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RANDOM FIELDS WITH PÓLYA CORRELATION STRUCTURE

RICHARD FINLAY * ** AND
EUGENE SENETA,* *** *University of Sydney*

Abstract

We construct random fields with Pólya-type autocorrelation function and dampened Pólya cross-correlation function. The marginal distribution of the random fields may be taken as any infinitely divisible distribution with finite variance, and the random fields are fully characterized in terms of their joint characteristic function. This makes available a new class of non-Gaussian random fields with flexible correlation structure for use in modeling and estimation.

Keywords: Random field; infinitely divisible distribution; Pólya autocorrelation

2010 Mathematics Subject Classification: Primary 60G60
Secondary 60G10; 60E07

1. Introduction

Our primary object of study in this paper is a class of random fields that are indexed in the temporal domain over $\mathbb{R}$ and in the spatial domain over $\{1, 2, \dots d\}$, $d \in \mathbb{N}^+$, that is, a multivariate stochastic process. We denote a random field $\{\boldsymbol{Z}(t)\}$, $t \in \mathbb{R}$, with spatial dimension defined on $\{1, 2, \dots, d\}$, $d \in \mathbb{N}^+$, by $\{\boldsymbol{Z}(t)\} = \{Z_1(t), \dots, Z_d(t)\}$. If all $\{Z_h(t)\}$ have second-order moments then the covariance matrix function of $\{\boldsymbol{Z}(t)\}$ is given by $C(t, t+s) = \operatorname{cov}(\boldsymbol{Z}(t), \boldsymbol{Z}(t+s)) = \mathbb{E}(\boldsymbol{Z}(t) - \mathbb{E}\boldsymbol{Z}(t))(\boldsymbol{Z}(t+s) - \mathbb{E}\boldsymbol{Z}(t+s))'$. The hth diagonal entry of $C(t, t+s)$ corresponds to the autocovariance (or direct covariance) between $Z_h(t)$ and $Z_h(t+s)$, while the (g, h)th off-diagonal entry corresponds to the so-called cross-covariance between $Z_g(t)$ and $Z_h(t+s)$, $g \neq h$. Thus, the diagonal entries of $C(t, t+s)$ are autocovariance functions and the off-diagonal entries are cross-covariance functions. If both $C(t, t+s)$ and $\mathbb{E}\boldsymbol{Z}(t)$ are independent of t, then $\{\boldsymbol{Z}(t)\}$ is said to be second-order stationary.

The class of admissible autocovariance and cross-covariance functions for Gaussian second-order stationary random fields, as well as other closely related elliptically contoured random fields, is well known. Here we consider elliptically contoured random fields constructed as Gaussian random fields multiplied by nonnegative random variables; the marginal distribution of the resulting random field is altered by the multiplication but the correlation structure is not. In this case the covariance matrix function may be taken as any function that satisfies $C(t, t+s) = C(t+s, t)'$ and

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i' C(t_i, t_j) a_j \geq 0 \tag{1}$$

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* Postal address: School of Mathematics and Statistics F07, University of Sydney, Sydney, NSW 2006, Australia.
** Email address: richardf@maths.usyd.edu.au
*** Email address: eseneta@maths.usyd.edu.au

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for all $n \in \mathbb{N}^+$, $t_k \in \mathbb{R}$, and $a_k \in \mathbb{R}^d$ for $k = 1, \ldots, n$ (see, for example, Cramér and Leadbetter (1967) and Gikhman and Skorokhod (1969), as well as Ma (2009), (2011a), (2011b), (2011c), (2011d), and the references therein). Du and Ma (2013), for example, constructed an elliptically contoured random field that may take any matrix function satisfying (1) as its covariance matrix function.

For random fields that are non-Gaussian, however, (1) is in general a necessary but not a sufficient condition for the covariance structure, and the range of admissible covariance structures must be investigated on a case-by-case basis. For example, for a log-Gaussian random field, (1) is a necessary but not sufficient condition for its covariance structure.

In this paper we construct second-order stationary random fields in both continuous and discrete time. In Section 3 we construct a random field in continuous time with Pólya-type autocorrelation function and dampened Pólya cross-correlation function. In Section 4 we construct a random field in discrete time, the more practically useful setting, with Young-type autocorrelation function and dampened Young cross-correlation function. In Section 5 we present a number of extensions (Pólya- and Young-type autocorrelation functions are defined in Section 2). Importantly, the marginal distribution of the random fields may be taken as any infinitely divisible distribution with finite variance, where by marginal distribution we mean the distribution of the random variable $Z_h(t)$ for fixed t. This extends the results from Finlay and Seneta (2007) and Finlay *et al.* (2011) to the multivariate setting, and makes available a new class of non-Gaussian second-order stationary random fields with flexible correlation structure for use in modeling and estimation.

Other authors have constructed non-Gaussian random fields. In addition to the above-mentioned papers, Marfè (2012), (2014), for example, constructed a multivariate Lévy process that can accommodate a flexible range of linear and nonlinear dependencies across the spatial dimension and for which the marginal distribution may approximate any Lévy type. Our construction has a number of advantages over alternatives in the literature, however. The marginal distribution of our random fields may be taken as any infinitely divisible distribution with finite variance, whereas, for example, the marginal distributions of the elliptically contoured random fields discussed above are restricted to be of normal variance-mixing type and so exclude any nonsymmetric distribution or any distribution that does not have support on $(-\infty, \infty)$, such as a distribution on the positive half-line. Furthermore, since the elliptically contoured random fields are constructed as Gaussian random fields multiplied by a nonnegative random variable, they revert to being Gaussian given a realization of that random variable. Our random fields can also be endowed with a rich and dynamic correlation structure across both the spatial and temporal dimensions. Although endowed with a rich dependence structure along the spatial dimension, the Lévy process constructed by Marfè has independent increments, so the increments lack a dependence structure along the time domain (it is the stationary increments of Marfè's process, rather than the process itself, that is most closely related to the processes that we construct). Finally, our method of construction, based on sums of independent and identically distributed (i.i.d.) random variables, lends itself particularly easily to numerical simulation, while the random fields are fully characterized in terms of their joint characteristic function, allowing for efficient estimation.

2. Pólya- and Young-type autocorrelation functions

Pólya (1949) provided a simple sufficient condition for the admissibility of a continuous-time autocorrelation function of a univariate Gaussian process, being essentially that a function $\rho(s)$ is admissible if it is real-valued, continuous, and symmetric about the origin, with $\rho(0) = 1$,

$\rho(s)$ convex for $s > 0$ and $\rho(s) \to 0$ as $s \to \infty$ (see also Lukacs (1960, Theorem 4.3.1), as well as Chung (2001) and Christakos (1984)). In fact the condition was originally stated in terms of characteristic functions, but a function is a real-valued characteristic function if and only if it is also an admissible autocorrelation function (see, for example, Finlay *et al.* (2011)).

This Pólya condition is useful in the univariate setting as it is reasonably flexible and, importantly, is easy to check in practice. There is a more general necessary and sufficient condition, being that $\rho(s)$ satisfy

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\rho(t_i - t_j)a_i\bar{a}_j \geq 0$$

for all $n \in \mathbb{N}^+$, $t_k \in \mathbb{R}$ and $a_k \in \mathbb{C}$ for $k = 1, \ldots, n$ (see, for example, Feller (1966, Section XIX.3)), but its practical use is limited, as for a given $\rho(s)$ it can be difficult to check.

A related theorem from Young (1913) gave an analogous result for the discrete-time setting. For $\rho(s),\ s \in \mathbb{N}$, Young's theorem essentially states that $\rho(s)$ is an admissible discrete-time autocorrelation function if it is real-valued and symmetric on $\{0 \pm 1, \pm 2, \ldots\}$, with $\rho(0) = 1$, $\rho(s) \to 0$ as $s \to \infty$, and $\rho(s) \geq 0,\ \rho(s+1) - \rho(s) \leq 0$, $\rho(s+2) - 2\rho(s+1) + \rho(s) \geq 0$ for $s = 0, 1, 2, \ldots$ (see also Zygmund (1968, Chapter V), as well as Kolmogoroff (1923)). Similar to the Pólya condition, the result was originally stated in the context of the Fourier series, but the Fourier series can be interpreted as a symmetric probability density function on $(-\pi, \pi)$ and, inverting the Fourier series, the $\rho(s)$ for $s \in \mathbb{N}$ (the Fourier coefficients) can be interpreted as the characteristic function of this probability density function evaluated at the integers. Being the characteristic function of a symmetric density function, and so real-valued, $\rho(s),\ s \in \mathbb{N}$ is also an admissible discrete-time autocorrelation function.

