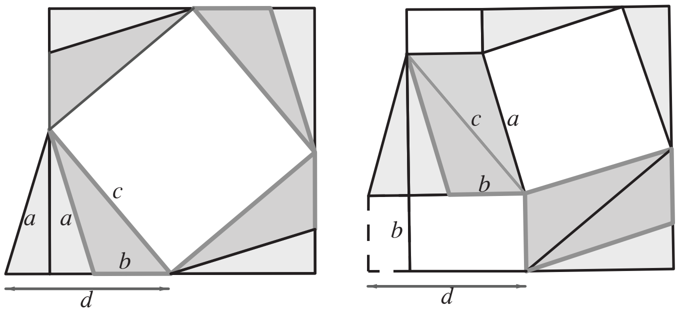

FIGURE 2

Acknowledgements

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Reference

1. F. Laudano, $c^2 = a^2 + bd$, a visual extension of the Pythagorean theorem, *Math. Gaz.* **105** (November 2021) pp. 520-521.

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107.39 Enomoto's problem in Wasan geometry

Japanese mathematics developed in the Edo era (1603-1868) is called *Wasan*. In this Note we consider a problem in Wasan geometry that appeared in a sangaku, which is a framed wooden board with geometric problems written on it. The figures of the problems were beautifully drawn in colour and the board was dedicated to a shrine or a temple. Today, sangaku is an iconic word for Wasan geometry. For a brief introduction of Wasan geometry, see [1]. In this Note, we consider the sangaku problem proposed by Enomoto (榎本信房) in 1807 [2], which can be stated as follows (see Figure 1):

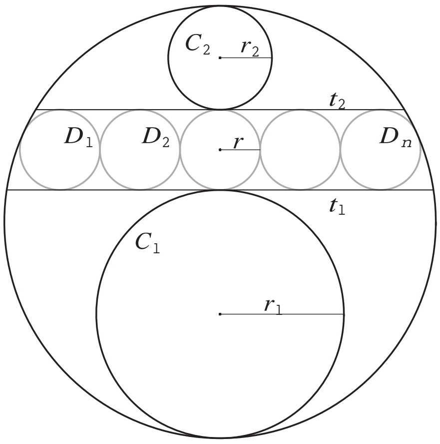

FIGURE 1: $4r_1r_2 = (n - 1)^2 r^2$

Problem 1: Let $D_1, D_2, \ldots, D_n$ be a chain of circles of radius r touching two parallel lines t_1 and t_2. A circle C_i of radius r_i touches t_i from the side opposite to t_j for $\{i, j\} = \{1, 2\}$ so that the line joining the centres of C_1 and C_2 is the perpendicular bisector of the segment joining the centres of D_1 and D_n. If a circle touches C_1, C_2, D_1 and D_n internally, then show that the following relation holds:

$$4r_1r_2 = (n - 1)^2 r^2. \tag{1}$$

Problem 1 can be generalised as follows.

Theorem 1: For a semicircle γ of diameter AB, let δ be a circle of centre H touching γ. If D is the foot of the perpendicular from H to the line AB and the two tangents of δ parallel to DH meet AB in points E and F so that $\overrightarrow{AB}$ and $\overrightarrow{EF}$ have the same direction, then the following statements hold.

(i) If δ touches γ internally, then $|AE||BF| = |DH|^2$.

(ii) If δ touches γ externally, then $|AF||BE| = |DH|^2$.

Proof: Assume that $r > 0$ and the points A, B, E and F have coordinates $(-r, 0)$, $(r, 0)$, $(2e, 0)$ and $(2f, 0)$, respectively, and C is the centre of γ, i.e., the origin.

Then D has coordinates $(e + f, 0)$, $|CD| = |e + f|$ and δ has radius $f - e$. If δ touches γ internally (see Figures 2 and 3), we get $|CH| = |r - (f - e)|$ and

$$|AE||BF| = |-r - 2e||r - 2f| = |CH|^2 - |CD|^2 = |DH|^2$$

by the right triangle CHD. This proves (i). The part (ii) is proved similarly, where we use $|CH| = |r + (f - e)|$ (see Figure 4).

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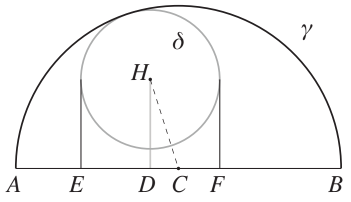

FIGURE 2: $|AE|\,|BF| = |DH|^2$

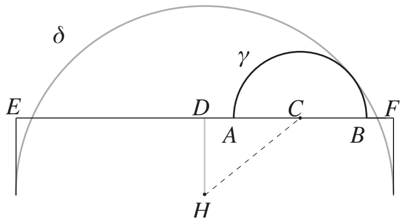

FIGURE 3: $|AE|\,|BF| = |DH|^2$

The theorem shows that the products $|AE|\,|BF|$ and $|AF|\,|BE|$ are constant if the segment DH and the circle δ are fixed while the points A and B vary. Problem 1 and its solution (1) are obtained if $|DH| = (n - 1)r$ in (i). The solution to this problem cited in both [3] (as Problem 4.9.2) and [4] (as Problem 8.9.3) states $r_1 r_2 = \left(\frac{1}{2}(2n - 1)\right)^2 r^2$, which is incorrect by (1). If the circle δ degenerates to the point H, we get the relation $|AD|\,|BD| = |DH|^2$, which shows the unsigned power of the point D with respect to the circle γ (see Figure 5). Therefore Theorem 1 is also a generalisation of this relation.

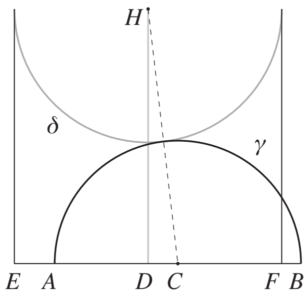

FIGURE 4: $|AF|\,|BE| = |DH|^2$

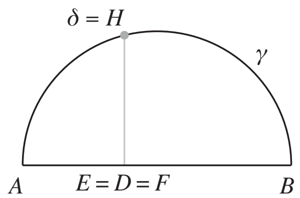

FIGURE 5: $|AD|\,|BD| = |DH|^2$

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References

1. H. Okumura, Wasan Geometry. In: B. Sriraman (eds), *Handbook of the Mathematics of the Arts and Sciences*, Springer Cham, 2020.
2. Nakamura (ed.), Saishi Shinsan, 1830, Tohoku University Digital Collection.
3. H. Fukagawa, J. Rigby, *Traditional Japanese Mathematics Problems of the 18th & 19th Centuries*, SCT publishing, Singapore, 2002.
4. H. Fukagawa, D. Sokolowsky, *Japanese mathematics; How many problems can you solve?* (Vol. 2) Morikita Shuppan 1994 (in Japanese).

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