoint being an identity element, the other being a zero element. The structure of standard threads is rather completely determined e.g. see [20]. Among the standard threads there are three which have a rather special rôle. These are as follows: A *unit thread* is a standard thread with only two idempotents and no nilpotent element. A unit thread is isomorphic to the usual unit interval [14]. A *nil thread* again has only two idempotents but has a non-zero nilpotent element. A nil thread is isomorphic with the interval $[\frac{1}{2}, 1]$, the multiplication being the maximum of $\frac{1}{2}$ and the usual product — or, what is the same thing, the Rees quotient of the usual $[0, 1]$ by the ideal $[0, \frac{1}{2}]$. Finally there is the *idempotent thread*, the multiplication being $x \circ y = \min(x, y)$. These three standard threads can often be considered separately and, in this paper, we reserve the symbols I_1, I_2 and I_3 to denote the unit, nil and idempotent threads respectively. Also, throughout this paper, by a homomorphism we mean a *continuous* homomorphism.

Now the classical monotone-light factorization theorem carries over intact to the theory of semigroups and homomorphisms. That is to say, if $f : S \to T$ is a continuous homomorphism from the compact semigroup S onto T then there exists a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & T \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle h} & \\ & M & \end{array}$$

where g is monotone, h is light, and M is the semigroup whose underlying

[1] With the support of the National Science Foundation NSF GP 4066.

space is the hyperspace of the upper semi-continuous decomposition of S formed by the components of the inverse images of f. From the time this was first observed by A. D. Wallace it has proved to be of considerable help in the study of homomorphisms in general. (See for example [12], [7] or [8].)

A good many of the results concerning homomorphisms of a compact semigroup have been related to standard threads in one way or another. One of the first, due to Cohen and Krule said, in particular, that a homomorphism on a standard thread was monotone [5].

At this point it would not be inappropriate to recall the example of Ursell [21] which indicates the possible paucity of homomorphisms for even an otherwise well behaved semigroup.

Let $A = I_2$, a nil thread, with zero a_0 and $B = I_3$, an idempotent thread, with zero b_0. In the product semigroup $B \times A$ the set

$$Q = \{b_0\} \times A \cup B \times \{a_0\}$$

is an ideal. Now form S — the Rees quotient of $B \times A$ by Q. Now S has the property that it contains a nil thread from zero to identity but also an idempotent thread from zero to identity. Two such subsemigroups cannot have a common homomorphic image. Thus, S admits no homomorphism onto a standard thread. Indeed, let T be any compact connected one dimensional semigroup with zero and identity. From [11] we know that T contains precisely one arc from zero to identity. It follows readily that there is no homomorphism of S onto T. Thus S has no dimension lowering homomorphism. (Actually, considerably more can be said about the restricted nature of the homomorphisms of S but a detailed discussion here might be out of place.)

From the above example it follows that a theory analogous to that of locally compact abelian groups, at least from the standpoint of "characters" is not to be expected. Of course, in a number of interesting cases there do exist homomorphisms of the semigroup in question onto a standard thread see, for example, [3], [7], [9], [10], [16], [17].

If f is a homomorphism from the semigroup S onto T we shall say that f has a *full cross section* if S contains a closed subsemigroup S_0 which meets each inverse image $f^{-1}(t)$ at precisely one point.

For a good part of what follows, we shall require some facts about the Green equivalences. These are defined for an arbitrary semigroup S as follows:

$$a \equiv b(\mathscr{L}) \leftrightarrows Sa \cup a = Sb \cup b$$
$$a \equiv b(\mathscr{R}) \leftrightarrows aS \cup a = bS \cup b$$
$$\mathfrak{H} = \mathscr{L} \cap \mathscr{R} \qquad \mathfrak{D} = \mathscr{L} \circ \mathscr{R} \; (= \mathscr{R} \circ \mathscr{L})$$
$$a \equiv b\mathfrak{J} \leftrightarrows a \cup aS \cup Sa \cup SaS = b \cup bS \cup Sb \cup SbS.$$

In a compact semigroup each of these defines an upper semicontinuous decomposition. Since a compact semigroup is stable all of the results of [1] and [18] apply. In particular $\mathfrak{D} = \mathfrak{J}$.