These Pólya and Young sufficient conditions turn out to define the set of autocorrelation and cross-correlation functions possible using the method that we employ here; our method essentially involves constructing random fields via carefully chosen sums of i.i.d. random variables, and the Pólya (in continuous time) and Young (in discrete time) conditions ensure that all sums that we consider are nonnegative.

3. A random field in continuous time

Assumption 1. *It holds that $\rho(s),\ s \in \mathbb{R}$, is a continuous function symmetric about $s = 0$ satisfying $\rho(0) = 1,\ \rho(s) \to 0$ as $s \to \infty$, and for $s \in [0, \infty)$ satisfying $\rho(s) \geq 0,\ \rho'(s) \leq 0$ and $\rho''(s) \geq 0$.*

Note that Assumption 1 implies that $\rho'(s) \to 0$ as $s \to \infty$.

Assumption 2. *It holds that $\kappa(s),\ s \in \mathbb{R}$ is a continuous function satisfying $0 \leq \kappa(s) \leq 1$.*

Theorem 1. *If $\rho(s)$ is a function satisfying Assumption 1 and $\kappa(s)$ is a function satisfying Assumption 2, then there exists a second-order stationary random field*

$$\{\boldsymbol{V}(t)\} = \{V_1(t), \ldots, V_d(t)\},\ t \in \mathbb{R},\ d \in \mathbb{N}^+$$

such that $\operatorname{corr}(V_h(t), V_h(t+s)) = \rho(s)$ *and* $\operatorname{corr}(V_g(t), V_h(t+s)) = \int_0^\infty\int_0^\infty \kappa(s+u+v)\rho''(s+u+v)\,\mathrm{d}v\,\mathrm{d}u$ *for $s > 0,\ g, h = 1, \ldots, d,\ g \neq h$. The marginal distribution of $V_h(t)$ can be taken as any infinitely divisible distribution with finite variance.*

Corollary 1. *If $\kappa(s) = K$, for $K \in [0, 1]$ a constant, then* $\operatorname{corr}(V_g(t), V_h(t+s))$ *reduces to $K\rho(s)$.*

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The rest of this section, and in particular Lemmas 1 to 3, constitute the proof of Theorem 1. Lemmas 1 to 3 generalize Lemmas 2 and 3 in Finlay *et al.* (2011), where the result was proved for the univariate case (see also Finlay and Seneta (2007), where the result was proved in the discrete-time univariate case for processes with gamma marginal distribution).

Let D_1 denote a given infinitely divisible distribution with finite variance, and $D_{1/n}$ the distribution of the n i.i.d. random variables whose sum has distribution D_1. Fix $n \in \mathbb{N}^+$ and set $Y^n_{i,j,h} \overset{\mathrm{D}}{=} D_{1/n}$, $i = 1, \dots, n$, $j = 0, \pm 1, \pm 2, \dots$, $h = 0, 1, \dots, d$ with all the $Y^n_{i,j,h}$ mutually independent, where '$\overset{\mathrm{D}}{=}$' denotes equality in distribution. Let $[x]$ denote the integer part, and to simplify notation define $\rho_n(x) = [n\rho(x/n)]$ and $f_n(x) = \rho_n(x) - \rho_n(x+1)$. For $\kappa(s)$ any continuous function such that $0 \le \kappa(s) \le 1$, define $f^\kappa_n(x+1) = f_n(x+1) + [\kappa(x/n)(\rho_n(x) - 2\rho_n(x+1) + \rho_n(x+2))]$.

Now using the convention that $\sum_{i=m+1}^{m} x_i = 0$ for any $m \ge 0$, define $V^n_h(t)$ for each $h = 1, \dots, d$ by

$$
\begin{aligned}
V^n_h(t) &= \sum_{j=-\infty}^{[nt]} \Bigg(\sum_{k=[nt]-j}^{\infty} \Bigg(\sum_{i=f_n(k+1)+1}^{f^\kappa_n(k+1)} Y^n_{i,j,0} + \sum_{i=f^\kappa_n(k+1)+1}^{f_n(k)} Y^n_{i,j,h} \Bigg)\Bigg) && (2)\\
&= \sum_{j=-\infty}^{[nt]} \Bigg(\sum_{k=[nt]-j}^{\infty} \Bigg(\sum_{i=\rho_n(k+1)-\rho_n(k+2)+1}^{\rho_n(k)-\rho_n(k+1)} \tilde{Y}^n_{i,j,h} \Bigg)\Bigg)\\
&= \sum_{j=-\infty}^{[nt]} \Bigg(\sum_{i=1}^{\rho_n([nt]-j)-\rho_n([nt]-j+1)} \tilde{Y}^n_{i,j,h} \Bigg). && (3)
\end{aligned}
$$

That is, the $\tilde{Y}^n_{i,j,h}$ are constructed such that for each j and h, for i between $\rho_n(k+1) - \rho_n(k+2) + 1$ and $\rho_n(k) - \rho_n(k+1)$, a fraction $\kappa(k/n)$ of the $\tilde{Y}^n_{i,j,h}$ are drawn from the $Y^n_{i,j,0}$ and the remaining fraction $1 - \kappa(k/n)$ are drawn from the $Y^n_{i,j,h}$. We define $\{\boldsymbol{V}^n(t)\}$ so that $V^n_h(t)$ for each h and t has marginal distribution D_1. This follows since $\rho(s) \to 0$ as $s \to \infty$ by Assumption 1, so that $\sum_{j=-\infty}^{[nt]} \rho_n([nt]-j) - \rho_n([nt]-j+1) = \rho_n(0) = n$ of the $\tilde{Y}^n_{i,j,h} \overset{\mathrm{D}}{=} D_{1/n}$ are summed in (3), ensuring that $V^n_h(t) \overset{\mathrm{D}}{=} D_1$. Furthermore, for each h, t, and s the number of $\tilde{Y}^n_{i,j,h}$ common to $V^n_h(t)$ and $V^n_h(t+s)$ is such that $\operatorname{corr}(V^n_h(t), V^n_h(t+s)) \to \rho(s)$ as $n \to \infty$, and similarly $\operatorname{corr}(V^n_g(t), V^n_h(t+s)) \to \int_0^\infty \int_0^\infty \kappa(s+u+v)\rho''(s+u+v)\,\mathrm{d}v\,\mathrm{d}u$ as $n \to \infty$ for each $g \ne h$, as shown in Lemmas 1 and 2 (correlation between $V^n_g(t)$ and $V^n_h(t+s)$ is created via the $Y^n_{i,j,0}$, which from (2) are common to the $V^n_h(t)$ for each $h = 1, 2, \dots, d$). Note that although (2) and (3) appear to involve infinite sums, for any fixed n all summands for k greater than some finite number and/or j less than some finite number are 0 (for both (2) and (3) at most n summands are nonzero since a total of n of the $Y^n_{i,j,0}$ or $Y^n_{i,j,h}$ are summed, and $\rho_n(x) - \rho_n(x+1) \in \mathbb{N}$ decreases and becomes 0 as x becomes large).

