INTRODUCTORY ADDRESS

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The aim of celestial mechanics is to derive all solutions of the differential equations of the three- and n-body problems and to reveal the interrelation among all such solutions with various values of the masses and initial conditions, not only to predict the positions of celestial bodies at any instant but to think over the cosmogony of stellar systems and to ascertain the law of universal gravitation in view of the time-space picture of physical theory.

The simplest way to obtain the solution is to find out the necessary number of the first integrals. By Bruns' and Poincaré's theorems there is no further integral, besides the classical integrals, in algebraic form of the momenta or in a uniform function of the orbital elements. Thus we must recourse to transcendental processes for deriving the solution. The perturbation is one of such a kind of successive approximations, such as classical methods of Newcomb, Lindstedt, Delaunay, Bohlin and Whittaker without error estimate and modern method of Krylov-Bogoliubov with error estimates. The formal solution in powers of the disturbing masses and the orbital parameters was shown by Bruns and Poincaré not to be uniformly convergent. All solutions, such that the mean motions — the coefficient of t in the arguments of trigonometric functions in the formal series expansions — are modified at each successive approximation, were shown not to be uniformly convergent. In the formal aspect such multiple Fourier series are in the form of quasi-periodic functions of Bohl and Esclangon. The error of truncating at a finite-order term should be estimated like in Krylov-Bogoliubov for the future.

The existence of periodic solutions and asymptotic solutions was proved under suitable choice of the initial conditions according to Poincaré, and also the existence of quasi-periodic solutions under suitable conditions, except the cases when the mean motions are Mahler's U numbers, according to Kolmogorov, Arnold and Moser. These theories are based on analytical continuation only for small values of the disturbing masses. For finite values of the masses and for arbitrarily chosen values of the initial conditions we have no means available at present to know in any analytical manner the behaviour of the solution, not to say the

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mutual relation among the various solutions with varying initial conditions. The condition for representability of the solution in quasi-periodic functions was my problem since 1924 but I could not succeed to solve it yet. It is connected with the stability of motion.

Owing to the failure of investigating the nature and behaviour of the motions in the planetary, satellite and the _n_-body problems we could study the qualitative characters, that is, the topological theory of the behaviour and interrelation among the solutions of the problems.

In order to apply Cauchy's existence theorem on the solution of a differential equation in an analytic function of the independent variable we must at first regularize the singularities. Painlevé discussed in detail the singularities in the three- and _n_-body problems. The condition for the singularities must be transcendental. According to the works of Sundman and later of Levi-Civita the singularities for a binary collision in the three-body problem can be regularized and the solution can be analytically continued beyond any binary collision. The solution in powers of the two-thirds' power of the time interval to or from the collision instant was shown to be convergent but the radius of convergence is so small to be used in usual ephemeris computation. For a triple collision the Weierstrass-Sundman condition that the total angular momentum of the three-body system should be zero was derived and the behaviour of the solution in the neighbourhood of the collision was studied by Sundman, Block and Siegel. The motion cannot be analytically continued beyond a triple collision except for very rare cases. Various kinds of imaginary collisions were discussed. The theory can be extended to the _n_-body problem.

The behaviour at infinity of the orbits was studied by Chazy. He concluded according to Poincaré and Schwarzschild that a distant comet could not be eternally captured by the Sun-Jupiter system. Merman criticized Chazy's work and pointed out that the probability of capture is positive if the total energy is negative. Merman derived complicated conditions for a capture.

To study the behaviour of the solution in the neighbourhood of a singularity the method of Briot-Bouquet was cultivated by Poincaré, Bendixon and Dulac. The nature of the solution-curves, called the characteristics, around a singularity, at the same time as various domains including the domains of ergodic character and with or without limit cycles were distinguished. In particular Poincaré's study on the characteristics on the surface of a torus was developed by Denjoy and they were shown to have different point-set theoretical behaviours according to the initial conditions. As a special case of characteristics the nature of geodesics on different surfaces was studied by Hadamard, Morse and Poincaré. Geodesics on surfaces of any genus were discussed by means of Fuchsian groups of transformations and non-Euclidean geometry.