If Q is an ideal of a compact semigroup S we shall refer to the semigroup formed by collapsing Q to a point as the Rees quotient and denote it by S/Q.

In general our terminology follows [4] and [22]. The symbols 0 and 1 will denote (if they exist) the zero and identity elements of the semigroup in question. Concerning decompositions we shall follow [23].

PROPOSITION 1. *Let S be a compact semigroup and f a homomorphism onto T where T is either a unit or a nil thread from 0 to 1. If $f^{-1}(0)$ is degenerate then S contains a unit or a nil thread M which meets both $f^{-1}(0)$ and $f^{-1}(1)$. Moreover, any such M defines a full cross-section.*

PROOF. Let N denote the minimal ideal of the compact semigroup $f^{-1}(1)$. Since $S\backslash f^{-1}(1)$ is not closed there is some point $r \in f^{-1}(1)$ which is also an element of $(S\backslash f^{-1}(1))^*$. Now if n is any element of N we have $nrn \in N$. Since N is the union of groups it follows that any such nrn is an element of some subgroup of N, say H_e where e is an idempotent. If we multiply nrn by its inverse with respect to e, we see, since multiplication is continuous and $S\backslash f^{-1}(1)$ is an ideal, that each open set about e meets $eSe\backslash f^{-1}(1)$. We now assert that eSe contains a continuum from $\theta = f^{-1}(0)$ to e. Suppose, on the contrary that this is not the case, so that we may write eSe as the sum of two separate sets A and B such that $\theta \in A$ and $e \in B$. Let x be a point of B such that $f(x)$ is the first point of $f(B)$ in the order from 0 to 1. We note that $\theta \neq x \notin f^{-1}(1)$. Indeed it follows immediately that N is a $\mathfrak{D}$-class of S and from [1] we know that $eSe \cap N = H_e$. We have already seen that each point of H_e lies in the closure of $eSe\backslash f^{-1}(1)$. Thus $x \notin f^{-1}(1)$. Now let V be an open set about x such that $V \cap A = \emptyset$. Let W be open about e such that $Wx \subseteqq V$. If p is any point of

$$W \cap (eSe\backslash f^{-1}(1))$$

then $px \in A \cup B = eSe$. But $px \notin A$ so that $px \in B$, and so, $f(px) < f(x)$ in the order from 0 to 1. Thus there must be a continuum from θ to e in eSe. It now follows that eSe is a compact connected semigroup with identity e and zero θ and no other idempotents. It then follows from [20] that eSe contains a standard thread M from zero to identity.

To see that M is a full cross-section let $x, y \in M$ such that $f(x) = f(y)$ and suppose that $x < y$ in the order in M taken from zero to identity. Then $x = ym$ for some $m \in M$ and so,

$$f(x) = f(ym) = f(y)f(m) = f(x)f(m).$$

It follows now that for any integer $n \geqq 1$,

$$f(x) = f(x)f(m^n).$$

It would then follow that $f(x) = f(x) \cdot q$ where q is an idempotent in T distinct from 0 and 1. Thus M is a full cross-section.

PROPOSITION 2. *Let S be a compact semigroup which is mapped homomorphically by f onto T which is either a unit or a nil thread from* 0 *to* 1. *Then S contains a compact connected semigroup M such that $f(M) = T$ and the Rees quotient $M/M \cap f^{-1}(0)$ is itself a unit or a nil thread. If each subgroup of $f^{-1}(0)$ is totally disconnected then M may be chosen as a unit or nil thread and hence a full cross-section of f.*

PROOF. We have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & T \\ & {\scriptstyle \alpha}\searrow \qquad \nearrow{\scriptstyle g} & \\ & S/f^{-1}(0) & \end{array}$$

Where α is the natural homomorphism associated with the Rees quotient by the ideal $f^{-1}(0)$ and g is induced by α and f in the natural way. Now Proposition 1 applies to g so that $S/f^{-1}(0)$ contains a standard thread $[k, e]$ where $k = \{f^{-1}(0)\}$, and $g([k, e]) = T$. Now let M be the closure of the semigroup generated by $\alpha^{-1}([k, e])$. It follows that M is a compact connected abelian semigroup with identity $\alpha^{-1}(e)$ and minimal ideal lying in $f^{-1}(0)$. It follows readily that $M/M \cap f^{-1}(0)$ is a standard thread. Now, since M is abelian, its minimal ideal is a group. If we then take each subgroup of $f^{-1}(0)$ to be totally disconnected we see that M has a zero, hence M is either a standard or a nil thread.