Lemma 1. *Under Assumption 1, for any $t \in \mathbb{R}$ and $s \ge 0$,*

$$\operatorname{corr}(V^n_h(t), V^n_h(t+s)) \to \rho(s) \quad \text{as } n \to \infty.$$

Proof. Using (3), consider any $V^n_h(t)$ and $V^n_h(t+s)$ for $t \in \mathbb{R}$, $s \ge 0$. Then, for any $j \le [nt]$, $V^n_h(t)$ contains the first $\rho_n([nt]-j) - \rho_n([nt]-j+1)$ of the $\tilde{Y}^n_{i,j,h}$, while $V^n_h(t+s)$ contains the first $\rho_n([nt+ns]-j) - \rho_n([nt+ns]-j+1)$ of the same $\tilde{Y}^n_{i,j,h}$. But, $s > 0$ so by Assumption 1 $\rho_n([nt+ns]-j) - \rho_n([nt+ns]-j+1) \le \rho_n([nt]-j) - \rho_n([nt]-j+1)$ for large n, so the number of $\tilde{Y}^n_{i,j,h}$ common to both $V^n_h(t)$ and $V^n_h(t+s)$ is simply $\rho_n([nt+ns] -$

$j) - \rho_n([nt+ns]-j+1)$ (recall that $\rho_n(x)$ is defined as $[n\rho(x/n)]$). For $j > [nt]$, $V_h^n(t)$ contains none of the $\tilde{Y}_{i,j,h}^n$, so the total number of $\tilde{Y}_{i,j,h}^n$ common to both $V_h^n(t)$ and $V_h^n(t+s)$ is given by

$$\sum_{j=-\infty}^{[nt]} \rho_n([nt+ns]-j) - \rho_n([nt+ns]-j+1) = \rho_n([nt+ns]-[nt])$$
$$= \left[n\rho\left(\frac{[nt+ns]-[nt]}{n}\right)\right]$$

and $\operatorname{corr}(V_h^n(t), V_h^n(t+s)) = [n\rho(([nt+ns]-[nt])/n)]/n \to \rho(s)$ as $n\to\infty$. This last step follows from Assumption 1 (see the conclusion of Lemma 2 of Finlay *et al.* (2011, p. 260)).

Lemma 2. *Under Assumptions 1 and 2, for any time $t\in\mathbb{R}$, $g\neq h$ and $s\geq 0$,*

$$\operatorname{corr}(V_g^n(t), V_h^n(t+s)) \to \int_0^\infty\int_0^\infty \kappa(s+u+v)\rho''(s+u+v)\,\mathrm{d}v\,\mathrm{d}u \quad \text{as } n\to\infty.$$

Proof. Using (2), consider any $V_g^n(t)$ and $V_h^n(t+s)$ for $g\neq h$, $s\geq 0$. Then, for any $j\leq [nt]$ and $k\geq [nt+ns]-j$, $V_g^n(t)$ and $V_h^n(t+s)$ both contain $[\kappa(k/n)(\rho_n(k)-2\rho_n(k+1)+\rho_n(k+2))]$ of the same $Y_{i,j,0}^n$. For $k<[nt+ns]-j$, $V_h^n(t+s)$ contains none of the $Y_{i,j,0}^n$, while for $j>[nt]$, $V_h^g(t)$ contains none of the $Y_{i,j,0}^n$. As such the number of $Y_{i,j,0}^n$ common to both $V_g^n(t)$ and $V_h^n(t+s)$ is $\sum_{j=-\infty}^{[nt]}\sum_{k=[nt+ns]-j}^{\infty}[\kappa(k/n)(\rho_n(k)-2\rho_n(k+1)+\rho_n(k+2))]$, and, ignoring rounding issues associated with taking the integer part, $\operatorname{corr}(V_g^n(t), V_h^n(t+s))$ is given by

$$\begin{aligned}
&\frac{1}{n}\sum_{j=-\infty}^{nt}\sum_{k=nt+ns-j}^{\infty}\kappa\left(\frac{k}{n}\right)(\rho_n(k)-2\rho_n(k+1)+\rho_n(k+2))\\
&\quad=\frac{1}{n^2}\sum_{j=-\infty}^{nt}\sum_{k=nt+ns-j}^{\infty}\kappa\left(\frac{k}{n}\right)\left(\frac{\rho(k/n)-2\rho(k/n+1/n)+\rho(k/n+2/n)}{1/n^2}\right)\\
&\quad=\frac{1}{n^2}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\kappa\left(s+\frac{u}{n}+\frac{v}{n}\right)\\
&\qquad\times\left(\frac{\rho(s+u/n+v/n)-2\rho(s+u/n+v/n+1/n)+\rho(s+u/n+v/n+2/n)}{1/n^2}\right),
\end{aligned} \tag{4}$$

where (4) follows by making the change of variable $u=nt-j$ and $v=k-u-ns$. Equation (4) converges to

$$\operatorname{corr}(V_g^n(t), V_h^n(t+s)) = \int_0^\infty\int_0^\infty \kappa(s+u+v)\rho''(s+u+v)\,\mathrm{d}v\,\mathrm{d}u \quad \text{as } n\to\infty \tag{5}$$

since $n^2(\rho(x)-2\rho(x+1/n)+\rho(x+2/n))\to\rho''(x)$ as $n\to\infty$. By noting that

$$\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} f(s+u+v) = \sum_{u=0}^{\infty}(u+1)f(s+u)$$

for any function f, we can also show that (4) converges to an expression equivalent to (5) given by

$$\int_0^\infty u\kappa(s+u)\rho''(s+u)\,\mathrm{d}u.$$

Note that if $\kappa(s) = K$ for $K \in [0, 1]$ is a constant, then (5) reduces to $K\rho(s)$.

Lemma 3. *Under Assumptions 1 and 2 there exists a process $\{\boldsymbol{V}(t)\}$, $t \in \mathbb{R}$ with finite dimensional distributions (and therefore marginal distribution and correlation structure) as implied by (2) and (3) as $n \to \infty$.*

Proof. First we show that the finite dimensional distributions of $\{\boldsymbol{V}^n(t)\}$, $t \in \mathbb{R}$, converge and define a proper set of random variables as $n \to \infty$. Fix $p \in \mathbb{N}^+$ and let $a_{1,1}, \dots, a_{1,p}, \dots, a_{d,1}, \dots, a_{d,p} \in \mathbb{R}$ and $-\infty < s_1 < s_2 < \cdots < s_p$, all in $\mathbb{R}$. To ease the notation, set $g(t, j) = \rho_n([ns_t] - j) - \rho_n([ns_t] - j + 1)$. Then starting from (3), we can show that $\sum_{h=1}^d \sum_{t=1}^p a_{h,t} V_h^n(s_t)$ is given by

$$\sum_{h=1}^{d}\biggl(\sum_{k=1}^{p}\biggl(\sum_{j=[ns_{k-1}]+1}^{[ns_k]}\biggl(\sum_{i=1}^{g(p,j)}\biggl(\biggl(\sum_{t=k}^{p}a_{h,t}\biggr)\tilde{Y}^n_{i,j,h}\biggr)\biggr)\biggr) + \sum_{k=1}^{p-1}\biggl(\sum_{l=1}^{p-k}\biggl(\sum_{j=[ns_{k-1}]+1}^{[ns_k]}\biggl(\sum_{i=g(p-l+1,j)+1}^{g(p-l,j)}\biggl(\biggl(\sum_{t=k}^{p-l}a_{h,t}\biggr)\tilde{Y}^n_{i,j,h}\biggr)\biggr)\biggr)\biggr)\biggr), \tag{6}$$

where we define $s_0 = -\infty$. The above expression reorders the summation of the $\tilde{Y}^n_{i,j,h}$ appearing in

$$\sum_{h=1}^{d}\sum_{t=1}^{p} a_{h,t} V_h^n(s_t)$$

so that any $\tilde{Y}^n_{i,j,h}$ appearing more than once in the sum are grouped together. But the $\tilde{Y}^n_{i,j,h}$ are not i.i.d. since by construction they are drawn from a set of (common) $Y^n_{i,j,0}$ and (unique) $Y^n_{i,j,h}$. Refining (6) to a grouping of all the $Y^n_{i,j,h}$ yields

