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SATURATION AND INVERSE THEOREMS FOR COMBINATIONS OF A CLASS OF EXPONENTIAL-TYPE OPERATORS

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1. Introduction. Rates of convergence, saturation theorems and the so-called "inverse problems" for Bernstein polynomials have been intensively studied (see, e.g., [**1**; **4**; **8**; **14**; **17**]). The same problems for some other positive operators have also been investigated by many authors. In this paper, we shall use a uniform approach to study the saturation and inverse problems for a class of linear combinations of operators including Bernstein polynomials, and Szász, Post-Widder, Gauss-Weierstrass and Baskakov operators.

In the literature, most saturation and inverse theorems for operators are on positive operators. Because of the Korovkin theorem, the optimal rate of convergence for a positive operator cannot be faster than that of C^2 functions. Therefore, the saturation classes for positive operators would generally contain functions with smoothness up to having second derivatives. In order to obtain more efficient approximation operators, one has to consider non-positive linear operators. In one of our previous papers [**9**], Ditzian and the author proved a saturation theorem for $B_n(f, k, t)$, a linear combination of Bernstein polynomials defined by Butzer [**6**]. In the present paper, we shall study a more general combination, which includes the combination discussed by Butzer. This new combination also answers a question of his [**6**] (c.f. Remark 2.3 later).

A saturation theorem for a class of operators under this new combination is proved in this paper. We would like to point out that for some operators of the class, e.g., the Post-Widder operator, no saturation theorems even for the positive case, i.e., without combinations, were known. Moreover, if we apply the saturation theorem to some known cases, e.g., the theory of Baskakov operators, stronger results may be achieved (c.f., Remark 9.7).

We also prove an inverse theorem for these combinations. For Bernstein polynomials (without combinations), a global inverse theorem was proved by Berens and Lorentz [**4**]. Our theorem is a local theorem for the aforementioned combinations. This result may apply to some other operators of the class for which no inverse theorems were known.

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2. The main results. We shall first define a few notations.

Definition 2.1. Let $S_\lambda(f, t) = \int_A^B W(\lambda, t, u)f(u)du$, where $W(\lambda, t, u) \geqq 0$ is a kernel of distribution, be a positive operator on $C(A, B)$ (A, B may be $\pm\infty$) into C^∞. $S_\lambda(\cdot, t)$ is said to be an *exponential-type operator* if

$$(1) \qquad \int_A^B W(\lambda, t, u)du = 1; \quad \text{and}$$

$$(2) \qquad \frac{\partial}{\partial t} W(\lambda, t, u) = \frac{\lambda}{p(t)} W(\lambda, t, u)(u - t),$$

where $p(t)$ is a polynomial of degree $\leqq 2$, $p(t) > 0$ on (A, B).

It is said to be *regular* if it further satisfies

$$(3) \qquad \int_A^B W(\lambda, t, u)dt = a(\lambda),$$

where $a(\lambda)$ is a rational function of λ, $a(\lambda) \to 1$ as $\lambda \to \infty$.

Note that the restrictions on $p(t)$ are not essential, but it is the simplest case that covers all the operators that we would like to discuss in this paper. On the other hand, as we shall see in Proposition 3.1, S_λ maps polynomials to polynomials with the same degree if and only if $p(t)$ is a polynomial of degree less than or equal to 2.

Definition 2.2. Let $d_0, d_1, \ldots, d_k$ be $k + 1$ arbitrary but fixed distinct positive integers. The operator $S_\lambda(f, k, t)$ ($\equiv S_\lambda(f, k, t; d_0, d_1, \ldots, d_k)$) is a linear combination of $S_{d_j\lambda}(f, t)$, defined by

$$(2.1) \qquad S_\lambda(f, k, t) = \sum_{j=0}^{k} c(j, k) S_{d_j\lambda}(f, t),$$

where

$$(2.2) \qquad c(j, k) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}, \quad k \neq 0, \quad \text{and} \quad c(0, 0) = 1.$$

Remark 2.3. (1) The operator $B_n(f, k, t)$ investigated by Butzer [**6**] is a special case of the operator defined in the above definition. In fact, in this case, $B_n(f, k, t) = S_n(f, k, t; 1, 2, 2^2, \ldots, 2^k)$, and $S_n(f, t) = B_n(f, t)$, the Bernstein polynomials.

(2) In [**6**] Butzer also asked if it is possible to define a linear combination $\bar{B}_n(f, k, t)$ of the Bernstein polynomials $B_n(f, t)$, such that $\bar{B}_n(f, k, t)$ and $B_n(f, k, t)$ are polynomials of the same degree but $\bar{B}_n(f, k, t)$ has faster rate of convergence than $B_n(f, k, t)$. His question can be answered by considering the combination $S_n(f, k, t)$ with $S_n(f, t) = B_n(f, t)$ for some properly chosen $d_0, d_1, \ldots, d_k$.

For example, when $d_j = j + 1$, $S_n(f, 2^k - 1, t)$ is a polynomial of degree $2^k n$, the same degree as $B_n(f, k, t)$. But we shall see in Proposition 3.6 that if $f^{(2^{k+1})}(t)$ exists then $S_n(f, 2^k - 1, t)$ converges to $f(t)$ at the rate of n^{-2^k}, while $B_n(f, k, t)$ converges to $f(t)$ at the rate $n^{-(k+1)}$. Therefore, we give a positive answer to Butzer's problem.

We are indebted to the referee for bringing the reference [**11**] to our attention, and for pointing out that the combination $S_\lambda(f, k, t)$ defined by (2.1) also covers the combination constructed by M. Frentiu in [**11**]. Frentiu's combination equals $S_\lambda(f, k, t)$ when $d_j = j + 1$. Therefore his result also answers Butzer's problem.

Definition 2.4. Let $\omega_k(f, h; a, b) = \sup\{|\Delta_t^k f(x)|; |t| \leqq h, x, x + kt \in [a, b]\}$ be the modulus of continuity. Then the *generalized Zygmund class* $\mathrm{Liz}(\alpha, k; a, b)$ is the class of functions such that $\omega_{2k}(f, h; a, b) \leqq Mh^{\alpha k}$.

Note that, when $k = 1$, $\mathrm{Liz}(\alpha, 1)$ reduces to the Zygmund class Lip $^*\alpha$.

Let $\Psi(x) > 0$ be a continuous function on (A, B) such that $S_\lambda(\Psi^2, t) < \infty$. Ψ is called a "growth-test function". In many examples, Ψ can be chosen as $Ne^{N|x|}$ for any $N > 0$. We shall see from Proposition 3.2 that Ψ can be chosen at least as $N(1 + x^2)^N$ for any $N > 0$. (In a later result of M. Ismail and the author, we proved that Ψ can always be chosen as $Ne^{N|x|}$.) Let $C_\Psi(A, B) = \{f \in C(A, B); |f(t)| \leqq M\Psi(t)$ for some $M > 0\}$, equipped with the norm $||f||_{C_\Psi} = \sup_{t \in (A,B)} |f(t)|\Psi^{-1}(t)$. In the following two theorems, assume $f \in C_\Psi(A, B)$ for some growth-test function Ψ, and $A < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < B$.

THEOREM 2.5 (Inverse theorem). *Let* $0 < \alpha < 2$. *If* $S_\lambda(f, k, t)$, *defined by* (2.1), *is a linear combination of exponential operators* $S_{d_j\lambda}(f, t)$, *then in the following, the implications* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *hold.*

(1) $||S_{\lambda_n}(f, k, t) - f(t)||_{C[a_1,b_1]} = O(\lambda_n^{-\alpha(k+1)/2})$, $\lambda_{n+1}/\lambda_n \leqq c$ *for some* $c > 0$;

(2) $f \in \mathrm{Liz}(\alpha, k + 1; a_2, b_2)$;

(3) (a) *For* $m < \alpha(k + 1) < m + 1, m = 0, 1, 2, \ldots, 2k + 1 : f^{(m)}$ *exists and* $f^{(m)} \in \mathrm{Lip}(\alpha(k + 1) - m; a_2, b_2)$,

(b) *For* $\alpha(k + 1) = m + 1, m = 0, 1, 2, \ldots, 2k : f^{(m)}$ *exists and* $f^{(m)} \in \mathrm{Lip}^*(1; a_2, b_2)$;

(4) $||S_\lambda(f, k, t) - f(t)||_{C[a_3,b_3]} = O(\lambda^{-\alpha(k+1)/2})$.

In the case $\alpha = 2$, we have the following result.

THEOREM 2.6 (Saturation Theorem). *Let* $S_\lambda(f, k, t)$ *be defined by* (2.1), *where* $S_\lambda(f, t)$ *are regular exponential-type operators. Denote* $I(f, \lambda, k, a, b) = \lambda^{k+1}||S_\lambda(f, k, t) - f(t)||_{C[a,b]}$. *Then in the following the implications* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *and* (4) $\Rightarrow$ (5) $\Rightarrow$ (6) *hold.*

(1) $I(f, \lambda_n, k, a_1, b_1) = O(1)$; $\lambda_{n+1}/\lambda_n \leqq c$;

(2) $f^{(2k+1)} \in A.C.[a_2, b_2]$ *and* $f^{(2k+2)} \in L_\infty[a_2, b_2]$;

(3) $I(f, \lambda, a_3, b_3) = O(1)$;

(4) $I(f, \lambda_n, k, a_1, b_1) = o(1)$; $\lambda_{n+1}/\lambda_n \leqq c$
(5) $f \in C^{2k+2}[a_2, b_2]$ *and* $\sum_{i=k+1}^{2k+2} Q(i, k, t) f^{(i)}(t) = 0, t \in [a_2, b_2]$,
where $Q(i, k, t)$ *are polynomials depending on* k;
(6) $I(f, \lambda, k, a_3, b_3) = o(1)$.

We remark that the restriction $\lambda_{n+1}/\lambda_n \leqq c$ on λ_n for the saturation theorem is only technical while for the inverse theorem it is essential.

Moreover, we believe that the saturation theorem also holds for all exponential operators.

3. Preliminaries. Let $S_\lambda(\cdot, t)$ be an exponential operator. We shall show that $S_\lambda(\cdot, t)$ and hence $S_\lambda(\cdot, k, t)$ are indeed approximation processes for functions f bounded by some growth-test function Ψ.