PROPOSITION 3. *Any standard thread, which is the homomorphic image of a compact totally disconnected semigroup, is an idempotent thread.*

PROOF. If T is such a semigroup and $f : S \to T$ is a homomorphism where S is as above, consider $T \backslash E$. If C^* is the closure of any component of $T \backslash E$ then C^* is a unit thread or a nil thread and hence raised back to S.

To show that the conditions of Proposition 3 may well obtain we recall the well known example of a dimension raising homomorphism first given by Koch: Let C be the usual Cantor set, taken from the unit interval. Let C have the multiplication $x \circ y = \min(x, y)$. Now take the interval $[0, 1]$ with the same multiplication i.e. $xy = \min(x, y)$. If f is any continuous order preserving mapping of C onto $[0, 1]$ then f will be a homomorphism. A simple example of such an f is one which identifies the endpoints of complementary domains, an inverse image is then one or two points. For other examples of dimension raising homomorphisms see [3] and [8].

LEMMA 1. *Let S be a compact connected abelian semigroup which is algebraically irreducible from its minimal ideal K to its identity. Then if S does not admit a homomorphism onto either a unit thread or a nil thread, S must be an idempotent thread.*

It is well known [13] that $S/\mathfrak{H}$ is a standard thread from say θ to e. Now if $S/\mathfrak{H}$ contains a unit or nil thread $[a, b]$ we have only to collapse the interval $[\theta, a]$ and the interval $[b, e]$ to obtain a homomorphism onto $[a, b]$. Thus if $S/\mathfrak{H}$ contains no such $[a, b]$ it must be entirely idempotent. But if $S/\mathfrak{H}$ is an idempotent thread S must be the same (see [10]). Indeed E and $S/\mathfrak{H}$ would then be canonically isomorphic so that $S \cong E \cong S/\mathfrak{H}$.

PROPOSITION 4. *Suppose S is a compact zero dimensional semigroup and f is a homomorphism of S onto T, a nondegenerate compact connected semigroup with identity. Then T is arcwise connected and in fact if t is any idempotent in T there is a point $k \in K(T)$, the minimal ideal of T, and an idempotent thread $[k, t]$ from k to t.*

PROOF. First of all, T is not a group. For suppose, on the contrary, that T is a group. Let s be a primitive idempotent in S. Then sSs is a group and it is immediate that f cut down to sSs is an epimorphism. But since we would then have a group homomorphism from a compact zero dimensional group onto a compact connected group and it would follow that T is degenerate [25]. Hence we may assume $K(T)$ to be proper. For the moment we shall assume that $K(T)$ is degenerate, which is to say T has a zero, and then reduce the general case to this. Let us first recall that E — the set of idempotents — is a compact space with a continuous partial order $\leqq$ defined by