$$\begin{aligned}
&\sum_{k=1}^{p}\biggl(\sum_{j=[ns_{k-1}]+1}^{[ns_k]}\biggl(\sum_{m=-\infty}^{j}\biggl(\sum_{i=g(p,m-1)+1}^{g^\kappa(p,m-1)}\biggl(\biggl(\sum_{h=1}^{d}\sum_{t=k}^{p}a_{h,t}\biggr)Y^n_{i,j,0}\biggr)\biggr)\biggr)\biggr)\\
&+\sum_{k=1}^{p-1}\biggl(\sum_{l=1}^{p-k}\biggl(\sum_{j=[ns_{k-1}]+1}^{[ns_k]}\biggl(\sum_{m=1}^{[ns_{p-l+1}]-[ns_{p-l}]}\biggl(\sum_{i=g(p-l+1,j+m-1)+1}^{g^\kappa(p-l+1,j+m-1)}\biggl(\biggl(\sum_{h=1}^{d}\sum_{t=k}^{p-l}a_{h,t}\biggr)Y^n_{i,j,0}\biggr)\biggr)\biggr)\biggr)\biggr)\\
&+\sum_{h=1}^{d}\biggl(\sum_{k=1}^{p}\biggl(\sum_{j=[ns_{k-1}]+1}^{[ns_k]}\biggl(\sum_{m=-\infty}^{j}\biggl(\sum_{i=g^\kappa(p,m-1)+1}^{g(p,m)}\biggl(\biggl(\sum_{t=k}^{p}a_{h,t}\biggr)Y^n_{i,j,h}\biggr)\biggr)\biggr)\biggr)\\
&\quad+\sum_{k=1}^{p-1}\biggl(\sum_{l=1}^{p-k}\biggl(\sum_{j=[ns_{k-1}]+1}^{[ns_k]}\biggl(\sum_{m=1}^{[ns_{p-l+1}]-[ns_{p-l}]}\biggl(\sum_{i=g^\kappa(p-l+1,j+m-1)+1}^{g(p-l+1,j+m)}\biggl(\biggl(\sum_{t=k}^{p-l}a_{h,t}\biggr)Y^n_{i,j,h}\biggr)\biggr)\biggr)\biggr)\biggr)\biggr),
\end{aligned} \tag{7}$$

where we define $g^\kappa(t, m-1) = g(t, m-1) + [\kappa(([ns_t] - m)/n)(g(t, m) - g(t, m-1))]$, so that $\sum_{i=g(t,m-1)+1}^{g^\kappa(t,m-1)}$ is the sum of the $\tilde{Y}^n_{i,j,h}$ between $g(t, m-1)$ and $g(t, m)$ which are drawn from the $Y^n_{i,j,0}$, of which there are $[\kappa(([ns_t] - m)/n)(g(t, m) - g(t, m-1))]$ in total, and $\sum_{i=g^\kappa(t,m-1)+1}^{g(t,m)}$ is the sum of the $\tilde{Y}^n_{i,j,h}$ between $g(t, m-1)$ and $g(t, m)$ which are drawn from the $Y^n_{i,j,h}$, of which there are $(g(t, m) - g(t, m-1)) - [\kappa(([ns_t] - m)/n)(g(t, m) - g(t, m-1))]$

in total. Each $Y^n_{i,j,h}$ is i.i.d. $D_{1/n}$ distributed, with characteristic function $\phi^D_{1/n}(t)$ say, so the characteristic function of $(\boldsymbol{V}^n(s_1), \dots, \boldsymbol{V}^n(s_p))$, defined as $\mathbb{E}\exp(\mathrm{i}\sum_{h=1}^d \sum_{t=1}^p a_{h,t} V^n_h(s_t))$, is given by

$$
\begin{aligned}
&\Phi^n_p(a_{1,1}, \dots, a_{d,p}) \\
&\quad = \Bigg(\prod_{k=1}^p \Bigg(\phi^D_{1/n}\Bigg(\sum_{h=1}^d \sum_{t=k}^p a_{h,t}\Bigg)\Bigg)^{\sum_{j=[ns_{k-1}]+1}^{[ns_k]} \sum_{m=-\infty}^{j} g^{\kappa}(p,m-1)-g(p,m-1)}\Bigg) \\
&\qquad \times \Bigg(\prod_{k=1}^{p-1}\Bigg(\prod_{l=1}^{p-k}\Bigg(\phi^D_{1/n}\Bigg(\sum_{h=1}^d \sum_{t=k}^{p-l} a_{h,t}\Bigg)\Bigg)^{\sum_{j=[ns_{k-1}]+1}^{[ns_k]} \sum_{m=1}^{[ns_{p-l+1}]-[ns_{p-l}]} \lambda_1}\Bigg)\Bigg) \\
&\qquad \times \Bigg(\prod_{h=1}^d\Bigg(\prod_{k=1}^{p}\Bigg(\phi^D_{1/n}\Bigg(\sum_{t=k}^{p} a_{h,t}\Bigg)\Bigg)^{\sum_{j=[ns_{k-1}]+1}^{[ns_k]} \sum_{m=-\infty}^{j} g(p,m)-g^{\kappa}(p,m-1)}\Bigg)\Bigg) \\
&\qquad \times \Bigg(\prod_{h=1}^d\Bigg(\prod_{k=1}^{p-1}\Bigg(\prod_{l=1}^{p-k}\Bigg(\phi^D_{1/n}\Bigg(\sum_{t=k}^{p-l} a_{h,t}\Bigg)\Bigg)^{\sum_{j=[ns_{k-1}]+1}^{[ns_k]} \sum_{m=1}^{[ns_{p-l+1}]-[ns_{p-l}]} \lambda_2}\Bigg)\Bigg)\Bigg), \qquad (8)
\end{aligned}
$$

where $\lambda_1 = g^{\kappa}(p-l+1, j+m-1) - g(p-l+1, j+m-1)$, $\lambda_2 = g(p-l+1, j+m) - g^{\kappa}(p-l+1, j+m-1)$, and we use the convention that $\prod_{i=m+1}^m x_i = 1$ for any $m \geq 0$.

As D_1 is infinitely divisible, $\phi^D_{1/n}(t) = (\phi^D_1(t))^{1/n}$. Now

$$
\begin{aligned}
&\frac{1}{n}\sum_{j=[ns_{k-1}]+1}^{[ns_k]} \sum_{m=-\infty}^{j} g^{\kappa}(p, m-1) - g(p, m-1) \\
&\quad = \frac{1}{n}\sum_{j=[ns_{k-1}]+1}^{[ns_k]} \sum_{m=-\infty}^{j} \Bigg[\kappa\Bigg(\frac{[ns_p]-m}{n}\Bigg)(\rho_n([ns_p]-m) - 2\rho_n([ns_p]-m+1) \\
&\qquad + \rho_n([ns_p]-m+2))\Bigg] \\
&\quad \to \int_{s_{k-1}}^{s_k}\int_{-\infty}^{y} \kappa(s_p - x)\rho''(s_p - x)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
$$

while

$$
\begin{aligned}
&\frac{1}{n}\sum_{j=[ns_{k-1}]+1}^{[ns_k]} \sum_{m=1}^{[ns_{p-l+1}]-[ns_{p-l}]} g^{\kappa}(p-l+1, j+m-1) - g(p-l+1, j+m-1) \\
&\quad = \frac{1}{n}\sum_{j=[ns_{k-1}]+1}^{[ns_k]} \sum_{m=1}^{[ns_{p-l+1}]-[ns_{p-l}]} \Bigg[\kappa\Bigg(\frac{[ns_{p-l+1}]-j-m}{n}\Bigg)(\rho_n([ns_{p-l+1}]-j-m) \\
&\qquad - 2\rho_n([ns_{p-l+1}]-j-m+1) + \rho_n([ns_{p-l+1}]-j-m+2))\Bigg] \\
&\quad \to \int_{s_{k-1}}^{s_k}\int_{0}^{s_{p-l+1}-s_{p-l}} \kappa(s_{p-l+1}-x-y)\rho''(s_{p-l+1}-x-y)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}
$$