PROPOSITION 3.1. *If* $e_1(x) = x$ *and* $e_2(x) = x^2$, *then*

$$S_\lambda(e_1, t) = t, \quad \textit{and} \tag{3.1}$$

$$S_\lambda(e_2, t) = t^2 + p(t)/\lambda. \tag{3.2}$$

In general, if f *is a polynomial, then* $S_\lambda(f, \cdot)$ *is also a polynomial of the same degree.*

Proof. Differentiating the identity $\int_A^B W(\lambda, t, u)du = 1$ on both sides, using the relation (2) of Definition 2.1, we obtain

$$\int_A^B W(\lambda, t, u)(u - t)du = 0, \quad \text{or} \quad \int_A^B W(\lambda, t, u)u\, du = t. \tag{3.3}$$

Similarly, differentiating (3.3), we obtain $\int_A^B W(\lambda, t, u)(u^2 - ut)du = p(t)/\lambda$. Relation (3.2) then follows by the linearity of integration and (3.1).

To prove the last statement of the proposition, let

$$A_m(\lambda, t) = \lambda^m \int_A^B W(\lambda, t, u)(u - t)^m du. \tag{3.4}$$

Then the property follows by the recursion relation

$$A_{m+1}(\lambda, t) = \lambda m p(t) A_{m-1}(\lambda, t) + p(t) \frac{d}{dt} A_m(\lambda, t), \tag{3.5}$$

the relations (3.1) and (3.2) and induction.

PROPOSITION 3.2. *Let* $A_m(\lambda, t)$ *be defined in* (3.4). *We have*
(1) $A_m(\lambda, t)$ *is a polynomial in* t *and* λ;
(2) *The degree of* $A_m(\lambda, t)$ *in* λ *is* $[m/2]$ *while in* t *is less than or equal to* m;
(3) *The coefficient of* λ^m *in the polynomial* $A_m(\lambda, t)$ *is* $(2m - 1)!!p(t)^m$, *while in the polynomial* $A_{2m+1}(\lambda, t)$ *is* $c_m p(t)^m p'(t)$ *for some constant* c_m, *where* $a!!$ *is the semifactorial of* a.

Proof. This proposition follows from the recursion relation (3.5), Proposition 3.1 and induction.

COROLLARY 3.3. *For any* $\delta > 0$ *and* $m > 0$, $A < a < b < B$, *we have,*

$$\int_{|u-t|\geqq\delta} W(\lambda, t, u)du = O(\lambda^{-m}), \quad \lambda \to +\infty, \tag{3.6}$$

uniformly on $[a, b] \subset (A, B)$.

Proof. Without loss of generality, we may assume m is an integer. Then (3.6) follows from the following estimate and the above proposition:

$$\int_{|u-t|\geqq\delta} W(\lambda, t, u)du \leqq \delta^{-2m} \int_{|u-t|\geqq\delta} W(\lambda, t, u)(u-t)^{2m}du \leqq \delta^{-2m}\lambda^{-2m}A_{2m}.$$

Using the above properties of $S_\lambda(\cdot, t)$ and the Cauchy-Schwarz inequality, the following proposition can be easily proved.

PROPOSITION 3.4. *If* $|f(t)| \leqq \Psi(t)$ *for some growth-test function* Ψ, *then the relation*

$$\lim_{\lambda\to\infty} S_\lambda(f, t) = f(t) \tag{3.7}$$

holds at each point of continuity of f. *If* f *is continuous on* $[a, b]$, *then* (3.7) *holds uniformly on every interial interval* $[a_1, b_1] \subset (a, b)$.

The following lemma concerning the properties of $c(j, k)$, the coefficients in $S_\lambda(f, k, t)$, is in fact well-known. For completeness, a short proof will be given.

LEMMA 3.5. *If* $c(j, k)$, $j = 0, 1, \ldots, k$ *are defined as in* (2.2), *then*

$$\sum_{j=0}^{k} c(j, k)d_j^{-m} = \begin{cases} 1 & m = 0 \\ 0 & m = 1, 2, \ldots, k. \end{cases} \tag{3.8}$$

Proof. Consider the Lagrange polynomial $L_{k,m}(x) = \sum_{i=0}^{k} x_i^m l_i(x) - x^m$, where

$$l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{k} \frac{x_j - x}{x_j - x_i}, \quad x_i = d_i^{-1}.$$

Since $L_{k,m}(x_i) = 0$ for $i = 0, 1, \ldots, k$, and $L_{k,m}(x)$ is a polynomial of degree k, we have $L_{k,m}(x) \equiv 0$. In particular,

$$\sum_{i=0}^{k} c(i, k)\frac{1}{d_i^m} = \begin{cases} L_{k,m}(0) = 0, & m \neq 0, \\ L_{k,m}(0) + 1 = 1, & m = 0. \end{cases}$$

Consequently, Proposition 3.4 also holds for $S_\lambda(f, k, t)$. Moreover, we have the following asymptotic relation.

PROPOSITION 3.6. *Let* $|f(t)| \leqq \Psi(t)$ *for some growth-test function* Ψ. *If* $f^{(2k+2)}(t)$ *exists, then*

$$(3.9)\qquad \lambda^{k+1}[S_\lambda(f,k,t) - f(t)] = \sum_{j=k+1}^{2k+2} Q(j,k,t)f^{(j)}(t) + o(1),$$

where $Q(j,k,t)$ *are polynomials in* t. *Moreover*, $Q(2k+2,k,t) = c_1 p(t)^{k+1}$, $Q(2k+1,k,t) = c_2 p(t)^k p'(t)$.

If $f \in C^{2k+2}[a,b]$, *then* (3.9) *is uniform in every interial interval* $[a_1,b_1] \subset (a,b)$.

As a special case, when $k = 0$, the above proposition reduces to the following:

COROLLARY 3.7 (Voronovskaja-type relation). *Let* $|f| \leqq \Psi$. *If* $f''(t)$ *exists, then*

$$(3.10)\qquad \lim_{\lambda\to\infty} \lambda[S_\lambda(f,t) - f(t)] = \tfrac{1}{2}p(t)f''(t).$$

Proof of Proposition 3.6. Suppose $f^{(2k+2)}(t)$ exists. By Taylor's expansion of f, we have

$$(3.11)\qquad \begin{aligned}\lambda^{k+1}[S_\lambda(f,k,t) - f(t)] &= \lambda^{k+1}\sum_{j=0}^{k+1} c(j,k)\int_A^B W(d_j\lambda,t,u)\\ &\quad\cdot\left[\sum_{m=1}^{2k+2}\frac{f^{(m)}(t)}{m!}(u-t)^m + \epsilon(u,t)(u-t)^{2k+2}\right]du,\end{aligned}$$

where $\epsilon(u,t) \to 0$ as $u \to t$; also, it is obvious that for some $M > 0$, $|\epsilon(u,t)(u-t)^{2k+2}| \leqq M(1+u^2)^{k+1}\Psi(u)$ for all u.

Using Proposition 3.2 and Lemma 3.5, we obtain the dominated part of (3.11)

$$(3.12)\qquad \begin{aligned}\lambda^{k+1}\sum_{j=0}^{k+1} c(j,k)\int_A^B W(d_j\lambda,t,u)\sum_{m=1}^{2k+2}\frac{f^{(m)}(t)}{m!}(u-t)^m du\\ = \sum_{j=0}^{2k+2} Q(j,k,t)f^{(j)}(t) + o(1).\end{aligned}$$

The remaining part can be estimated as follows: For any $\epsilon > 0$, let $\delta > 0$ be such that $|\epsilon(u,t)| < \epsilon$, whenever $|u-t| \leqq \delta$. Then

$$(3.13)\qquad \left|\int_A^B W(\lambda,t,u)\epsilon(u,t)(u-t)^{2k+2}du\right| = \int_{|u-t|<\delta} + \int_{|u-t|\geqq\delta} = I_1 + I_2.$$

By the Cauchy-Schwarz inequality and Corollary 3.3,

$$|I_2| = o(\lambda^{-(k+1)}).$$

On the other hand,

$$|I_1| = O(\lambda^{-(k+1)})\cdot\sup_{|u-t|<\delta}|\epsilon(u,t)| \leqq \epsilon\cdot O(\lambda^{-(k+1)}).$$

Since $\epsilon > 0$ is arbitrary, these estimates together with (3.12) prove (3.9).

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When $f \in C^{2k+2}[a, b]$, $f^{(2k+2)}$ is uniformly continuous in $[a, b]$. Let $[a_1, b_1] \subset (a, b)$. We observe for $t \in [a_1, b_1]$: (1) the $o(1)$ term in (3.12) is uniform (in fact, it is equal to a polynomial in λ^{-1} and t); (2) in estimating (3.13), we can choose $\delta \leqq \min\{a_1 - a, b - b_1\}$ independent of t, by the uniform continuity of $f^{(2k+2)}$ in $[a, b]$, and so the uniformity of (3.9) follows.

4. The space $C_0(\alpha, k; a, b)$**.** We shall prove Theorem 2.5 in this and the two following sections. The equivalence of (2) and (3) of the theorem is well known (c.f., [**20**, pp. 257, 333 and 337]). The rest of the proof will be divided into two parts. We first prove the special case when f is of compact support strictly contained inside the open interval (a, b), and then pass to the general case. We shall show, under the restriction that $\operatorname{supp} f \subset (a, b)$, that the conditions (1) and (2) in the theorem, with (a_i, b_i) being replaced by (a, b), are both equivalent to the fact that f belongs to an intermediate space $C_0(\alpha, k; a', b')$, which will be defined in the following.

Definition 4.1. Let $[a, b]$ be a fixed interval, and let $[a', b'] \subset (a, b)$. Denote $\mathscr{G} = \{g; g \in C_0^{2k+2}, \operatorname{supp} g \subset [a', b']\}$. For functions $f \in C_0$ with $\operatorname{supp} f \subset [a', b']$, define

$$K(\xi, f) = \inf_{g \in \mathscr{G}} \{||f - g|| + \xi(||g|| + ||g^{(2k+2)}||)\}$$

where $0 < \xi \leqq 1$, and the norms are the supremum-norms on $[a', b']$. A function $f \in C_0$ with $\operatorname{supp} f \subset [a', b']$ is said to belong to $C_0(\alpha, k + 1; a', b')$, $0 < \alpha < 2$ if $||f||_\alpha \equiv \sup_{0<\xi\leqq 1} \xi^{-\alpha/2} K(\xi, f) < \infty$.

For other properties and further discussion of J. Peetre's K-functionals, see, for example, [**7**].

We begin with some estimates involving the K-functions defined above.