$$e \leqq g \leftrightarrows eg = ge = e.$$

Now let e be an arbitrary idempotent in T and V any open set about e. Suppose that there are no idempotents in $eTe \cap V$ except e. It would then follow from [19] that eTe contains a one parameter local semigroup $[x, e]$ at e such that $[x, e] \cap H_e = \{e\}$. We know from [1] that in fact $eTe \cap D_e = H_e$ and thus each point of $[x, e]$ lies in $TeT \backslash D_e = I(e)$ which is an ideal of T. Thus one can use this one-parameter sub-semigroup to generate an abelian compact connected sub-semigroup A which has an identity, namely, e and is itself not a group. Indeed, A is not the union of groups since it contains points of eTe arbitrary close to e whose $\mathfrak{H}$-classes (of T and of A) do not contain idempotents. Now from Lemma 1 it follows that A contains a compact connected semigroup A_0 which can be mapped homomorphically onto either a unit thread or a nil thread. However, it would then follow that S contains a compact sub-semigroup S_0 which can be mapped homomorphically onto a unit or a nil thread. This is ruled out by Proposition 1. Thus, in the partially ordered set E there are no local minima and that for each idempotent e the set eEe satisfies the conditions of Theorem 1 of [15]. It now follows that $E(T)$ is a continuum. To see this, let C be any component of $E(T)$. If C did not contain the zero element of T then let c_0 be a minimal element in the sense of $\leqq$. Now the set $c_0E(T)c_0$ contains, from the above considerations, a non-degenerate continuum of idempotents. But since c_0 is minimal in $E(T)$ we have $c_0E(T)c_0 \cap C = \{c_0\}$. But this is a contradiction since C can then be extended and so is not a component. This means that any component of $E(T)$ contains the zero element of T and so $E(T)$ is a continuum. Now, in $E(T)$ the set of elements x such that $x \leqq e$, where $e^2 = e$, is precisely $eE(T)e$ which is a continuum. Thus $E(T)$ satisfies corollary 1 of [15] and so there is a compact connected chain of idempotents from zero to e. That is to say, there is an idempotent thread from zero to e in $E(T)$.

Now we return to the case in which $K(T)$ is non-degenerate. The above discussion applies to the Rees quotient $T/K(T)$. Let $\alpha : T \to T/K(T)$ be the natural homomorphism and let $[0, e]$ be an idempotent thread in $T/K(T)$. We claim that $\{\alpha^{-1}((0, e])\}^*$ is an idempotent thread. To see this one uses lemma 2.6 of [11] which says that under these conditions the set $Q = \{\alpha^{-1}((0, e])\}^* \backslash \alpha^{-1}((0, e])$ is either a point or a compact connected group. If the latter, the group must be trivial since $\alpha^{-1}(0, e])$ is composed entirely of idempotents. Thus in either case Q is degenerate and we have the desired idempotent thread between e, where e was an arbitrary idempotent, and some point in K.

PROPOSITION 5. *Let S be a compact connected semigroup with identity and f a homomorphism of S onto the standard thread T. If no closed ideal which is a complete inverse image separates S then f is non-alternating.*

PROOF. Letting as usual 0 and 1 denote the zero and identity of T suppose $x, y \in T$ are much that $S \backslash f^{-1}(x) = A \cup B$ mutually separate and that $f^{-1}(y)$ meets A and B. Consider first the case in which $y < x$. Now $f^{-1}(0)$ is an ideal and as such contains K the minimal ideal of S. Since K does not meet $f^{-1}(x)$ we have either $A \supset K$ or $B \supset K$. Suppose, say, that $K \subset A$. If there were an element $b \in f^{-1}(y) \cap B$ then bS would necessarily meet $f^{-1}(x)$ since bS contains b and meets K. But then, $f(bS) = f(b)f(S) = yT = [0, y]$ so that $x \in [0, y]$ and $x \leqq y$ which contradicts $y < x$. On the other hand suppose that $x < y$. In this case we note that $x \neq 0$ since $f^{-1}(0)$ is an ideal and $f^{-1}(x)$ separates S. Hence we again may take, say, $K \subset A$. Now $f^{-1}([0, x])$ is an ideal which is a complete inverse image and does not meet $f^{-1}(y)$ since $x < y$. Hence $f^{-1}([0, x])$ does not contain A and does not contain B. It follows then that $f^{-1}([0, x])$ separates which is contrary to hypothesis.

To see that the condition placed on the ideals which are inverse images is necessary one has only to map the usual interval $[-1, 1]$ onto the interval $[0, 1]$ using $x \to |x|$. This homomorphism fails to be non-alternating.

The following proposition indicates that if a standard thread can be raised from T to S then it is embedded in a rather special way.

PROPOSITION 6. *Let I be a standard thread which is a sub-semigroup of a compact semigroup S. If I meets a $\mathfrak{D}$-class D of S it does so in at most one point.*

PROOF. It suffices to prove the result when S has an identity since its adjunction does not disturb the Green equivalences.