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Similarly,

$$\frac{1}{n}\sum_{j=[ns_{k-1}]+1}^{[ns_k]}\sum_{m=-\infty}^{j} g(p,m)-g^{\kappa}(p,m-1)\to\int_{s_{k-1}}^{s_k}\int_{-\infty}^{y}(1-\kappa(s_p-x))\rho''(s_p-x)\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\frac{1}{n}\sum_{j=[ns_{k-1}]+1}^{[ns_k]}\sum_{m=1}^{[ns_{p-l+1}]-[ns_{p-l}]} g(p-l+1,j+m)-g^{\kappa}(p-l+1,j+m-1)$$
$$\to\int_{s_{k-1}}^{s_k}\int_{0}^{s_{p-l+1}-s_{p-l}}(1-\kappa(s_{p-l+1}-x-y))\rho''(s_{p-l+1}-x-y)\,\mathrm{d}y\,\mathrm{d}x.$$

Hence, $\Phi_p^n(a_{1,1},\dots,a_{d,p})$ converges to a function $\Phi_p(a_{1,1},\dots,a_{d,p})$ given by

$$\begin{aligned}\Phi_p(a_{1,1},\dots,a_{d,p})
&=\Bigg(\prod_{k=1}^{p}\Bigg(\phi_1^D\Bigg(\sum_{h=1}^{d}\sum_{t=k}^{p}a_{h,t}\Bigg)\Bigg)^{\int_{s_{k-1}}^{s_k}\int_{-\infty}^{y}\kappa(s_p-x)\rho''(s_p-x)\,\mathrm{d}x\,\mathrm{d}y}\Bigg)\\
&\quad\times\Bigg(\prod_{k=1}^{p-1}\Bigg(\prod_{l=1}^{p-k}\Bigg(\phi_1^D\Bigg(\sum_{h=1}^{d}\sum_{t=k}^{p-l}a_{h,t}\Bigg)\Bigg)^{\int_{s_{k-1}}^{s_k}\int_0^{s_{p-l+1}-s_{p-l}}\Psi_1\,\mathrm{d}y\,\mathrm{d}x}\Bigg)\Bigg)\\
&\quad\times\Bigg(\prod_{h=1}^{d}\Bigg(\prod_{k=1}^{p}\Bigg(\phi_1^D\Bigg(\sum_{t=k}^{p}a_{h,t}\Bigg)\Bigg)^{\int_{s_{k-1}}^{s_k}\int_{-\infty}^{y}(1-\kappa(s_p-x))\rho''(s_p-x)\,\mathrm{d}x\,\mathrm{d}y}\Bigg)\Bigg)\\
&\quad\times\Bigg(\prod_{h=1}^{d}\Bigg(\prod_{k=1}^{p-1}\Bigg(\prod_{l=1}^{p-k}\Bigg(\phi_1^D\Bigg(\sum_{t=k}^{p-l}a_{h,t}\Bigg)\Bigg)^{\int_{s_{k-1}}^{s_k}\int_0^{s_{p-l+1}-s_{p-l}}\Psi_2\,\mathrm{d}y\,\mathrm{d}x}\Bigg)\Bigg)\Bigg),\end{aligned}\tag{9}$$

where $\Psi_1=\kappa(s_{p-l+1}-x-y)\rho''(s_{p-l+1}-x-y)$, $\Psi_2=(1-\kappa(s_{p-l+1}-x-y))\rho''(s_{p-l+1}-x-y)$, and which is continuous about the origin so long as $\phi_1^D(t)$ is. Weak convergence of the finite dimensional distributions of $\{\boldsymbol{V}^n(t)\}$ to proper random variables follows from Billingsley (1968, Theorem 7.6). (Noting that for $s_p\ge s_j>s_k$, $\int_{-\infty}^{s_k}\int_{-\infty}^{y}f(s_p-x)\,\mathrm{d}x\,\mathrm{d}y=\int_0^{\infty}\int_{s_p-s_j}^{\infty}f(s_j-s_k+x+y)\,\mathrm{d}x\,\mathrm{d}y$ and $\int_{-\infty}^{s_k}\int_0^{s_p-s_j}f(s_p-x-y)\,\mathrm{d}x\,\mathrm{d}y=\int_0^{\infty}\int_0^{s_p-s_j}f(s_j-s_k+x+y)\,\mathrm{d}x\,\mathrm{d}y$, we can also use (9) to verify that $V_h(t)\overset{\mathrm{D}}{=}D_1$ and that the desired correlation structure holds.)

Hence, the finite dimensional distributions as $n\to\infty$ of $\{\boldsymbol{V}^n(t)\}$ as defined by (3) are consistent, and so by Kolmogorov's existence theorem there exists a random field $\{\boldsymbol{V}(t)\}$, $t\in\mathbb{R}$ with these same finite dimensional distributions (see, for example, Khoshnevisan (2002)).

Kolmogorov's continuity theorem provides a sufficient condition for $\{\boldsymbol{V}(t)\}$ to have a modification with almost surely continuous sample paths, being that there exist constants $C>0$, $p>0$ and $\gamma>d$ such that $\mathbb{E}|\boldsymbol{V}(t)-\boldsymbol{V}(t+s)|^p\le C|s|^{\gamma}$ (here $\{\tilde{\boldsymbol{V}}(t)\}$ is said to be a modification of $\{\boldsymbol{V}(t)\}$ if $\mathrm{P}(\tilde{\boldsymbol{V}}(t)=\boldsymbol{V}(t))=1$ for all t; see, for example, Khoshnevisan (2002) and Øksendal (2003)). Taking $p=2$ and $|\boldsymbol{x}|=\sqrt{x_1^2+\cdots+x_d^2}$ we have $\mathbb{E}|\boldsymbol{V}(t)-\boldsymbol{V}(t+s)|^2=\mathbb{E}(V_1(t)-V_1(t+s))^2+\cdots+\mathbb{E}(V_d(t)-V_d(t+s))^2=d\mathbb{E}(V_1(t)-V_1(t+s))^2=d\operatorname{var}(V_1(t)-V_1(t+s))$.

Now

$$\begin{aligned}
V_1^n(t) - V_1^n(t+s) &= \sum_{j=-\infty}^{[nt]} \left(\sum_{i=1}^{\rho_n([nt]-j)-\rho_n([nt]-j+1)} \tilde{Y}_{i,j,1}^n \right) \\
&\quad - \sum_{j=-\infty}^{[nt+ns]} \left(\sum_{i=1}^{\rho_n([nt+ns]-j)-\rho_n([nt+ns]-j+1)} \tilde{Y}_{i,j,1}^n \right) \\
&= \sum_{j=-\infty}^{[nt]} \left(\sum_{i=\rho_n([nt+ns]-j)-\rho_n([nt+ns]-j+1)+1}^{\rho_n([nt]-j)-\rho_n([nt]-j+1)} \tilde{Y}_{i,j,1}^n \right) \\
&\quad - \sum_{j=[nt]+1}^{[nt+ns]} \left(\sum_{i=1}^{\rho_n([nt+ns]-j)-\rho_n([nt+ns]-j+1)} \tilde{Y}_{i,j,1}^n \right) \qquad (10)
\end{aligned}$$

and since the $\tilde{Y}_{i,j,1}^n$ are i.i.d., $\operatorname{var}(V_1^n(t) - V_1^n(t+s))$ is given by the number of $\tilde{Y}_{i,j,1}^n$ included in the sums that constitute (10), multiplied by $\operatorname{var}(\tilde{Y}_{1,1,1}^n) = \sigma^2/n$, where we define σ^2 as the variance of $V_1^n(t) \overset{\mathrm{D}}{=} D_1$. The number of $\tilde{Y}_{i,j,1}^n$ included in (10) is given by $2(\rho_n(0) - \rho_n([nt+ns]-[nt])) \leq 2n(1-\rho(s+2/n)+1/n)$, so that $\operatorname{var}(V_1^n(t) - V_1^n(t+s)) \leq 2(1-\rho(s+2/n)+1/n)\sigma^2 \to 2(1-\rho(s))\sigma^2$ as $n \to \infty$, and, therefore, $\mathbb{E}|\boldsymbol{V}(t) - \boldsymbol{V}(t+s)|^2 \leq 2d\sigma^2(1-\rho(s))$. As such, if there exists a $C > 0$ and $\gamma > d$ such that for any s, $1 - \rho(s) \leq C|s|^\gamma$, or, equivalently, if $\rho(s) \geq 1 - C|s|^\gamma$, then $\{\boldsymbol{V}(t)\}$ will have a continuous modification.