Lemma 4.2. *Let* $a < a' < a'' < b'' < b' < b$. *If* $f \in C_0$ *with* $\operatorname{supp} f \subset [a'', b'']$, *and* $||S_{\lambda_n}(f, k, t) - f(t)||_{C[a,b]} \leqq M\lambda_n^{-\alpha(k+1)/2}$, *where* $\lambda_{n+1}/\lambda_n \leqq c$, *then*

$$(4.1) \qquad K(\xi, f) \leqq M_0[\lambda^{-\alpha(k+1)/2} + \lambda^{k+1}\xi K(\lambda^{-(k+1)}, f)].$$

Proof. First note that it is enough to show (4.1) when λ is replaced by λ_n. That is,

$$(4.1') \qquad K(\xi, f) \leqq M_0[\lambda_n^{-\alpha(k+1)/2} + \lambda_n^{k+1}\xi K(\lambda_n^{-(k+1)}, f)].$$

This is because for each λ we can choose λ_{n-1} and λ_n such that $\lambda_{n-1} < \lambda \leqq \lambda_{n+1}$, and (4.1′) yields

$$(4.2) \qquad K(\xi, f) \leqq M_0[\lambda^{-\alpha(k+1)/2} + c^{k+1}\lambda^{k+1}\xi K(\lambda^{-(k+1)}, f)]$$
$$\leqq M'_0[\lambda^{-\alpha(k+1)/2} + \lambda^{(k+1)}\xi K(\lambda^{-(k+1)}, f)].$$

Next, since supp $f \subset [a'', b'']$, there exists an $h \in \mathcal{G}$ such that

$$\left\| h^{(i)}(t) - \frac{d^i}{dt^i} S_{\lambda_n}(f, k, t) \right\|_{C[a,b]} \leqq M' \lambda_n^{-(k+1)},$$

$i = 0$ and $2k + 2$ [Corollary 3.3].

Therefore,

$$K(\xi, f) \leqq 3M'\lambda_n^{-(k+1)} + ||f(t) - S_{\lambda_n}(f, k, t)||_{C[a,b]} + \xi\left[||S_{\lambda_n}(f, k, t)||_{C[a,b]} + \left\| \frac{d^{2k+2}}{dt^{2k+2}} S_{\lambda_n}(f, k, t) \right\|_{C[a,b]} \right].$$

Hence it is sufficient to show that there exists an M, such that, for each $g \in \mathcal{G}$,

$$\left\| \frac{d^{2k+2}}{dt^{2k+2}} S_\lambda(f, k, t) \right\|_{C[a,b]} \leqq M\lambda^{k+1}\{||f - g|| + \lambda^{-(k+1)}||g^{(2k+2)}||\}.$$

In fact, we have

$$\left\| \frac{d^{2k+2}}{dt^{2k+2}} S_\lambda(f, k, t) \right\|_{C[a,b]}$$

$$\leqq \sum_{j=0}^{k} |c(j, k)| \left\| \frac{d^{2k+2}}{dt^{2k+2}} \int W(d_j\lambda, t, u)[f(u) - g(u)]du \right\|_{C[a,b]} \tag{4.3}$$

$$+ \sum_{j=0}^{k} |c(j, k)| \left\| \frac{d^{2k+2}}{dt^{2k+2}} \int W(d_j\lambda, t, u)g(u)du \right\|_{C[a,b]} \equiv I_1 + I_2.$$

We first estimate I_1. From the relation (2) of Definition 2.1, we can easily obtain by induction that

$$\frac{\partial^{2m}}{\partial t^{2m}} W(\lambda, t, u) = \sum_{i=0}^{m} \lambda^{m+i} W(\lambda, t, u)(u - t)^{2i} q_{i,2m}(u, t)$$

$$\frac{\partial^{2m+1}}{\partial t^{2m+1}} W(\lambda, t, u) = \sum_{i=0}^{m} \lambda^{m+i+1} W(\lambda, t, u)(u - t)^{2i+1} q_{i,2m+1}(u, t) + \lambda^m W(\lambda, t, u) q_{2m+1}(u, t) \tag{4.4}$$

where $q_{i,j}(u, t)$ and $q_{2m+1}(u, t)$ are polynomials in u (and $1/\lambda$) and are bounded with respect to t for $t \in [a, b]$. Hence, we have [by Proposition 3.2],

$$\left\| \frac{d^{2k+2}}{dt^{2k+2}} \int_A^B W(d_j\lambda, t, u)(f(u) - g(u))du \right\|_{C[a,b]} \leqq M_j\lambda^{k+1}||f - g|| \tag{4.5}$$

(supp $f \cup$ supp $g \subset [a, b]$), where M_j is independent of g.

To estimate I_2, first notice that $\int_A^B W(\lambda, t, u)u^i du$ is a polynomial in t with degree i [Proposition 3.1], it follows that $d^k/dt^k \int W(\lambda, t, u)u^i du = 0$ for $k > i$. Therefore, as a linear combination of these equations, we have

$$\int_A^B \left[\frac{\partial^k}{\partial t^k} W(\lambda, t, u) \right] (u - t)^i du = 0 \quad \text{for } k > i. \tag{4.6}$$

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Now for $g \in C_0^{2k+2}$, we first form

$$\frac{d^{2k+2}}{dt^{2k+2}} \int_A^B W(\lambda, t, u) g(u) du = \int_A^B \left[\frac{\partial^{2k+2}}{\partial t^{2k+2}} W(\lambda, t, u)\right] g(u) du.$$

Then for the Taylor expansion

$$g(u) = \sum_{l=0}^{2k+1} \frac{g^{(l)}(t)}{l!} (u - t)^l + \frac{g^{(2k+2)}(\xi)}{(2k+2)!} (u - t)^{2k+2},$$

and using (4.6), we have

$$\left\|\frac{\partial^{2k+2}}{\partial t^{2k+2}} S_\lambda(g, t)\right\|_{C[a,b]}$$
$$\leqq \frac{1}{(2k+2)!} \|g^{(2k+2)}\| \cdot \left\| \int \left|\frac{\partial^{2k+2}}{\partial t^{2k+2}} W(\lambda, t, u)\right| (u - t)^{2k+2} du \right\|_{C[a,b]}.$$

Applying (4.4) and using Proposition 3.2, we calculate that

$$\left\|\frac{d^{2k+2}}{dt^{2k+2}} S_\lambda(g, t)\right\|_{C[a,b]} \leqq M\|g^{(2k+2)}\|.$$

Substituting into (4.3), we see (4.1) is true.

LEMMA 4.3. *Under the same assumption as in Lemma 4.2, there holds* $K(\xi, f) \leqq M'\xi^{\alpha/2}$, *i.e.*, $f \in C_0(\alpha, k+1; a', b')$.

The proof of this lemma is standard and can be found in [**4**].

The previous two lemmas give one direction of the following theorem.

THEOREM 4.4. *Let* $a < a' < a'' < b'' < b' < b$. *If* $f \in C_0$ *with* $\operatorname{supp} f \subset [a'', b'']$, *then*

$$\|S_{\lambda_n}(f, k, t) - f(t)\|_{C[a,b]} \leqq M\lambda_n^{-\alpha(k+1)/2}, \quad \lambda_{n+1}/\lambda_n \leqq c$$

implies $f \in C_0(\alpha, k+1; a', b')$, *and the latter implies*

$$\|S_\lambda(f, k, t) - f(t)\|_{C[a,b]} \leqq M\lambda^{-\alpha(k+1)/2}.$$

Proof. It remains to show the second implication. For this, it is enough to show $\|S_\lambda(f, k, t) - f(t)\|_{C[a,b]} \leqq MK(\lambda^{-(k+1)}, f)$.

For $g \in \mathcal{G}$, we have

$$\|S_\lambda(f, k, t) - f(t)\|_{C[a,b]} \leqq \|S_\lambda(f - g, k, t)\|_{C[a,b]} + \|S_\lambda(g, k, t) - f(t)\|_{C[a,b]}.$$

Clearly, the first term is bounded by $M_1\|f - g\|$ since $\operatorname{supp}(f - g) \subset [a, b]$. To estimate the second term, expand $g(u)$ by Taylor's Formula, and use

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Proposition 3.2 and Lemma 3.5; we see it is bounded by

$$||g - f|| + M_2\lambda^{-(k+1)} \sum_{m=k+1}^{2k+2} ||g^{(m)}|| \leqq ||g - f|| + M_2'\lambda^{-(k+1)}(||g|| + ||g^{(2k+2)}||)$$

where M' is an absolute constant. In other words,

$$||S_\lambda(f, k, t) - f(t)||_{C[a,b]} \leqq MK(\lambda^{-(k+1)}, f).$$

5. Proof of Theorem 2.5 when supp $f \subset (a, b)$. In this section we shall show that $f \in C_0(\alpha, k + 1; a', b')$ is equivalent to $f \in \mathrm{Liz}(\alpha, k + 1; a, b)$ for functions satisfying $\mathrm{supp} f \subset [a'', b'']$. In combination with the result of Section 4, this yields that $||S_\lambda(f, k, t) - f(t)||_{C[a,b]} = O(\lambda^{-\alpha(k+1)/2})$ is equivalent to $f \in \mathrm{Liz}(\alpha, k + 1; a, b)$ whenever $\mathrm{supp} f \subset [a'', b'']$.

THEOREM 5.1. *Let $a < a' < a'' < b'' < b' < b$. If $f \in C_0$ with* $\mathrm{supp} f \subset [a'', b'']$, *then $f \in C_0(\alpha, k + 1; a', b')$ if and only if $f \in$* $\mathrm{Liz}(\alpha, k + 1; a, b)$.

This theorem is probably well-known. For completeness, we shall give an outline of the proof:

If $f \in C_0(\alpha, k + 1; a', b')$, in order to show $\omega_{2k+2}(f, h) = O(h^{\alpha(k+1)})$, let $|\delta| < h$ and let $g \in \mathscr{G}$. It is easy to see $|\Delta_\delta^{2k+2}f(t)| \leqq 2^{2k+2}||f - g|| + \delta^{2k+2}||g^{(2k+2)}||$. Hence $|\Delta_\delta^{2k+2}f(t)| \leqq 2^{2k+2}K(\delta^{2k+2}, f) \leqq 2^{2k+2}Mh^{\alpha(k+1)}$. Conversely, assume $f \in \mathrm{Liz}(\alpha, k + 1; a, b)$. It is enough to show $K(\xi, f) \leqq M\xi^{\alpha/2}$ when ξ is sufficiently small. If we define $g_0 \in \mathscr{G}$ by

$$g_0(x) = \frac{1}{\binom{2k+2}{k+1}\eta^{2k+2}} \cdot \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left[(-1)^k \bar{\Delta}^{2k+2}_{\sum_{i=1}^{2k+2} u_i} f(x) + \binom{2k+2}{k+1} f(x) \right] du_1 \ldots du_{2k+2},$$

where $(k + 1)\eta < \min(a'' - a', b' - b'', k + 1)$ and $\bar{\Delta}_h^m$ is the symmetric difference, it is straightforward to show

$$||f - g_0|| \leqq M'\eta^{\alpha(k+1)} \quad \text{and} \quad ||g_0^{(2k+2)}|| \leqq M''\eta^{-(2k+2)}\omega_{2k+2}(f, \eta).$$

From this, it is easy to derive $K(\eta^{2k+2}, f) \leqq M\eta^{(k+1)\alpha}$.