Suppose, on the contrary, that $I \cap D$ is non-degenerate. Since D is compact we may let y be the last point, and x the first point of $I \cap D$, the order in I being from zero to identity. First of all, we assert that $[x, y] \subseteqq D$, where $[x, y] \subseteqq I$. Let $t \in [x, y]$. Then since $t \in SyS$ and $x \in StS$ we have $SxS \subseteqq StS \subseteqq SyS$ and since x and y are $\mathfrak{D}$ (and $\mathscr{J}$) equivalent we have $t \in D$. Now $t \in Sy \cap yS$. From [1] we know that $Sy \cap D = L_y =$ the $\mathfrak{L}$-class of y, and $yS \cap D = R_y =$ the $\mathfrak{R}$-class of y. It follows then that $t \in H_y =$ the $\mathfrak{H}$-class of y. Thus $[x, y] \subseteqq H_y$. Now if the arc $[x, y]$ contains an idempotent, H_y is a topological group. In this case $I \cap H_y = [x, y]$ would be a compact sub-semigroup and hence a subgroup of H_y. This is manifestly impossible unless $[x, y]$ is degenerate. Suppose now that $[x, y]$ contains no idempotent. In this case we know that $xy < x$. In particular if q is the least idempotent of I such that $y < q$ then there is a point $s \in [x, q]$ such that $ys = x$. In particular Hs meets H. Now, from [4], we know that $Hs \cap H \neq \emptyset$ implies $Hs = H$. Thus $xs \in H$. Since $xs < x$ in the order from zero to identity in I, (there being no idempotent in the segment $[x, q]$), and x is the first point of $I \cap D$ we have a contradiction.

PROPOSITION 7. *Let S be a compact semigroup and let $\{S_\alpha \,|\alpha \in \Gamma\}$ be a continuous decomposition of S with each S_α a compact subsemigroup of S. Then the minimal ideals K_α of the elements S_α form a continuous decomposition of the compact space $\cup \{K_\alpha \,|\alpha \in \Gamma\}$.*

PROOF. Let S_0 be an element of the decomposition having K_0 as minimal ideal. Let W be any open set about K_0. There exist open sets U about K_0 and V about S_0 such that $VUV \subseteqq U$ since $S_0 K_0 S_0 = K_0$ and the sets in question are compact. We may take $U \subseteqq W$, and $U^* \cap S_0 \neq \emptyset$. Now let S_i be a sequence of elements of the decomposition having S_0 as sequential limiting set. Since $\lim S_i = S$, for large enough j, $S_j \cap U \neq \emptyset$ and $S_j \subseteqq V$. It follows readily that $U^* \cap S_j$ is a closed ideal of S_j and as such must contain K_j

https://doi.org/10.1017/S1446788700004158 Published online by Cambridge University Press

so that in particular $K_j \subset W$. This shows that the minimal ideals form at least an upper semi-continuous decomposition. To see that the sets K_α form a continuous collection let K_i be a sequence which converges to say K'. (We know from the above that K' is contained in the minimal ideal of the element S' in the original decomposition.) We claim that K' is the minimal ideal of S' and to verify this we need only show that K' is an ideal of S'. Let s' be an arbitrary point of S' and suppose s_i converges to s' where $s_i \in S_i \supseteqq K_i$. Then if k_i converges to k' where $k_i \in K_i$ then $k_i s_i$ converges to $k's'$. Since $k_i s_i \in K_i$ for each i it follows that $k's' \in K'$ which means K' is a right ideal. Likewise K' is a left ideal and the Proposition is proved.

The following example shows that the union of the minimal ideals discussed in Proposition 7 need not form a sub-semigroup even when the decomposition in question is a congruence.

Let S be the semigroup whose multiplication is given below

	a	b	c	d	e
a	a	b	b	d	d
b	b	b	b	d	d
c	b	b	c	d	e
d	d	d	d	d	d
e	d	d	e	d	e

Let C be the congruence whose single non-degenerate class is $\{d, b\}$. The collection of minimal ideals for this congruence is $\{a, c, e, d\}$. However $ac = b$.

The above construction can easily be used to give an example of a compact connected semigroup (with identity if desirable) having a continuous decomposition which is a congruence such that the minimal ideals of the elements do not form a sub-semigroup.