4. A random field in discrete time

Assumption 3. *It holds that $\rho(s)$, $s \in \mathbb{N}$ is a function symmetric about $s = 0$ satisfying $\rho(0) = 1$, $\rho(s) \to 0$ as $s \to \infty$, and for $s > 0$ satisfying $\rho(s) \geq 0$, $\rho(s+1) - \rho(s) \leq 0$, $\rho(s+2) - 2\rho(s+1) + \rho(s) \geq 0$. This is the discrete time analogue of Assumption 1.*

Note that Assumption 3 implies $\rho(s+1) - \rho(s) \to 0$ as $s \to \infty$.

Assumption 4. *It holds that $\kappa(s)$, $s \in \mathbb{N}$ is such that $0 \leq \kappa(s) \leq 1$.*

Theorem 2. *If $\rho(s)$ is a function satisfying Assumption 3 and $\kappa(s)$ is a function satisfying Assumption 4, then there exists a second-order stationary random field $\{\boldsymbol{X}(t)\} = \{X_1(t), \ldots, X_d(t)\}$, $t \in \mathbb{N}$, $d \in \mathbb{N}^+$ such that $\operatorname{corr}(X_h(t), X_h(t+s)) = \rho(s)$ and $\operatorname{corr}(X_g(t), X_h(t+s)) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \kappa(s+j+k)(\rho(s+j+k) - 2\rho(s+j+k+1) + \rho(s+j+k+2))$ for $s \in \mathbb{N}^+$, $g, h = 1, \ldots, d$, $g \neq h$. The marginal distribution of $X_h(t)$ can be taken as any infinitely divisible distribution with finite variance.*

Corollary 2. *If $\kappa(s) = K$, for $K \in [0, 1]$ a constant, then $\operatorname{corr}(X_g(t), X_h(t+s))$ reduces to $K\rho(s)$.*

Proof. Redefining $f_n(x)$ as $[n\rho(x)] - [n\rho(x+1)]$ and

$$f_n^\kappa(x+1) = f_n(x+1) + [\kappa(x)([n\rho(x)] - 2[n\rho(x+1)] + [n\rho(x+2)])],$$

and defining

$$X_h^n(t) = \sum_{j=-\infty}^{t} \left(\sum_{k=t-j}^{\infty} \left(\sum_{i=f_n(k+1)+1}^{f_n^\kappa(k+1)} Y_{i,j,0}^n + \sum_{i=f_n^\kappa(k+1)+1}^{f_n(k)} Y_{i,j,h}^n \right) \right)$$

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$$
\begin{aligned}
&= \sum_{j=-\infty}^{t} \left(\sum_{k=t-j}^{\infty} \left(\sum_{i=[n\rho(k+1)]-[n\rho(k+2)]+1}^{[n\rho(k)]-[n\rho(k+1)]} \tilde{Y}^n_{i,j,h} \right) \right) \\
&= \sum_{j=-\infty}^{t} \left(\sum_{i=1}^{[n\rho(t-j)]-[n\rho(t-j+1)]} \tilde{Y}^n_{i,j,h} \right) \\
&\stackrel{\mathrm{D}}{=} D_1
\end{aligned}
$$

we can show, via an almost identical argument to that used in Lemmas 1 and 2, that $\operatorname{corr}(X_h^n(t), X_h^n(t+s)) \to \rho(s)$ and $\operatorname{corr}(X_g^n(t), X_h^n(t+s)) \to \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\kappa(s+j+k)(\rho(s+j+k) - 2\rho(s+j+k+1)+\rho(s+j+k+2))$ as $n \to \infty$. We can further show that $\sum_{h=1}^{d}\sum_{t=1}^{p} a_{h,t}X_h^n(s_t)$ is given by (7) where we replace $[ns_k]$ wherever it appears by s_k, replace $\rho_n(x)$ by $[n\rho(x)]$, replace $\kappa(x/n)$ by $\kappa(x)$, redefine $g(t, j)$ as $g(t, j) = [n\rho(s_t - j)] - [n\rho(s_t - j + 1)]$, and redefine $g^{\kappa}(t, m-1)$ as $g^{\kappa}(t, m-1) = g(t, m-1) + [\kappa(s_t - m)(g(t, m) - g(t, m-1))]$, where as before $\sum_{i=g(t,m-1)+1}^{g^{\kappa}(t,m-1)}$ is the sum of the $\tilde{Y}^n_{i,j,h}$ between $g(t, m-1)$ and $g(t, m)$ which are drawn from the $Y^n_{i,j,0}$, of which there are $[\kappa(s_t - m)(g(t, m) - g(t, m-1))]$ in total, and $\sum_{i=g^{\kappa}(t,m-1)+1}^{g(t,m)}$ is the sum of the $\tilde{Y}^n_{i,j,h}$ between $g(t, m-1)$ and $g(t, m)$ which are drawn from the $Y^n_{i,j,h}$, of which there are $(g(t, m) - g(t, m-1)) - [\kappa(s_t - m)(g(t, m) - g(t, m-1))]$ in total. Using these same redefinitions, the characteristic function of $(\boldsymbol{X}^n(s_1), \dots, \boldsymbol{X}^n(s_p))$ is given by (8). Equation (8) then converges to an expression similar to (9) as $n \to \infty$, where we now replace $\int_{s_{k-1}}^{s_k}\int_{-\infty}^{y}\kappa(s_p - x)\rho''(s_p - x)\,\mathrm{d}x\,\mathrm{d}y$ with $\sum_{j=s_{k-1}+1}^{s_k}\sum_{m=-\infty}^{j}\kappa(s_p - m)(\rho(s_p - m) - 2\rho(s_p - m + 1) + \rho(s_p - m + 2))$, replace $\int_{s_{k-1}}^{s_k}\int_0^{s_{p-l+1}-s_{p-l}}\kappa(s_{p-l+1} - x - y)\rho''(s_{p-l+1} - x - y)\,\mathrm{d}y\,\mathrm{d}x$ with $\sum_{j=s_{k-1}+1}^{s_k}\sum_{m=1}^{s_{p-l+1}-s_{p-l}}\kappa(s_{p-l+1} - j - m)(\rho(s_{p-l+1} - j - m) - 2\rho(s_{p-l+1} - j - m + 1) + \rho(s_{p-l+1} - j - m + 2))$, and similarly for the expressions involving $1 - \kappa$. Weak convergence of the finite dimensional distributions of $\{\boldsymbol{X}^n(t)\}$ follows from Billingsley (1968, Theorem 7.6; see also the second paragraph on p. 30), which in the discrete-time case is enough to prove that our process $\{\boldsymbol{X}^n(t)\}$ converges weakly to the limit of process $\{\boldsymbol{X}(t)\}$.

5. Possible extensions

To keep the exposition as simple as possible we imposed a number of constraints on our construction which can be relaxed, as detailed below.

5.1. Extending the domain

We constructed $\{\boldsymbol{V}(t)\}$ as a random field where the time dimension was defined on $\mathbb{R}$ and the spatial dimension was defined on $\{1, 2, \dots d\}$. By considering $\sum_{h=1}^{d}\sum_{t=1}^{p} a_{h,t}V^n_{\eta_h}(s_t)$ for $d, p \in \mathbb{N}^+$ and $\eta_h \in \mathbb{N}$ or $\eta_h \in \mathbb{R}$ in Lemma 3, instead of $\sum_{h=1}^{d}\sum_{t=1}^{p} a_{h,t}V^n_h(s_t)$, we can use the argument put forward in Lemma 3 to show that the finite dimensional distributions of $\{V^n_{\eta_1}(t), \dots, V^n_{\eta_d}(t)\}$ are well defined for any $\eta_h \in \mathbb{N}$ or $\eta_h \in \mathbb{R}$, and, therefore, a limit random field with spatial dimension defined on $\mathbb{N}$ or $\mathbb{R}$, instead of just on $\{1, 2, \dots, d\}$, exists. The characteristic function of the finite dimensional distributions of the new process, $\mathbb{E}\exp(\mathrm{i}\sum_{h=1}^{d}\sum_{t=1}^{p} a_{h,t}V^n_{\eta_h}(s_t))$, is unchanged from (9).