6. Proof of Theorem 2.5—The general case. In this section, we shall prove the Inverse Theorem 2.5 for the general case. The proof will be divided into two parts: The implication (2) to (4) and the implication (1) to (3). The equivalence of (2) and (3) is known.

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6.1. *The implication* (2) $\Rightarrow$ (4). Assume $f \in \text{Liz}(\alpha, k+1; a_2, b_2)$. Choose a', a'', b', b'' in such a way that $a_2 < a' < a'' < a_3 < b'' < b' < b_2$. Let $g \in C_0^\infty$ be such that $g(x) = 1$ for $x \in [a'', b'']$, and $\text{supp}\, g \subset [a', b']$. Then fg has compact support strictly in (a_2, b_2), and $fg \in \text{Liz}(\alpha, k+1; a_2, b_2)$ since f does. Hence by Theorems 4.4 and 5.1,

$$||S_\lambda(fg, k, t) - fg(t)||_{C[a_2,b_2]} = O(\lambda^{-\alpha(k+1)/2}).$$

But, for $t \in [a_3, b_3]$,

$$S_\lambda(fg, k, t) - f(t)g(t) = \sum_{j=0}^{k} c(j,k) \int_{a''}^{b''} W(\lambda, t, u)(f(u) - f(t))du$$
$$+ o(\lambda^{-(k+1)}) = S_\lambda(f, k, t) - f(t) + o(\lambda^{-(k+1)}),$$

where the remainders $o(\lambda^{-(k+1)})$ are uniform for $t \in [a_3, b_3]$ (Corollary 3.3). Consequently, $||S_\lambda(f, k, t) - f(t)||_{C[a_3,b_3]} = O(\lambda^{-\alpha(k+1)/2})$.

6.2. *The implication* (1) $\Rightarrow$ (3). Now assume $||S_{\lambda_n}(f, k, t) - f(t)||_{C[a_1,b_1]} = O(\lambda_n^{-\alpha(k+1)/2})$, $\lambda_n/\lambda_{n-1} < c$. We shall prove the implication by induction on $\tau \equiv \alpha(k+1)$. The induction progresses as follows:

First, we prove it for the case when $0 < \tau \leqq 1$. Then, for any $\delta \in (0, 1)$, we prove the case when $1 - \delta < \tau < 2 - \delta$. In general, assume the proposition holds for $0 < \tau \leqq m - \delta$, $m = 1, 2, \ldots, 2k+1$, $0 < \delta < \frac{1}{2}$, and then prove the case when $m - \delta \leqq \tau < m + 1 - 2\delta$. Since $\delta > 0$ can be chosen arbitrarily small, the proposition holds for all $\tau \in (0, 2k+2)$.

6.2.1. *The case* $0 < \tau \leqq 1$. Let a', a'', b', b'' be chosen so that $a_1 < a' < a'' < a_2$ and $b_2 < b'' < b' < b_1$. Also, let $g \in C_0^\infty$ be such that $\text{supp}\, g \subset [a'', b'']$ and $g(x) = 1$ on $[a_2, b_2]$.

LEMMA 6.1. *Let* g *be chosen as above. If*

$$||S_{\lambda_n}(f, k, t) - f(t)||_{C[a_1,b_1]} = O(\lambda_n^{-\tau/2}), \quad 0 < \tau \leqq 1$$

then

$$||S_{\lambda_n}(fg, k, t) - fg(t)||_{C[a',b']} = O(\lambda_n^{-\tau/2}).$$

Proof. For $t \in [a', b']$, we have

$$S_{\lambda_n}(t, g, k, t) - f(t)g(t)$$
$$= \sum_{j=0}^{k} c(j,k) \int_{a_1}^{b_1} W(d_j\lambda_n, t, u)(f(u)g(u) - f(t)g(t))du + o(\lambda^{-(k+1)})$$
$$(6.1) \qquad = g(t) \sum_{j=0}^{k} c(j,k) \int W(d_j\lambda_n, t, u)[f(u) - f(t)]du$$
$$+ \sum_{j=0}^{k} c(j,k) \int_{a_1}^{b_1} W(d_j\lambda_n, t, u)f(u)(g(u) - g(t))du + o(\lambda_n^{-(k+1)})$$
$$\equiv I_1(t) + I_2(t) + o(\lambda_n^{-(k+1)})$$

where the o-terms are uniform for $t \in [a', b']$ (Corollary 3.3).

The assumption $||S_{\lambda_n}(f, k, t) - f(t)||_{C[a_1, b_1]} = O(\lambda_n^{-\tau/2})$ yields the estimate

$$||I_1(t)||_{C[a',b']} \leqq ||g|| \cdot ||S_{\lambda_n}(f, k, t) - f(t)||_{C[a',b']} \leqq M_1 \lambda_n^{-\tau/2}. \tag{6.2}$$

Next, by the mean value theorem, we reduce $I_2(t)$ to

$$I_2(t) = \sum_{j=0}^{k} c(j, k) \int_{a_1}^{b_1} W(d_j\lambda_n, t, u) f(u) \{g'(\xi)(u - t)\} du.$$

Hence

$$\begin{aligned} ||I_2(t)||_{C[a',b']} &\leqq \sum_{j=0}^{k} |c(j,k)| \\ &\quad \times \left\{ ||g'|| \cdot \left\| \int_{a_1}^{b_1} W(d_j\lambda_n, t, u) \cdot |f(u)|\, |u - t| du \right\|_{C[a',b']} \right\} \\ &\leqq ||f||_{C[a',b']} ||g'|| \cdot \left(\sum_{j=0}^{k} |c(j,k)| \right) \\ &\qquad \times \max_{0 \leqq j \leqq k} \left\| \int W(d_j\lambda_n, t, u) |u - t| du \right\|_{C[a',b']}. \end{aligned} \tag{6.3}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &||I_2(t)||_{C[a',b']} \\ &\leqq ||f||_{C[a',b']} ||g'|| \left(\sum_{j=0}^{k} |c(j,k)| \right) \\ &\quad \times \max_{0 \leqq j \leqq k} \left\| \int W(d_j\lambda_n, t, u)(u - t)^2 du \right\|^{1/2}_{C[a',b']} \\ &\leqq ||f||_{C[a',b']} ||g'|| \left(\sum_{j=0}^{k} |c(j,k)| \right) ||p(t)||^{1/2}{}_{C[a',b']} \lambda^{-1/2} \\ &\qquad = O(\lambda^{-1/2}) \leqq O(\lambda^{-\tau/2}). \end{aligned} \tag{6.4}$$

Combining (6.1), (6.2) and (6.4) we conclude that

$$||S_{\lambda_n}(fg, k, t) - fg(t)||_{C[a',b']} = O(\lambda^{-\tau/2}).$$

LEMMA 6.2. *Let* $a_1 < a_2 < b_2 < b_1$. *If* $||S_{\lambda_n}(f, k, t) - f(t)||_{C[a_1,b_1]} = O(\lambda_n^{-\tau/2})$, $\lambda_{n+1}/\lambda_n \leqq c$, $0 < \tau \leqq 1$, *then* $f \in \mathrm{Lip}(\tau; a_2, b_2)$ *if* $\tau < 1$ *and* $f \in \mathrm{Lip}^*(1; a_2, b_2)$ *if* $\tau = 1$.

Proof. Let a', a'', b', b'' and g be chosen as above. Since $\operatorname{supp} fg \subset [a'', b''] \subset (a', b')$, it follows from Theorems 4.4 and 5.1 and Lemma 6.1 that $fg \in \mathrm{Lip}(\tau; a', b')$ if $\tau < 1$ and $fg \in \mathrm{Lip}^*(1; a', b')$ if $\tau = 1$. Noticing $g(t) = 1$ for $t \in [a_2, b_2]$, this reduces to the required result.

6.2.2. *The induction process*. Assume that the proposition holds for

$0 < \tau \leqq m - \delta (m = 1, 2, \ldots, 2k+1; 0 < \delta < \frac{1}{2})$ and suppose that

$$||S_{\lambda_n}(f, k, t) - f(t)||_{C[a_1,b_1]} \leqq M_\tau \lambda_n^{-\tau/2}$$

where $m - \delta \leqq \tau \leqq m + 1 - 2\delta$, $\lambda_{n+1}/\lambda_n \leqq c$. We must deduce (3) of Theorem 2.5 for that τ.

Let $x_i, y_i, i = 1, 2, 3$ be chosen such that $a_1 < x_i < x_{i+1} < a_2 < b_2 < y_{i+1} < y_i < b_1$. Let $g \in C_0^\infty$ with supp $g \subset (x_3, y_3)$ and $g(x) = 1$ on $[a_2, b_2]$.

LEMMA 6.3. *Let g be chosen as above. If* $||S_{\lambda_n}(f, k, t) - f(t)||_{C[a_1,b_1]} = O(\lambda_n^{-\tau/2})$, *then*

$$||S_{\lambda_n}(fg, k, t) - fg(t)||_{C[x_2,y_2]} = O(\lambda_n^{-\tau/2}),$$

where $m - \delta \leqq \tau \leqq m + 1 - 2\delta$.

Proof. First notice that, by the induction hypothesis, the condition of the lemma with $\tau = m - \delta$ implies that $f^{(m-1)}$ exists and

$$f^{(m-1)} \in \text{Lip}(1 - \delta; x_1, y_1).$$

Next, for $t \in [x_2, y_2]$, we form

$$\begin{aligned} (6.5)\quad & S_{\lambda_n}(fg, k, t) - f(t)g(t) \\ &= \sum_{j=0}^{k} c(j,k) \int W(d_j\lambda_n, t, u) f(u)g(u)du - f(t)g(t) \\ &= \sum_{j=0}^{k} c(j,k) \int_{x_1}^{y_1} W(d_j\lambda_n, t, u)(f(u) - f(t))(g(u) - g(t))du \\ &\quad + \sum_{j=0}^{k} c(j,k) \int W(d_j\lambda_n, t, u)(g(u) - g(t))du \cdot f(t) \\ &\quad + \sum_{j=0}^{k} c(j,k) \int W(d_j\lambda_n, t, u)(f(u) - f(t))du \cdot g(t) + o(\lambda^{-(k+1)}) \\ &\equiv I_1 + I_2 + I_3 + o(\lambda^{-(k+1)}) \end{aligned}$$

where the o-term is uniform for $t \in [x_2, y_2]$ (Corollary 3.3).