Proposition 8. *Let f be an open homomorphism of the compact semigroup S onto an idempotent thread T. Then f has a full cross-section.*

Proof. Let A be the subset of S which is the union of the minimal ideals of the inverse images of f. First of all A is a sub-semigroup of S. For let $a_1, a_2 \in A$ and, say $f(a_1) \leqq f(a_2)$. Then $a_1 a_2 \in f^{-1}(a_1)$. Since a_1 is an

https://doi.org/10.1017/S1446788700004158 Published online by Cambridge University Press

element of the minimal ideal of $f^{-1}(a_1)$ there is a primitive idempotent e such that $a_1 e = a_1$. Now, $a_1 a_2 = (a_1 e) a_2 = a_1(e a_2) \in a_1 f^{-1}(a_1)$ which is an element of the minimal ideal of $f^{-1}(a_1)$. Now let $\hat{f} : A \to T$ be f cut down to A. From Proposition 7, we known that $\hat{f}$ is an open homomorphism. Thus, if A_0 is any component of A we have $\hat{f}(A_0) = T$. By taking A_0 to be the component of an idempotent we may suppose that A_0 is a sub-semigroup. Since any compact sub-semigroup of a compact completely simple semigroup is again completely simple, A_0 satisfies all of the following:

(1) A_0 is mapped homomorphically onto T
(2) Each inverse is a completely simple semigroup
(3) A_0 is compact, connected and the union of groups.

Now let e be any idempotent in $A_0 \cap f^{-1}(1)$. From condition (3) it follows, using [10], that A contains an idempotent thread I from e to the minimal ideal of A. Since the minimal ideal of A is contained in $f^{-1}(0)$ it follows that I meets both $f^{-1}(0)$ and $f^{-1}(1)$ and hence $\hat{f}(I) = T$. Thus, the only thing left to show is that I cannot meet an inverse image in more than one point. Suppose a and x are elements of I such that, say, $a \leqq x$ and $f(a) = f(x)$. Since $I \subseteqq A$ the points a and x lie in the same minimal ideal of $f^{-1}f(a) = f^{-1}f(x)$. However, a and x commute, and two idempotents of a completely simple semigroup commute if and only if they are equal. Thus $a = x$.

Proposition 9. *There exists a compact connected abelian two-dimensional semigroup with identity which contains no isomorphic copy of the usual unit interval and yet can be mapped with an open monotone homomorphism onto the usual unit interval.*

Proof. The semigroup, S, in question will be a sub-semigroup of the cartesian product of C — the usual circle group and I — the usual unit interval. Let $f : [0, \infty) \to C \times I$ be defined by $f(x) = (e^{2\pi i x}, e^{-x})$ and set

$$S = f([0, 1)) \cup \{(c, t) \mid c = e^{2\pi i x}\}$$

where $1 \leqq x < 2$ and $0 \leqq t \leqq e^{-x}$ (see figure 1). Clearly, S is a compact connected semigroup. The open homomorphism, π, will be the projection of $C \times I$ onto I, cut down to S. We easily see that π is a continuous homomorphism. To show that π is open we need only show that if $\{t_n\}$ is a sequence in I with $\{t_n\} \to t_0$ and if $s_0 \in \pi^{-1}(t_0)$, then there is a sequence $\{s_n\}$ in S with $s_n \in \pi^{-1}(t_n)$ such that $\{s_n\} \to s_0$. To this end, choose $(c_0, t_0) \in S$ and $\{t_n\}$ a sequence in I with $\{t_n\} \to t_0$. If $t_0 < e^{-2}$ or $t_0 > e^{-1}$ we easily find a sequence $\{c_n\}$ in C such that $\{c_n\} \to c_0$ and $(c_n, t_n) \in \pi^{-1}(t_n)$.

In case $e^{-2} \leqq t_0 \leqq e^{-1}$ we have $c_0 = e^{2\pi i x_0}$ for some x_0 where $x_0 \leqq -\log t_0$ and $1 \leqq x_0 \leqq 2$. Now choose a sequence $\{x_n\}$ so that $x_n \leqq -\log t_n$ and $\{x_n\} \to x_0$, then $(e^{2\pi i x_n}, t_n) \to (c_0, t_0)$ and $(e^{2\pi i x_n}, t_n) \in \pi^{-1}(t_n)$.