5.2. Altering the marginal distribution

In our construction, each $\{V_h(t)\}$, $h = 1, \dots, d$, has the same marginal distribution. This is not necessary: the marginal distribution of $\{V_h(t)\}$ for each h is determined by the number of $\tilde{Y}^n_{i,j,h}$ that are summed in (2) and (3), and this can be varied. For example, by summing in (3)

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from $i = 1$ to $0.7(\rho_n([nt]-j) - \rho_n([nt]-j+1))$ for each j for $h = 1$, instead of from $i = 1$ to $\rho_n([nt]-j) - \rho_n([nt]-j+1)$, $\{V_1(t)\}$ will have marginal distribution $D_{0.7}$, while $\{V_h(t)\}$, $h \neq 1$ will have marginal distribution D_1. Note that we still require that all $\{V_h(t)\}$ belong to the same class of infinitely divisible distribution.

5.3. Allowing $\rho(s) \to \delta > 0$ as $s \to \infty$

In our construction we assumed that $\rho(s) \to 0$ as $s \to \infty$, which ensures that a total of n of the $\tilde{Y}^n_{i,j,h}$ are summed in (2) and (3). Let $\rho^*(s)$ satisfy all the requirements of Assumption 1 except for $\rho^*(s) \to 0$ as $s \to \infty$, and instead let $\rho^*(s) \to \delta > 0$ as $s \to \infty$. We can construct a random field $\{\boldsymbol{V}^*(t)\}$ which has marginal distribution D_1 for any infinitely divisible distribution D_1 with finite variance, an autocorrelation function $\operatorname{corr}(V^*_h(t), V^*_h(t+s)) = \rho^*(s)$, and cross-correlation function $\operatorname{corr}(V^*_g(t), V^*_h(t+s)) = \int_0^\infty \int_0^\infty \kappa(s+u+v)\rho^{*\prime\prime}(s+u+v)\,\mathrm{d}v\,\mathrm{d}u$ as follows. Define $\rho(s) = (\rho^*(s) - \delta)/(1-\delta)$ so that $\rho(s)$ satisfies all requirements of Assumption 1, including that for $\rho(s) \to 0$ as $s \to \infty$, and construct $\{\boldsymbol{V}(t)\}$ as per Section 3 except taking $Y^n_{i,j,h} \stackrel{\mathrm{D}}{=} D_{(1-\delta)/n}$ in (2) and (3) instead of $Y^n_{i,j,h} \stackrel{\mathrm{D}}{=} D_{1/n}$, so that $V_h(t) \stackrel{\mathrm{D}}{=} D_{1-\delta}$ for each h and t. We now define a set of random variables $\{\boldsymbol{V}^\delta\} = \{V^\delta_1, \dots, V^\delta_d\}$ such that each $V^\delta_h \stackrel{\mathrm{D}}{=} D_\delta$ and is independent of $\{\boldsymbol{V}(t)\}$, and define $\{\boldsymbol{V}^*(t)\} = \{\boldsymbol{V}(t)\} + \{\boldsymbol{V}^\delta\}$. Then

$$
\begin{aligned}
\operatorname{corr}(V^*_h(t), V^*_h(t+s)) &= \operatorname{corr}(V_h(t) + V^\delta_h, V_h(t+s) + V^\delta_h) \\
&= (1-\delta)\operatorname{corr}(V_h(t), V_h(t+s)) + \delta \operatorname{corr}(V^\delta_h, V^\delta_h) \\
&= (1-\delta)\rho(s) + \delta \\
&= \rho^*(s) - \delta + \delta \\
&= \rho^*(s),
\end{aligned}
$$

while

$$
\begin{aligned}
&\operatorname{corr}(V^*_g(t), V^*_h(t+s)) \\
&\quad= \operatorname{corr}(V_g(t) + V^\delta_g, V_h(t+s) + V^\delta_h) \\
&\quad= (1-\delta)\operatorname{corr}(V_g(t), V_h(t+s)) + \delta \operatorname{corr}(V^\delta_g, V^\delta_h) \\
&\quad= (1-\delta)\int_0^\infty \int_0^\infty \kappa(s+u+v)\rho''(s+u+v)\,\mathrm{d}v\,\mathrm{d}u + \delta \operatorname{corr}(V^\delta_g, V^\delta_h) \\
&\quad= \int_0^\infty \int_0^\infty \kappa(s+u+v)\rho^{*\prime\prime}(s+u+v)\,\mathrm{d}v\,\mathrm{d}u + \delta \operatorname{corr}(V^\delta_g, V^\delta_h)
\end{aligned}
$$

since $\rho''(s) = \rho^{*\prime\prime}(s)/(1-\delta)$. Constructing $\{\boldsymbol{V}^\delta\}$ such that each V^δ_h is i.i.d. yields

$$
\operatorname{corr}(V^*_g(t), V^*_h(t+s)) = \int_0^\infty \int_0^\infty \kappa(s+u+v)\rho^{*\prime\prime}(s+u+v)\,\mathrm{d}v\,\mathrm{d}u,
$$

but $\{\boldsymbol{V}^\delta\}$ may be constructed so that $\operatorname{corr}(V^\delta_g, V^\delta_h)$ takes any value between 0 and 1.

5.4. Allowing the cross-correlation function to vary

The cross-correlation between $\{V_g(t)\}$ and $\{V_h(t)\}$ is determined by the degree of overlap between the $\tilde{Y}^n_{i,j,g}$ and $\tilde{Y}^n_{i,j,h}$ in (3). For simplicity, in our construction we choose to have all overlaps occur via the $Y^n_{i,j,0}$, and to have the same degree of overlap, and, thus, the same cross-correlation function, between all g and h. This is not necessary. For example, create additional random variables $Y^n_{i,j,h} \stackrel{\mathrm{D}}{=} D_{1/n}$ for $h = -1, -2, \dots$, where as before $i = 1, \dots, n$ and

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$j = 0, \pm 1, \pm 2, \dots$, again with all the $Y^n_{i,j,h}$ mutually independent. Now for each j, for $h = 1, \dots, d$ and for $c(a)$ any function satisfying Assumption 3, define $\tilde{Y}^n_{i,j,h}$ such that for $k = 0, 1, 2 \dots$ and for $a = 0, 1, 2 \dots$, $\tilde{Y}^n_{i,j,h} = Y^n_{i,j,-(h+a)}$ for $i = e^\kappa_n(k+1, a)+1, \dots, e^\kappa_n(k+1, a+1)$, and $\tilde{Y}^n_{i,j,h} = Y^n_{i,j,h}$ for $i = f^\kappa_n(k+1)+1, \dots, f_n(k)$, where for a given j and h we adopt the convention that as k and a increase, if $\tilde{Y}^n_{i,j,h}$ for some i has already been assigned a value then we do not reassign it a new value, and where $e^\kappa_n(x+1, a) = \rho_n(x+1) - \rho_n(x+2) + [(1-c(a))\kappa(x/n)(\rho_n(x) - 2\rho_n(x+1) + \rho_n(x+2))]$, $f_n(x) = \rho_n(x) - \rho_n(x+1)$ and $f^\kappa_n(x+1) = f_n(x+1) + [\kappa(x/n)(\rho_n(x) - 2\rho_n(x+1) + \rho_n(x+2))]$. That is, the $\tilde{Y}^n_{i,j,h}$ are constructed such that for each j and h, for i between $\rho_n(k+1) - \rho_n(k+2) + 1$ and $\rho_n(k) - \rho_n(k+1)$ a fraction $(c(a) - c(a+1))\kappa(k/n)$ of the $\tilde{Y}^n_{i,j,h}$ are drawn from the $Y^n_{i,j,-(h+a)}$ for $a = 0, 1, 2, \dots$, and the remaining fraction $1 - \kappa(k/n)$ are drawn from the $Y^n_{i,j,h}$.