The estimates for I_2 and I_3 can be made immediately:

$$(6.6)\qquad ||I_3||_{C[x_2,y_2]} \leqq ||g|| \cdot ||S_{\lambda_n}(f, k, t) - f(t)||_{C[x_2,y_2]} = O(\lambda_n^{-\tau/2})$$

by the assumption of the lemma (and $[x_2, y_2] \subset (a_1, b_1)$); and

$$(6.7)\qquad ||I_2||_{C[x_2,y_2]} \leqq ||f||_{C[x_2,y_2]} ||S_{\lambda_n}(g, k, t) - g(t)||_{C[x_2,y_2]} = O(\lambda_n^{-(k+1)/2}) = O(\lambda_n^{-\tau/2})$$

as $g \in C_0^\infty \subset C^{2k+2}$.

In the estimating of $||I_1||_{C[x_2,y_2]}$, we see that by the induction hypothesis,

$f^{(m-1)}$ exists on $[x_1, y_1]$. So that, for $t \in [x_2, y_2]$, by Taylor's expansion,

$$\begin{aligned}
&\sum_{j=0}^{k} c(j,k) \int_{x_1}^{y_1} W(d_j\lambda_n, t, u)(f(u) - f(t))(g(u) - g(t))du \\
&\quad = \sum_{i=1}^{m-1} \frac{f^{(i)}(t)}{i!} \sum_{j=0}^{k} c(j,k) \\
&\qquad \times \int_{x_1}^{y_1} W(d_j\lambda_n, t, u)(u-t)^i (g(u) - g(t))du \\
&\qquad + \frac{1}{(m-1)!} \sum_{j=0}^{k} c(j,k) \\
&\qquad \times \int_{x_1}^{y_1} W(d_j\lambda_n, t, u)(u-t)^{m-1}(f^{(m-1)}(\xi) - f^{(m-1)}(t)) \\
&\qquad\qquad \cdot g'(\eta)(u-t)du \equiv I_4 + I_5
\end{aligned} \tag{6.8}$$

where ξ and η are between u and t.

Clearly, $||I_4||_{C[x_2,y_2]} = O(\lambda_n^{-(k+1)/2}) = O(\lambda_n^{-\tau/2})$.

Also, since

$$f^{(m-1)} \in \operatorname{Lip}(1-\delta, x_1, y_1), \ |f^{(m-1)}(\xi) - f^{(m-1)}(t)| \leqq M|\xi - t|^{1-\delta} \leqq M|u-t|^{1-\delta}$$

for some $M > 0$. Therefore

$$\begin{aligned}
&||I_5||_{C[x_2,y_2]} \\
&\quad \leqq M||g'|| \cdot \frac{\sum_{j=0}^{k}|c(j,k)|}{(m-1)!} \\
&\qquad \times \max_{0 \leqq j \leqq k} \left\| \int_{x_1}^{y_1} W(d_j\lambda_n, t, u)|u-t|^{m+1-\delta} du \right\|_{C[x_2,y_2]} \\
&\quad \leqq M' \max_{0 \leqq j \leqq k} \left\| \int_{x_1}^{y_1} W(d_j\lambda_n, t, u)|u-t|^{2(m+1)} du \right\|_{C[x_2,y_2]}^{(m+1-\delta)/2(m+1)}
\end{aligned} \tag{6.9}$$

by Jensen's inequality. Thus $||I_5||_{C[x_2,y_2]} = O(\lambda_n^{-(m+1-\delta)/2}) = O(\lambda_n^{-\tau/2})$ by Proposition 3.2.

Combining the above estimates, we obtain the lemma.

LEMMA 6.4. *Let $a_1 < a_2 < b_2 < b_1$. If $||S_{\lambda_n}(f, k, t) - f(t)||_{C[a_1,b_1]} = O(\lambda_n^{-\tau/2})$, $m - \delta < \tau < m + 1 - 2\delta$, then*

(1) *$f^{(m-1)}$ exists, $f^{(m-1)} \in \operatorname{Lip}(\tau - m + 1; a_2, b_2)$, if $m - \delta < \tau < m$;*

(2) *$f^{(m-1)}$ exists, $f^{(m-1)} \in \operatorname{Lip}^*(1; a_2, b_2)$, if $\tau = m$;*

(3) *$f^{(m)}$ exists, $f^{(m)} \in \operatorname{Lip}(\tau - m; a_2, b_2)$, if $m < \tau < m + 1 - 2\delta$.*

Proof. Let $x_i, y_i, i = 1, 2, 3$ and g be defined as in Lemma 6.3. As an intermediate result, Lemma 6.3 yields $||S_{\lambda_n}(fg, k, t) - fg(t)||_{C[x_2,y_2]} = O(\lambda_n^{-\tau/2})$. Furthermore, since fg has compact support in $[x_3, y_3] \subset (x_2, y_2)$, Theorems 4.4

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and 5.1 imply (by virtue of the equivalence (2) $\Leftrightarrow$ (3) of Theorem 2.5) that the conclusions of the lemma hold for the function (fg). However, $g(x) = 1$ on $[a_2, b_2]$, so by restricting ourselves to this interval, we can replace fg by f in the results, which gives the desired relations of the lemma.

7. Outline of the proof of the saturation theorem. From now on, assume $S_\lambda(\cdot, t)$ is regular.

Using the properties proved in the previous sections, the proof of the saturation theorem is considerably simple. First, by Theorem 2.5, the condition $\|S_{\lambda_n}(f, k, t) - f(t)\|_{C[a_1,b_1]} = O(\lambda_n^{-(k+1)})$ implies $f^{(2k)} \in C[a_2, b_2]$. Moreover it is easy to see that the conditions $\|S_{2\lambda_n}(f, k, t) - S_{\lambda_n}(f, k, t)\|_{C[a_1,b_1]} = O(\lambda_n^{-k-1})$ and $\|S_{\lambda_n}(f, k, t) - f(t)\|_{C[a_1,b_1]} = O(\lambda_n^{-k-1})$ are equivalent. However, dealing with $\{S_{2\lambda}(f, k, t) - S_\lambda(f, k, t)\}$ instead of $\{S_\lambda(f, k, t) - f(t)\}$ will simplify much of the proof of one later step, Lemma 7.1.

Thus we may assume that $\{\lambda_n^{k+1}(S_{\lambda_n}(f, k, t) - S_{\lambda_n}(f, k, t))\}$ is bounded in $C[a_1, b_1]$ and hence in $L_\infty[a_1, b_1]$. Because $L_\infty[a_1, b_1]$ is the dual space of $L_1[a_1, b_1]$, by weak*-compactness, there is an $h \in L_\infty[a_1, b_1]$ and a subnet $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_i}^{k+1}(S_{\lambda_{n_i}}(f, k, t) - S_{\lambda_{n_i}}(f, k, t))$ converges to h in the weak* topology. In particular, for any $g \in C_0^\infty$ with supp $g \subset (a_1, b_1)$, we have

$$\lambda_{n_i}^{k+1}\langle S_{2\lambda_{n_i}}(f, k, t) - S_{\lambda_{n_i}}(f, k, t), g(t)\rangle \to \langle h(t), g(t)\rangle, \tag{7.1}$$

where

$$\langle h, g\rangle = \int_A^B h(t)g(t)dt.$$

In the case when $f \in C^{2k+2}[a_1, b_1]$, by Proposition 3.6, we have

$$\lim_{\lambda_{ni}\to\infty} \lambda_{n_i}^{k+1}[S_{2\lambda_{n_i}}(f, k, t) - S_{\lambda_{n_i}}(f, k, t)] = -\left(1 - \left(\frac{1}{2}\right)^{k+1}\right) \sum_{j=k+1}^{2k+2} Q(j, k, t)_J{}^{(j)}(t) \equiv P_{2k+2}(D)f(t). \tag{7.2}$$

Therefore, for functions $f \in C^{2k+2}$ and $g \in C_0^\infty$, we have

$$\lim_{\lambda_{n_i}\to\infty} \lambda_{n_i}^{k+1}\langle S_{2\lambda_{n_i}}(f, k, t) - S_{\lambda_{n_i}}(f, k, t), g(t)\rangle = \langle P_{2k+2}(D)f(t), g(t)\rangle = \langle f(t), P^*_{2k+2}(D)g(t)\rangle, \tag{7.3}$$

where $P^*_{2k+2}(D)$ is the dual operator of $P_{2k+2}(D)$. (In this case, in fact, it is a result of integration by parts.)

Since $C_\Psi(A, B) \cap C^{2k+2}[a_1, b_1]$ is dense in $C_\Psi(A.B)$ with respect to $\|\cdot\|_{C_\Psi}$, there exists a sequence $\{f_\sigma\}$ in $C_\Psi(A, B) \cap C^{2k+2}[a_1, b_1]$, converging to f in the $\|\cdot\|_{C_\Psi}$-norm.

In considering the expression of

$$\lim_{\sigma\to\infty}\lim_{\lambda_{ni}\to\infty} \lambda_{ni}{}^{k+1}\langle S_{2\lambda_{n_i}}(f_\sigma, k, t) - S_{\lambda_{n_i}}(f_\sigma, k, t), g(t)\rangle$$

for such a sequence f_σ, we need the following lemma.

LEMMA 7.1. *Let* $f \in C_\Psi(A, B)$, $g \in C_0{}^\infty$ *with* supp $g \subset (a, b)$. *Then*

$$(7.4) \qquad |\lambda^{k+1} < S_{2\lambda}(f, k, t) - S_\lambda(f, k, t), g(t) > | \leqq M||f||_{C_\Psi},$$

where M *depends on* g *(and its derivatives).*

We reserve the proof for the next section.

Thus, for $f_\sigma \in C_\Psi(A, B) \cap C^{2k+2}[a_1, b_1]$, converging to f in the $||\cdot||_{C_\Psi}$ norm, we have

$$(7.5) \qquad \begin{aligned}&\lim_{\sigma\to\infty}\lim_{\lambda_{n_i}\to\infty} \lambda_{ni}{}^{k+1}\langle S_{2\lambda_{n_i}}(f_\sigma, k, t) - S_{\lambda_{n_i}}(f_\sigma, k, t), g(t)\rangle \\ &\qquad = \lim_{\lambda_{n_i}\to\infty} \lambda_{ni}{}^{k+1}\langle S_{2\lambda_{n_i}}(f, k, t) - S_{\lambda_{n_i}}(f, k, t), g(t)\rangle.\end{aligned}$$

Combining (7.1), (7.3) and (7.5), we get $\langle h(t), g(t)\rangle = \langle f(t), P^*{}_{2k+2}(D)g(t)\rangle$ for all $g \in C_0{}^\infty$ with supp $g \subset (a_1, b_1)$.