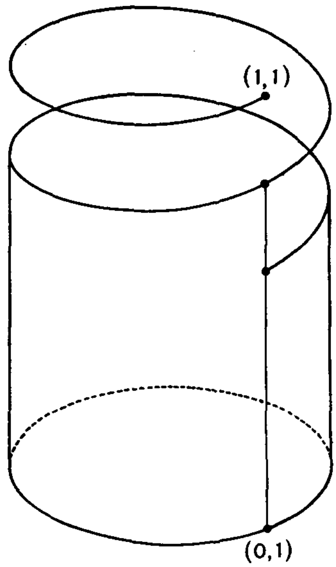

Figure 1

Since $C \times I$ contains a unique copy of I which was not retained when S was chosen, S contains no copy of I.

PROPOSITION 10. *There exists a compact connected abelian semigroup with identity which can be mapped with an open monotone homomorphism onto a nil thread but contains no isomorphic copy of a nil thread.*

PROOF. Let S be the semigroup constructed for Proposition 9. Define upon S the congruence which collapses each set of the form $\{z\} \times [0, \frac{1}{5}]$ where $z \in C$, to a point. There is induced an open homomorphism onto the Rees quotient $[0, 1]/[0, \frac{1}{5}]$.

In both Propositions 9 and 10 one can take the semigroups so that each inverse image of the homomorphism is non degenerate. This is done by using two one-parameter semigroups and performing a similar construction. (More simply one can form cartesian products and use the projection).

Using Propositions 9 and 10 it can be seen that if T is a standard thread which is not an idempotent thread there is a compact connected abelian semigroup S with identity and an open monotone homomorphism of S onto T for which there is no cross-section.

PROPOSITION 11. *Let S be a compact semigroup and f an open light homomorphism onto T, a standard thread, then S contains a standard thread I from $f^{-1}(0)$ to $f^{-1}(1)$, (which is necessarily a full cross section for f). Moreover I may be taken to contain any particular idempotent in $f^{-1}(1)$.*

PROOF. Let e be an idempotent in $f^{-1}(1)$. Let C be the component of

https://doi.org/10.1017/S1446788700004158 Published online by Cambridge University Press

e in S. Since f is open $f(C) = T$ and thus $f(eCe) = T$. Then f cut down to eCe is a light mapping of eCe onto T. It follows that each subgroup of eCe is totally disconnected. From [17] eCe contains a thread I from its identity e to some point in its minimal ideal.

To see that I is a full cross section for f we note that $f(I) = T$ and $f|I$ is light. By a result in [5], $f|I$ is also monotone, hence $f|I$ is one-to-one and so I is a full cross section for f.

Example. Let S be the cartesian product of I_3 and I_2 and let S_0 be those points (x, y) with $y \leqq x$. Then the projection onto I_3 cut down to S_0 defines an open monotone homomorphism of S_0 onto I_3. Clearly I_3 cannot be raised to S_0 in such a way as to define copy of I_3 meeting both the identity and zero of S_0.

We use the term *local thread* in the following sense: An arc A is called a local thread at an idempotent e if e is one endpoint and an identity for A, and there exists an open set 0 about e such that 0 contains an open set V satisfying

$$(V \cap A)^2 \subseteqq (0 \cap A).$$

PROPOSITION 12. *Let S be a non degenerate compact connected semigroup with identity* 1 *which is embeddable in the plane. Then S contains a local thread at* 1.

PROOF. From [2] we may assume without loss of generality that the maximal subgroup at 1 is precisely $\{1\}$. Let V be an open set about 1 such that $V^* \cap K = \emptyset$. Now if for each idempotent $e \in V$, H_e is totally disconnected then the Rees quotient S/SQS, where $Q = (S \backslash V)^*$, has all of its subgroups totally disconnected and hence contains a thread from zero to identity. Thus in this case there is, in particular, a local thread in S at 1. If, on the other hand, for every such V there is an idempotent e with H_e not totally disconnected then consider C_e the component of e in H_e. Now C_e is a circle group. Let e_i be a sequence of such idempotents converging to 1. Now it follows from [24] that given any C_{e_i} and C_{e_j} then one is contained in the bounded complementary domain of the other. Thus we may suppose that 1 is in the bounded complementary domains of all such C_{e_i} or is in the unbounded complementary domains of all C_{e_i}. In the former case it follows readily that S would contain the bounded complementary domains of the C_{e_i} making S locally euclidean at 1 which is impossible. In the latter case H_1 would have to be non degenerate, again a contradiction.