Poincaré's method of finding out periodic solutions is based on the minimum principle and the analytic continuation. Birkhoff invented the minimax method. Morse defined the type number in contrast to the rotation number of Poincaré and Birkhoff. The conjugate points in the calculus of variations play an important role. Morse went further and developed his calculus of variations in the large and the functional topology and could prove the existence of closed geodesics on the surface of an ellipsoid of any dimension. The analysis situs of Poincaré was sketched with the idea of homology, homotopy, Betti numbers, torsion coefficients, Heegaard diagrams and the fundamental groups. These ideas were applied to the manifolds of motion regarded as topological spaces. The restricted three-body problem was discussed on such analysis situs of the manifolds of motion.

The existence of an invariant point in a surface transformation on the surface of section shows the existence of a periodic solution according to the study of Poincaré and Birkhoff. The behaviour of the transformed points by successive application of the surface transformation reveals the stability character of the motion. The existence of invariant points in a ring transformation conjectured by Poincaré was proved by Birkhoff and extended in several aspects even to a functional space, by Birkhoff, Kerékjartó, Kellog and Nielsen. The theorem was applied to prove the existence of periodic solutions, not by the method of analytic continuation, in the restricted three-body problem by Birkhoff. The invariant points under surface transformations were classified were discussed on the point of view of stability. Birkhoff discovered the ring of instability and remarkable closed curves. He studied the distribution of various types of motion, periodic, asymptotic and recurrent, by means of the surface transformations.

Birkhoff extended the periodic solutions to the recurrent motions on Poincaré's idea of regional recurrence. The theorem was sharpend by Carathéodory, Khintchine and Hopf. The idea was further generalized to the central motions by Birkhoff on point-set theoretical ground and to fleeing points by Hopf. The idea of metrical transitivity was introduced by Birkhoff. Classical forms of the ergodic theorem was discussed by Ehrenfest and others as the foundation of the gas-kinetic theory since Maxwell, Boltzmann and Gibbs. The mean ergodic theorem was proved by von Neumann in a Hilbert space for quantum mechanics. The individual ergodic theorem was proved by Birkhoff. The ideas of mixture and homogeneous chaos were introduced by Hopf and Wiener. The ergodic theorem was recently generalized to functional spaces.

The proof of the existence of mean motions in the theory of secular perturbations was carried out by Bohl and Weyl. It is connected with the distribution of numbers modulo one. Also statistical methods in Diophantine approximation and asymptotic distribution functions were applied to prove the existence of mean motions. The secular constant and the zero-points of an analytical almost periodic functions were studied. Almost periodic motions on a circle and on a surface were studied by Bohr, Jenssen and Fenchel. The relation between the recurrent motion of Birkhoff and the almost periodic motion was studied in connection with the Liapounov stability. Strongly and weakly stable motions were defined in relation to Liapounov stability and Poisson stability.

These topological theories, in combination with the numerical computation of the trajectories with concrete numerical values of the parameters, such as the works of Hénon, Rabe, Bartlett and Szebehely, if a sufficient number of numerical solutions are derived, will throw light on a rough idea on the behaviour and interrelation of the motions of the proposed dynamical problem. Hénon, Deprit and Contopoulos have embarked in such research of their individual particular problems. With the accumulation of the results of such numerical computations the topological theories are expected to reveal the key to know the interrelation and interlacement of various kinds of trajectories as the solutions of the problem. Hénon discovered by numerical computer analysis the domains of periodicity and ergodicity as islands and seas on the surface of section. This result reveals the existence of a non-uniform integral in the three-body problem in contrast to Poincaré's theorem on the non-existence of uniform integrals other than the classical integrals. The problem is to derive the explicit expressions for the boundaries of those domains.

Now the study of topological theories and the development of powerful computers are showing the way to lead to the main problem of celestial mechanics, long-cherished in the heart of human culture.