This implies $P_{2k+2}(D)f(t) = h(t)$ since they are equal as generalized functions. However, as a first order linear differential equation for $f^{(2k+1)}$, with the non-homogeneous term, which can be represented in terms of $f^{(i)}$, $i \leqq 2k$ (in $C[a_2, b_2]$), and h (in $L_\infty[a_1, b_1]$), we deduce that $f^{(2k+1)} \in A.C.[a_2, b_2]$ and hence $f^{(2k+2)} \in L_\infty[a_2, b_2]$.

The "little o" part is similar with only one difference: instead of $\langle f(t), P^*{}_{2k+2}(D)g(t)\rangle = \langle h(t), g(t)\rangle$, we have $\langle f(t), P^*{}_{2k+2}(D)g(t)\rangle = 0$.

The implications (2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (6) in the theorem are slightly stronger than Proposition 3.6. But as $f^{(2k+1)} \in A.C.[a_2, b_2]$ and $f^{(2k+2)} \in L_\infty[a_2, b_2]$, we have $f^{(2k+1)} \in \text{Lip}(1; a_2, b_2)$. The rest of the proofs are computational and will be omitted.

8. Proof of Lemma 7.1. To complete the proof of Theorem 2.6, it remains to prove Lemma 7.1.

The proof of the lemma is divided into three parts.

8.1. Let $f \in C_\Psi(A, B)$, $g \in C_0{}^\infty$ with supp $g \subset (a, b)$. Denote

$$(8.1) \qquad S_{2\lambda}(f, k, t) - S_\lambda(f, k, t) = \sum_{j=1}^{2k+2} \alpha(j, k) S_{e_j\lambda}(f, t),$$

where $e_j \in \{d_0, d_1, \ldots, d_k, 2d_0, 2d_1, \ldots, 2d_k\}$. By Lemma 3.5 we have

$$(8.2) \qquad \sum_{j=1}^{2k+2} \alpha(j, k) e_j{}^{-m} = 0, \quad m = 0, 1, 2, \ldots, k.$$

We first get some integration in the representations of (7.4) over finite intervals, then use Taylor's expansion on g, and break the integration into several parts which are easier to estimate. That is, we have

$$\begin{aligned}(8.3)\quad &\lambda^{k+1}\langle S_{2\lambda}(f,k,t) - S_\lambda(f,k,t), g(t)\rangle \\ &= \lambda^{k+1}\int_A^B\int_A^B\left\{\sum_{j=1}^{2k+2}\alpha(j,k)W(e_j\lambda,t,u)f(u)g(t)\right\}dudt \\ &= \lambda^{k+1}\int_{\text{supp } g}\int_A^B\{\ldots\}\,dudt \\ &= \lambda^{k+1}\int_{\text{supp } g}\int_a^b\{\ldots\}\,dudt + o(1)\,||f||_{C_\Psi} \\ &= \lambda^{k+1}\int_A^B\int_a^b\{\ldots\}\,dudt + o(1)\,||f||_{C_\Psi}.\end{aligned}$$

By Fubini's theorem, this expression can be rewritten as

$$\begin{aligned}(8.4)\quad &\lambda^{k+1}\int_a^b\int_A^B\{\ldots\}\,dtdu + o(1)\,||f||_{C_\Psi} \\ &= \lambda^{k+1}\int_a^b\int_A^B\sum_{\gamma=0}^{2k+2}\sum_{j=1}^{2k+2}\frac{1}{\gamma!}\alpha(j,k)W(e_j\lambda,t,u)f(u) \\ &\quad\times g^{(\gamma)}(u)(t-u)^\gamma dtdu \\ &\quad+\lambda^{k+1}\int_a^b\int_A^B\sum_{j=1}^{2k+2}\alpha(j,k)W(e_j\lambda,t,u)f(u)\epsilon(t,u)(t-u)^{2k+2}dtdu \\ &\quad+o(1)\,||f||_{C_\Psi} \\ &= \sum_{\gamma=0}^{2k+2}\lambda^{k+1}\int_a^b\int_A^B\sum_{j=1}^{2k+2}\frac{1}{\gamma!}\alpha(j,k)W(e_j\lambda,t,u)f(u) \\ &\quad\times g^{(\gamma)}(u)(t-u)^\gamma dtdu \\ &\quad+\lambda^{k+1}\int_A^B\int_a^b\sum_{j=1}^{2k+2}\alpha(j,k)W(e_j\lambda,t,u)f(u)\epsilon(t,u)(t-u)^{2k+2}dudt \\ &\quad+o(1)\,||f||_{C_\Psi} \\ &= \sum_{\gamma=0}^{2k+2}\lambda^{k+1}\int_a^b\int_A^B\sum_{j=1}^{2k+2}\alpha(j,k)W(e_j\lambda,t,u)\,\phi_\gamma(u)t^\gamma dtdu \\ &\quad+\lambda^{k+1}\int_A^B\int_a^b\sum_{j=1}^{2k+2}\alpha(j,k)W(e_j\lambda,t,u)f(u)\epsilon(t,u)(t-u)^{2k+2}dudt \\ &\quad+o(1)\,||f||_{C_\Psi} \\ &\equiv \sum_{\gamma=0}^{2k+3}I_\gamma + o(1)\,||f||_{C_\Psi};\end{aligned}$$

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where

$$\phi_\gamma(u) = \sum_{m=\gamma}^{2k+2} (-1)^{m-\gamma} \frac{1}{\gamma!(m-\gamma)!} f(u) g^{(m)}(u) u^{m-\gamma}.$$

8.2. In order to estimate I_γ, $\gamma \leqq 2k + 2$, we need the following lemma:

LEMMA 8.1. *For every non-negative integer m, there holds*

$$(8.5) \qquad \int_A^B W(\lambda, t, u) t^m dt = P(u, \lambda) + O(\lambda^{-k-2}),$$

where $P(u, \lambda)$ is a polynomial in u and λ^{-1}, the degree of P in u is m, and the remainder $O(\lambda^{-k-2})$ is uniform for $u \in [a, b]$.

Proof. When $m = 0$, the lemma follows from condition (2) of Definition 2.1. For the induction step, use integration by parts and (2) of Definition 2.1 to derive

$$(8.6) \qquad \begin{aligned} &\int_A^B W(\lambda, t, u) t^{m+1} dt \\ &= u \int_A^B W(\lambda, t, u) t^m dt - \int_A^B W(\lambda, t, u) t^m (u - t) dt \\ &= u \int_A^B W(\lambda, t, u) t^m dt - \frac{1}{\lambda} \int_A^B \left[\frac{\partial}{\partial t} W(\lambda, t, u)\right] t^m p(t) dt \\ &= u \int_A^B W(\lambda, t, u) t^m dt + \frac{1}{\lambda} \int_A^B W(\lambda, t, u)(t^m p(t))' dt. \end{aligned}$$

Since degree $p(t) \leqq 2$, degree $(t^m p(t))' \leqq m + 1$, (8.6) is indeed a recursion relation. Hence (8.5) holds for all m.

Now the estimates of I_γ, $\gamma \leqq 2k + 2$ are easy: applying Lemma 8.1 ,we get

$$(8.7) \qquad I_\gamma = \lambda^{k+1} \sum_{j=0}^{k+1} \alpha(j, k) \int_a^b \phi_\gamma(u) P(u, e_j\lambda) du + o(1).$$

Since $P(u, \lambda)$ is a polynomial in u and $1/\lambda$, by relation (8.2), we have

$$(8.8) \qquad I_\gamma = O(1), \quad \gamma \leqq 2k + 2.$$

8.3. It remains to estimate I_{2k+3}. First notice that

$$|\epsilon(t, u)| = \frac{2}{(2k+2)!} |g^{(2k+2)}(\xi)| \leqq \frac{2}{(2k+2)!} ||g^{(2k+2)}||_{C[a,b]} < \infty$$

and $|f(u)| \leqq ||\Psi||_{C[a,b]} ||f||_{C_\Psi}$. It follows that

$$(8.9) \qquad |I_{2k+3}| \leqq M||f||_{C_\Psi} \lambda^{k+1} \int_A^B \int_a^b \sum_{j=1}^{2k+2} |\alpha(j, k)| W(e_j\lambda, t, u)(t - u)^{2k+2} du dt.$$

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In order to show $I_{2k+3} = O(1)$, it is sufficient to show

$$\text{(8.10)} \quad F_\lambda(k) = \lambda^k \int_A^B \int_a^b W(\lambda, t, u)(u-t)^{2k} du dt = O(1).$$

Let c, d be two finite numbers such that $[a, b] \cup \{0\} \subset (c, d) \subset (A, B)$. $F_\lambda(k)$ can be rewritten as

$$\text{(8.11)} \quad F_\lambda(k) = \int_c^d \int_a^b + \int_{(A,B)\setminus(c,d)} \int_a^b \equiv J_1 + J_2.$$

The fact that $J_1 = O(1)$ follows from Lemma 3.2, as the integration in t is only over a finite interval. Since $J_2 = (\int_A^c + \int_d^B) \int_a^b$, we shall only estimate $\int_d^B \int_a^b$. Notice that $t \geqq d$, $0 < t - b \leqq t - u$. Hence

$$\begin{aligned} J_3 \equiv \int_d^B \int_a^b &\leqq \lambda^k \int_d^B \int_a^b W(\lambda, t, u) \frac{(u-t)^{2(k+N)}}{(t-b)^{2N}} du dt \\ \text{(8.12)} \qquad &\leqq \lambda^k \int_d^B \frac{dt}{(t-b)^{2N}} \left\{ \int_a^b W(\lambda, t, u)(u-t)^{4k+2} du \right\}^{1/2} \\ &\qquad \cdot \left\{ \int_a^b W(\lambda, t, u)(u-t)^{4N-2} du \right\}^{1/2}. \end{aligned}$$

Recall that (Lemma 3.2) $\lambda^{-m} \int_A^B W(\lambda, t, u)(u-t)^m dt$ is a polynomial in t with degree less than or equal to m; moreover, $\lambda^{-m} A_m(\lambda, t) = O(\lambda^{-[(m+1)/2]})$. Therefore,

$$\text{(8.13)} \quad \left[\int_a^b W(\lambda, t, u)(u-t)^{4N-2} du \right]^{1/2} \leqq M\lambda^{-(N-1/2)} |P_{4N-2}(t)|^{1/2}$$

where $P_{4N-2}(t)$ is a polynomial in t of degree $4N - 2$. Hence

$$\text{(8.14)} \quad J_3 \leqq M\lambda^{k-N+1/2} \int_d^B \frac{|P_{4N-2}(t)|^{1/2}}{(t-b)^{2N}} \left\{ \int_a^b W(\lambda, t, u)(u-t)^{4k+2} du \right\}^{1/2} dt.$$

By the Cauchy-Schwarz inequality, we can further estimate

$$\begin{aligned} J_3 &\leqq M\lambda^{k+1/2-N} \left\{ \int_d^B \frac{|P_{4N-2}(t)|}{(t-b)^{4N}} dt \right\}^{1/2} \\ &\quad \times \left\{ \int_d^B \int_a^b W(\lambda, t, u)(u-t)^{4k+2} du dt \right\}^{1/2} \\ &\equiv M\lambda^{k+1/2-N} \cdot L_1^{1/2} \cdot L_2^{1/2}. \end{aligned}$$

It is sufficient to show L_1 and L_2 are finite integrals, for then by choosing $N \geqq k + 1$, $J_3 = O(1)$.