PROPOSITION 13. *Suppose S is a compact semigroup with identity and f a light open homomorphism onto T a compact connected semigroup with identity embeddable in the plane. If T is not a group then S contains a local thread at* 1 *which meets each inverse image in at most one point.*

Bibliography

[1] L. Anderson, R. P. Hunter and R. J. Koch, 'Some results on stability in semigroups'. *Trans. A.M.S.*, 117 (1965), 521—529.

[2] L. W. Anderson and R. P. Hunter, 'Sur les demi-groupes compacts et connexes'. *Fundamenta Math.* LVI (1964).

[3] L. W. Anderson and R. P. Hunter, 'Dimension and homomorphisms'. *Mathematische Annalen* 147 (1962), 248—268 (1962).

[4] A. H. Clifford and G. B. Preston, 'The algebraic theory of semi-groups'. *Amer. Math. Soc. Surveys Number* 7.

[5] H. Cohen and I. Krule, 'Continuous homomorphic images of real semigroups with zero'. *Proc. A.M.S.* 10 (1959), 106—109.

[6] J. A. Green, 'On the structure of semigroups'. *Annals of Math.* 54 (1951), 163—172.

[7] R. P. Hunter, 'On homogroups and their applications to compact connected homogroups'. *Fund. Math.* L11 (1963), 69—102.

[8] R. P. Hunter, 'On one dimensional semigroups'. *Mathematische Annalen* 146 (1962), 383—396.

[9] R. P. Hunter, 'Certain upper semi-continuous decompositions of a semigroup'. *Duke Math. J.* 27 (1960), 283—290.

[10] R. P. Hunter, 'Note on arcs in semigroups'. *Fund. Math.* XLIX (1961), 23—245.

[11] R. P. Hunter, 'On the semigroup structure of continua'. *Trans. A.M.S.* 93, (1959), 356—368.

[12] R. P. Hunter, 'Certain homomorphisms of compact connected semigroups'. *Duke Math. J.* 28, 1 (1961), 83—88.

[13] R. P. Hunter and N. J. Rothman, 'Characters and cross-sections in semigroups'. *Duke Math. J.*, 29 (1962), 347—366.

[14] W. M. Faucatt, 'Compact semigroups irreducibly connected between two idempotents'. *Proc. Amer. Math. Soc.*, 6 (1955), 741—747.

[15] R. J. Koch, 'Arcs in partially ordered spaces'. *Pacific Journal of Math.* 9 (1959), 723—728.

[16] R. J. Koch, 'Ordered semigroups in partially ordered semigroups'. *Pacific Journal of Math.* 10 (1960), 1333—1338.

[17] R. J. Koch, 'Threads in compact semigroups'. *Math. Zeit.* 86 (1964), 312—316.

[18] R. J. Koch and A. D. Wallace, 'Stability in semigroups'. *Duke Math. J.* 24 (1957), 193—196.

[19] P. S. Mostert and A. L. Shields, 'One parameter semigroups in a semigroup'. *Trans. A.M.S.* 96 (1960), 510—517.

[20] P. S. Mostert and A. L. Shields, 'On the structure of semigroups on a compact manifold with boundary'. *Annals of Math.* 65 (1957), 117—143.

[21] J. Ursell, 'On two problems of Mostert and Shields'. *Proc. A.M.S.* 14 (1963), 633.

[22] A. D. Wallace, 'The Structure of topological semigroups'. *Bull. A.M.S.* 61 (1955), 95—112.

[23] G. T. Whyburn, 'Analytic topology', *Colloquium publications, A.M.S.* (1951).

[24] L. Anderson and R. P. Hunter, 'The $\mathfrak{H}$-equivalence in compact semigroups'. *Bulletin Soc. Math. Belgique XIV* (1962), 274—296.

[25] D. Montgomery and L. Zippin, *Topological transformation groups.* (Interscience, New York 1955).