The estimate L_1 is trivial, since the integrand $|P_{4N-2}(t)|/(t-b)^{4N}$ is dominated by Mt^{-2} for some $M > 0$ and $\int_d^\infty t^{-2} dt$ is convergent. To estimate L_2, following Lemma 8.1,

$$L_2 \leqq \int_a^b \int_A^B W(\lambda, t, u)(u-t)^{4k+2} dt\, du$$

$$= \int_a^b P_{4k+4}(u, \lambda) du + O(\lambda^{-k-2}),$$

where $P_{4k+4}(u, \lambda)$ is a polynomial in u and λ^{-1}. Hence $L_2 = O(1)$. This proves $I_{2k+3} = O(1)$.

Combining this with the estimates (8.8), the proof of Lemma 7.1 (and hence the proof of the saturation theorem) is completed.

9. Applications. The theorems proved in the previous sections can be applied to a number of operators, for example, Bernstein polynomials, Szász operators, Post-Widder operators etc. For certain operators, some modifications may be necessary.

9.1. *Post-Widder operators and Gauss-Weierstrass operators.* The Post-Widder operator S_n^1 and the Gauss-Weierstrass operator S_λ^2 are defined as follows:

$$(9.1) \qquad S_n^1(f, t) = \frac{1}{(n-1)!}\left(\frac{n}{t}\right)^n \int_0^\infty e^{-nu/t} u^{n-1} f(u) du$$

and

$$(9.2) \qquad S_\lambda^2(f, t) = \sqrt{\frac{\lambda}{2\pi}} \int_{-\infty}^\infty e^{-\lambda(u-t)^2/2} f(u) du.$$

(In the literature, the Gauss-Weierstrass operator is often defined for $\lambda = 1/2\tau$ and $\tau \to 0^+$.) It is easy to see that S_n^1 and S_λ^2 satisfy Definition 1.1: for condition (2), $p_1(t) = t^2$ and $p_2(t) = 1$; for condition (3), $a_1(\lambda) = n/(n-1)$

$$\left(\text{in particular, } \int_0^\infty W_1(n, t, u) t^j dt = \frac{n^{j+1}(n-2-j)!}{(n-1)!} u^j\right)$$

and $a_2(\lambda) = 1$. Also, for the underlining space, $A = 0$, $B = +\infty$ for S_n^1 and $A = -\infty$, $B = +\infty$ for S_λ^2.

Moreover, in the application, S_n^1 generally apply on functions in $C[0, \infty)$ instead of $C(0, \infty)$. Therefore, for both operators, their growth-test functions $\Psi_1(t)$ and $\Psi_2(t)$ can be chosen as $e^{N|t|}$ for any $N > 0$.

Note that in both inverse and saturation theorems which we have just proved, the theorems are applicable to the case when λ runs through integers only, for in particular, $\lambda_{n+1}/\lambda_n = (n+1)/n \leqq 2$.

THEOREM 9.1. *The saturation and inverse theorems corresponding to Theorems* 2.5 *and* 2.6 *hold for the Post-Widder operators and the Gauss-Weierstrass operators (where the growth-test functions* $\Psi_1(t) = \Psi_2(t) = e^{N|t|}$, *for any* $N > 0$).

9.2. *Bernstein polynomials and the Szász operators.* The two theorems proved

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in the previous sections hold for Bernstein polymonials

$$S_n^3(f,t) \equiv B_n(f,t) = \sum_{k=0}^{n} \binom{n}{k} t^k(1-t)^{n-k} f\left(\frac{k}{n}\right)$$

and the Szász operators

$$S_\lambda^4(f,t) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{1}{k!}(\lambda t)^k f\left(\frac{k}{\lambda}\right).$$

It is straightforward to verify conditions (1) and (2) of Definition 2.1 for both operators (for S_n^3,

$$W_3(n,t,u) = \sum_{k=0}^{n} \binom{n}{k} t^k(1-t)^{n-k}\delta\left(u-\frac{k}{n}\right),$$

$\delta(x)$ being the delta function, $A = 0$, $B = 1$, $p_3(t) = t(1-t)$; similarly, for S_λ^4, $A = 0$, $B = \infty$, $p_4(t) = t$). We only remark that now the relation (2) of the definition is in the sense of distribution.

Hence they are operators of exponential type. As for condition (3), they satisfy a modified version:

(3*) If $h \in C_0^{2k+1}$, supp $h \subset (a,b)$, then

$$\int_a^b h(u) \int_A^B W(\lambda,t,u)dtdu = a(\lambda) \int_a^b h(u)du + O(\lambda^{-k-1})\, ||h||_{C^{2k+1}} \tag{9.3}$$

where $a(\lambda) \to 1$ is a rational function of λ, and $||h||_{C^{2k+1}} = ||h|| + ||h^{(2k+1)}||$.

This would be enough for proving Lemma 7.1, which is the only place that condition (3) has been used. The reason is as follows: If $||S_{\lambda_n}(f,k,t) - f(t)||_{C[a,b]} = O(\lambda_n^{-k-1})$, by Theorem 2.5, we get $f \in C^{2k+1}(a,b)$. Therefore the functions

$$\phi_\gamma(u)\left(= \sum_{m=\gamma}^{2k+2} (-1)^{m-\gamma} \frac{1}{\gamma!(m-\gamma)!} f(u)g^{(m)}(u)u^{m-\gamma}\right)$$

defined in (8.4) are in $C_0^{2k+1}(a,b)$. Now by (3*) and following similar procedures as in Section 8.2, Lemma 7.1 can be proved.

The fact that the Bernstein polynomials satisfy (3*) is not difficult to see. First we calculate

$$\int_0^1 W_3(n,t,u)dt = \sum_{m=0}^{n} \frac{1}{n+1}\delta\left(u-\frac{m}{n}\right). \tag{9.4}$$

Therefore, for $h \in C_0^{2k+1}$, with supp $h \subset (a,b)$,

$$J \equiv \int_a^b h(u) \int_0^1 W_3(n,t,u)dtdu = \sum_{m=0}^{n} \frac{1}{n+1} h\left(\frac{m}{n}\right). \tag{9.5}$$

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By the Euler-Maclaurin formula ([**5**, pp. 268–275]), we obtain

$$(9.6) \qquad J = \frac{n}{n+1}\left(\int_a^b h(u)du + R\right),$$

where

$$R = -\frac{(b-a)^2}{2n} h'(\xi), \quad a \leqq \xi \leqq b, \quad \text{for } k = 0, \quad \text{and}$$

$$R = \frac{1}{n^{2k+1}} \sum_{m=k_0}^{n_0} \int_0^1 P_{2k}(t) h^{(2k)}(a + n^{-1}(t+m))dt, \quad n_0 - k_0 \leqq n - 1, \quad \text{for } k \neq 0.$$

For $k \neq 0$, using the fact that

$$||P_{2k}||_{C[0,1]} = (-1)^k P_{2k}(\tfrac{1}{2}) = (-1)^k \left[\frac{B_{2k}(\frac{1}{2}) - B_{2k}}{(2k)!}\right],$$

where $B_{2k}(\frac{1}{2})$ is the value of the Bernoulli polynomial at $\frac{1}{2}$ and B_{2k} is the Bernoulli number, R is estimated as

$$(9.7) \qquad |R| \leqq \frac{|P_{2k}(\frac{1}{2})|}{(2k)! n^{2k}} \int_a^b |h^{(2k)}(t)| dt.$$

The condition (3*) for the Szász operators can be verified similarly.

Since the Bernstein polynomials are usually defined on $C[0, 1]$ functions while the Szász operators are defined on $C[0, \infty)$ functions, the growth test function for the Bernstein polynomials is not necessary and for the Szász operators can be chosen as e^{Nt} for any $N > 0$.

THEOREM 9.2. *The corresponding saturation and inverse theorems hold for Bernstein polynomials and Szász operators* (*where* $\Psi_4(t) = e^{Nt}$, $N > 0$).

9.3. *Baskakov operators.* The Baskakov operator $S_\lambda^5(f, t)$ is a generalization of the Bernstein polynomial and the Szász operator. However, our result is only applicable to functions with growth not faster than some $\Psi_5(t) = (1+t)^N$, $N > 0$.

Definition 9.3. The Baskakov operator S_λ^5 is defined as

$$(9.8) \qquad S_\lambda^5(f, t) = \sum_{k=0}^{\infty} (-1)^k \frac{\phi_\lambda^{(k)}(t)}{k!} t^k f\left(\frac{k}{\lambda}\right),$$

where $\{\phi_\lambda\}$ is a family of real-valued functions such that (1) $\phi_\lambda(x)$ can be expanded in Taylor's series in $[0, \beta)$ (β may be equal to ∞); (2) $\phi_\lambda(0) = 1$; (3) $(-1)^k \phi_\lambda^{(k)}(x) \geqq 0$ $(k = 0, 1, 2, \ldots)$ for $x \in [0, \beta)$; (4) $-\phi_\lambda^{(k)}(x) = \lambda \phi_{\lambda+c}^{(k-1)}(x)$ $(k = 1, 2, \ldots)$, $x \in [0, \beta)$, for some constant c; (5) For any fixed constant M, $\lim_{x\to\infty} \phi_\lambda(x) x^k = 0$ for $k = 0, 1, 2, \ldots, M$.

The kernel for the Baskakov operator is

$$W_5(\lambda, t, u) = \sum_{k=0}^{\infty} (-1)^k \frac{\phi_\lambda^{(k)}(t)}{k!} t^k \delta\left(u - \frac{k}{\lambda}\right).$$

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It is trivial that S_λ^5 satisfies condition (1) of Definition 2.1. However, to see that it also satisfies the other two conditions is not so easy. We show it satisfies condition (2) in the following lemma.

LEMMA 9.4. *The kernel W_5 satisfies the following differential equation:*

$$\frac{\partial}{\partial t} W_5(\lambda, t, u) = \frac{\lambda}{p_5(t)} W_5(\lambda, t, u)(u - t), \tag{9.9}$$

where $p_5(t) = t(1 + ct)$.

Proof. Recall that $\phi_\lambda(x)$ can be expanded in a Taylor's series. Hence we have

$$\phi_\lambda(x) = \sum_{k=0}^{\infty} \frac{\phi_\lambda^{(k)}(0)}{k!} x^k, \quad \text{and} \quad \phi_{\lambda+c}(x) = \sum_{k=0}^{\infty} \frac{\phi_{\lambda+c}^{(k)}(0)}{k!} x^k.$$

From (4) and (2) of Definition 9.3, we have

$$\begin{aligned}\phi_{\lambda+c}^{(k)}(0) &= (-1)^k(\lambda + c)(\lambda + 2c) \ldots (\lambda + kc) \\ &= \phi_\lambda^{(k)}(0)(\lambda + kc)/\lambda.\end{aligned}$$

Hence,

$$\begin{aligned}\phi_{\lambda+c}(x) &= \sum_{k=0}^{\infty} \frac{\phi_\lambda^{(k)}(0)}{k!} \left(1 + \frac{k}{\lambda} c\right) x^k \\ &= \phi_\lambda(x) - cx \sum_{k=1}^{\infty} \frac{\phi_{\lambda+c}^{(k-1)}(0)}{(k-1)!} x^{k-1} = \phi_\lambda(x) - cx\, \phi_{\lambda+c}(x).\end{aligned}$$

Or,

$$\phi_{\lambda+c}(x) = \frac{1}{1 + cx} \phi_\lambda(x).$$

Now

$$\begin{aligned}\phi_{\lambda+c}^{(k)}(x) &= (-1)^k(\lambda + c) \ldots (\lambda + kc)\phi_{\lambda+(k+1)c}(x) \\ &= (-1)^k\lambda(\lambda + c) \ldots (\lambda + (k-1)c)\phi_{\lambda+kc}(x) \frac{1}{1 + cx} \left(\frac{\lambda + kc}{\lambda}\right) \\ &= \phi_\lambda^{(k)}(x)\left(1 + \frac{kc}{\lambda}\right) \frac{1}{1 + cx}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\partial}{\partial t} W_5(\lambda, t, u) &= \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \frac{k}{t} \phi_\lambda^{(k)}(t)\delta\left(u - \frac{k}{\lambda}\right) \\ &\quad - \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \lambda\, \phi_{\lambda+c}^{(k)}(t)\delta\left(u - \frac{k}{\lambda}\right) \\ &= \frac{\lambda}{t} W_5(\lambda, t, u)u - \frac{\lambda}{1 + ct} \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \left(1 + \frac{k}{\lambda} c\right) \\ &\quad \times \phi_\lambda^{(k)}(t)\delta\left(u - \frac{k}{\lambda}\right) \\ &= \frac{\lambda}{t} W_5(\lambda, t, u)u - \frac{\lambda}{1 + ct} W_5(\lambda, t, u) - \frac{\lambda c}{1 + ct} W_5(\lambda, t, u)u \\ &= \frac{\lambda}{t(1 + ct)} W_5(\lambda, t, u)(u - t).\end{aligned}$$

To show S_λ^5 satisfies (3*), we can proceed similarly as in Section 9.2. Therefore, it is sufficient to show the following lemma. (The stronger form we prove in the lemma is useful in practice.)

LEMMA 9.5. *If $f \in C_0(a, b)$, and $M > 0$ is a fixed integer, then for $\gamma = 0, 1, 2, \ldots, M$, the following equation holds*

$$(9.10)\quad \int_a^b \int_0^B W_5(\lambda, t, u) f(u) t^\gamma dt\, du = \sum_{m/n \in (a,b)} f\left(\frac{m}{n}\right) \times \frac{(m+\gamma)!}{m!} \frac{1}{(\lambda - c)(\lambda - 2c)\ldots(\lambda - (\gamma+1)c)}.$$

Proof. First, we prove the following assertion:

$$(9.11)\quad (-1)^m \int_0^\beta \phi_\lambda^{(m)}(t) t^k dt = \frac{1}{(\lambda - c)(\lambda - 2c)\ldots(\lambda - (k - m + 1)c)}$$

for non-negative integers k and m such that $k - m \leqq M$.

The proof is by induction. First, from

$$\phi_\lambda^{(k-1)}(t) = -\frac{1}{\lambda - c} \phi_{\lambda-c}^{(k)}(t),$$

we have

$$\int_0^\beta \phi_\lambda(t) dt = \int_0^\beta -\frac{1}{\lambda - c} \phi_{\lambda-c}'(t) dt = \frac{1}{\lambda - c} \phi_\lambda(0) = \frac{1}{\lambda - c}.$$

Hence, (9.11) holds for $m = 0$, $k = 0$. For $m = 0$ fixed, we proceed by induction on k. Suppose $k + 1 \leqq M$, using conditions (4), (2) and (5) in Definition 9.3 and integration by parts, we have

$$\int_0^\beta \phi_\lambda(t) t^{k+1} dt = \int_0^\beta -\frac{1}{\lambda - c} \phi_{\lambda-c}'(t) t^{k+1} dt$$
$$= \frac{k+1}{\lambda - c} \int_0^\beta \phi_{\lambda-c}(t) t^k dt = \frac{(k+1)!}{(\lambda - c)\ldots(\lambda - (k+2)c)},$$

by the induction hypothesis. Therefore, (9.11) is valid for $m = 0$ and $k = 0, 1, \ldots, M$.

Now, assume (9.11) holds for $(m - 1)$ and all positive integers $k \leqq M$. We first show by induction that (9.11) holds for m and $k - m \leqq M - 1$. By using a technique similar to the above, we have

$$\int_0^\beta \phi_\lambda^{(m)}(t) t^k dt = -\int_0^\beta \lambda\, \phi_{\lambda+c}^{(m-1)}(t) t^k dt$$
$$= (-1)^m \lambda \frac{k!}{(\lambda - c)\ldots(\lambda - (k - m + 1)c)}.$$

Next, for m and $k - m = M$, by using conditions (4), (2) and (5) of Defini-

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tion 9.3, integration by parts, and the induction hypothesis, we obtain

$$\int_0^\beta \phi_\lambda^{(m)}(t)t^{m+M}dt = -\int_0^\beta \lambda\ \phi_{\lambda-c}^{(m-1)}(t)t^{m+M}dt$$

$$= \lambda(m+M)\int_0^\beta \phi_{\lambda+c}^{(m)}(t)t^{m+M-1}dt$$

$$= (-1)^m\lambda(m+M)\frac{(m+M-1)!}{\lambda(\lambda-c)\ldots(\lambda-(M-1)c)}.$$

Hence the proof of equation (9.11) is completed.

Using relation (9.11) for $k = m + \gamma\ \gamma = 0, 1, \ldots, M$, we have

$$\int_0^\beta W_5(\lambda, t, u)t^\gamma dt = \sum_{m=0}^\infty \frac{(-1)^m}{m!}\int_0^\beta \phi_\lambda^{(m)}(t)t^{m+\gamma}\delta\left(u-\frac{m}{\lambda}\right)dt$$

$$= \sum_{m=0}^\infty \frac{1}{m!}\frac{(m+\gamma)!}{(\lambda-c)(\lambda-2c)\ldots(\lambda-(\gamma+1)c)}\delta\left(u-\frac{m}{\lambda}\right).$$

Hence,

$$\int_a^b\int_0^\beta W_5(\lambda, t, u)f(u)t^\gamma dt\, du$$

$$= \sum_{m/\lambda\in(a,b)} f\left(\frac{m}{\lambda}\right)\frac{1}{m!}\frac{(m+\gamma)!}{(\lambda-c)(\lambda-2c)\ldots(\lambda-(\gamma+1)c)}.$$

Therefore, following the same proof as in Section 9.2, using the Euler-Maclaurin formula, we prove the saturation and inverse theorems for Baskakov operators.

THEOREM 9.6. *The corresponding saturation and inverse theorems hold for Baskakov operators.*

Remark 9.7. (1). The conditions in Definition 9.3 are slightly different from those in the original definitions (c.f. [**2**] and [**17**]). The differences are, the corresponding intervals in conditions (1), (3) and (4) have been changed to $[0, \beta)$, where β satisfies condition (5). These modifications are based on the concrecated examples (e.g., if $S_n^5(f, t) = B_n(f, t)$, then $\phi_n(t) = (1-t)^n$, $\beta = 1$. In this case, $\phi_n(x)$ does not satisfy the conditions in the original definition).

(2) The condition (5) in Definition 9.3 is in fact equivalent to the condition

$$(9.12)\qquad \int_0^{E(c)} (-1)^l\frac{\phi_\lambda^{(l)}(x)}{l!}x^l dx = \frac{1}{\lambda-c},\quad l = 0, 1, 2, \ldots$$

used by Suzuki in his saturation theorem for Baskakov operator ([**17**, p. 441]). (The technique of the proof for the equivalence is similar to that used in this section. We shall omit the proof here.)

The condition (9.12), as being equivalent to condition (5) of Definition 9.3, is enough for proving the saturation theorem, while the other conditions, i.e., conditions (20), part of (21), and (22) of [**17**], are redundant for this purpose.

We would like to point out that the notation $E(c)$ used by Suzuki is equal to the β in our definition. The value of $E(c)$ is equal to ∞ when $c \geqq 0$ and is equal to $1/|c|$ when $c < 0$. The formula that

$$E(c) = \frac{5c^2 - c - 2}{2c(c-1)} \quad \text{if } c \neq 0, 1$$

given by Suzuki is valid only if $c = -1$.

(3) Let us recall some of the known results for this operator. Baskakov's original work [**2**] investigated the convergence theorems of bounded continuous functions; Suzuki [**17**] studied saturation classes for continuous functions with compact supports; a result of Berens [**3**] is also for bounded continuous functions. Such restrictions on f limit the applications. Our results are for functions with growth less than some $(1 + x)^N$ for some $N > 0$. (From the remark on $\Psi(t)$ in Section 2, we see that the results are actually true for functions with growth less than e^{Nx} for some $N > 0$.) This class is considerably wider.

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