In this case, again adopting the convention that $\sum_{i=m+1}^m x_i = 0$ for any $m \ge 0$, we have

$$\begin{aligned}
V^n_h(t) &= \sum_{j=-\infty}^{[nt]} \Bigg(\sum_{k=[nt]-j}^{\infty} \Bigg(\sum_{a=0}^{\infty} \Bigg(\sum_{i=e^\kappa_n(k+1,a)+1}^{e^\kappa_n(k+1,a+1)} Y^n_{i,j,-(h+a)} \Bigg) + \sum_{i=f^\kappa_n(k+1)+1}^{f_n(k)} Y^n_{i,j,h} \Bigg)\Bigg) \\
&= \sum_{j=-\infty}^{[nt]} \Bigg(\sum_{k=[nt]-j}^{\infty} \Bigg(\sum_{i=\rho_n(k+1)-\rho_n(k+2)+1}^{\rho_n(k)-\rho_n(k+1)} \tilde{Y}^n_{i,j,h} \Bigg)\Bigg) \\
&= \sum_{j=-\infty}^{[nt]} \Bigg(\sum_{i=1}^{\rho_n([nt]-j)-\rho_n([nt]-j+1)} \tilde{Y}^n_{i,j,h} \Bigg).
\end{aligned}$$

The construction above ensures that, for $g < h$, the number of $Y^n_{i,j,-a}$, $a = 1, 2, \dots$, common to both the $\tilde{Y}^n_{i,j,g}$ and the $\tilde{Y}^n_{i,j,h}$ for i between $\rho_n(k+1) - \rho_n(k+2) + 1$ and $\rho_n(k) - \rho_n(k+1)$ is given by

$$\begin{aligned}
&\sum_{a=h}^{\infty} \min\Bigg((c(a-h) - c(a-h+1))\kappa\left(\frac{k}{n}\right)(\rho_n(k) - 2\rho_n(k+1) + \rho_n(k+2)), \\
&\qquad\qquad (c(a-g) - c(a-g+1))\kappa\left(\frac{k}{n}\right)(\rho_n(k) - 2\rho_n(k+1) + \rho_n(k+2)) \Bigg) \\
&\quad = \sum_{a=h}^{\infty} (c(a-g) - c(a-g+1))\kappa\left(\frac{k}{n}\right)(\rho_n(k) - 2\rho_n(k+1) + \rho_n(k+2)) \\
&\quad = c(h-g)\kappa\left(\frac{k}{n}\right)(\rho_n(k) - 2\rho_n(k+1) + \rho_n(k+2)).
\end{aligned}$$

Using an argument similar to that used in Lemma 2, we can show that, for any $g \ne h$,

$$\operatorname{corr}(V^n_g(t), V^n_h(t+s)) \to c(|h-g|) \int_0^\infty \int_0^\infty \kappa(s+u+v)\rho''(s+u+v)\,\mathrm{d}v\,\mathrm{d}u \quad \text{as } n \to \infty.$$

That is, the cross-correlation function is now dampened by $c(|h-g|)$ and so decreases as g and h move further apart. The characteristic function of this new process is given by

$$\begin{aligned}
&\Phi_p(a_{1,1}, \dots, a_{d,p}) \\
&\quad = \Bigg(\prod_{k=1}^{p} \Bigg(\phi^D_1 \Bigg(\sum_{h=1}^{d} \sum_{t=k}^{p} a_{h,t} \Bigg)\Bigg)^{c(d-1) \int_{s_{k-1}}^{s_k} \int_{-\infty}^{y} \kappa(s_p - x)\rho''(s_p - x)\,\mathrm{d}x\,\mathrm{d}y} \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\times \Bigg(\prod_{j=2}^{d}\Bigg(\prod_{k=1}^{p}\Bigg(\phi_1^D\Bigg(\sum_{h=j}^{d}\sum_{t=k}^{p} a_{h,t}\Bigg)\Bigg)^{-c'(d-j)\int_{s_{k-1}}^{s_k}\int_{-\infty}^{y}\kappa(s_p-x)\rho''(s_p-x)\,\mathrm{d}x\,\mathrm{d}y}\Bigg)\Bigg)\\
&\times \Bigg(\prod_{j=2}^{d-1}\Bigg(\prod_{i=2}^{j}\Bigg(\prod_{k=1}^{p}\Bigg(\phi_1^D\Bigg(\sum_{h=i}^{j}\sum_{t=k}^{p} a_{h,t}\Bigg)\Bigg)^{c''(j-i)\int_{s_{k-1}}^{s_k}\int_{-\infty}^{y}\kappa(s_p-x)\rho''(s_p-x)\,\mathrm{d}x\,\mathrm{d}y}\Bigg)\Bigg)\Bigg)\\
&\times \Bigg(\prod_{j=1}^{d-1}\Bigg(\prod_{k=1}^{p}\Bigg(\phi_1^D\Bigg(\sum_{h=1}^{j}\sum_{t=k}^{p} a_{h,t}\Bigg)\Bigg)^{-c'(j-1)\int_{s_{k-1}}^{s_k}\int_{-\infty}^{y}\kappa(s_p-x)\rho''(s_p-x)\,\mathrm{d}x\,\mathrm{d}y}\Bigg)\Bigg)\\
&\times \Bigg(\prod_{k=1}^{p-1}\Bigg(\prod_{l=1}^{p-k}\Bigg(\phi_1^D\Bigg(\sum_{h=1}^{d}\sum_{t=k}^{p-l} a_{h,t}\Bigg)\Bigg)^{c(d-1)\int_{s_{k-1}}^{s_k}\int_{0}^{s_{p-l+1}-s_{p-l}}\Psi_1\,\mathrm{d}y\,\mathrm{d}x}\Bigg)\Bigg)\\
&\times \Bigg(\prod_{j=2}^{d}\Bigg(\prod_{k=1}^{p-1}\Bigg(\prod_{l=1}^{p-k}\Bigg(\phi_1^D\Bigg(\sum_{h=j}^{d}\sum_{t=k}^{p-l} a_{h,t}\Bigg)\Bigg)^{-c'(d-j)\int_{s_{k-1}}^{s_k}\int_{0}^{s_{p-l+1}-s_{p-l}}\Psi_1\,\mathrm{d}y\,\mathrm{d}x}\Bigg)\Bigg)\Bigg)\\
&\times \Bigg(\prod_{j=2}^{d-1}\Bigg(\prod_{i=2}^{j}\Bigg(\prod_{k=1}^{p-1}\Bigg(\prod_{l=1}^{p-k}\Bigg(\phi_1^D\Bigg(\sum_{h=i}^{j}\sum_{t=k}^{p-l} a_{h,t}\Bigg)\Bigg)^{c''(j-i)\int_{s_{k-1}}^{s_k}\int_{0}^{s_{p-l+1}-s_{p-l}}\Psi_1\,\mathrm{d}y\,\mathrm{d}x}\Bigg)\Bigg)\Bigg)\Bigg)\\
&\times \Bigg(\prod_{j=1}^{d-1}\Bigg(\prod_{k=1}^{p-1}\Bigg(\prod_{l=1}^{p-k}\Bigg(\phi_1^D\Bigg(\sum_{h=1}^{j}\sum_{t=k}^{p-l} a_{h,t}\Bigg)\Bigg)^{-c'(j-1)\int_{s_{k-1}}^{s_k}\int_{0}^{s_{p-l+1}-s_{p-l}}\Psi_1\,\mathrm{d}y\,\mathrm{d}x}\Bigg)\Bigg)\Bigg)\\
&\times \Bigg(\prod_{h=1}^{d}\Bigg(\prod_{k=1}^{p}\Bigg(\phi_1^D\Bigg(\sum_{t=k}^{p} a_{h,t}\Bigg)\Bigg)^{\int_{s_{k-1}}^{s_k}\int_{-\infty}^{y}(1-\kappa(s_p-x))\rho''(s_p-x)\,\mathrm{d}x\,\mathrm{d}y}\Bigg)\Bigg)\\
&\times \Bigg(\prod_{h=1}^{d}\Bigg(\prod_{k=1}^{p-1}\Bigg(\prod_{l=1}^{p-k}\Bigg(\phi_1^D\Bigg(\sum_{t=k}^{p-l} a_{h,t}\Bigg)\Bigg)^{\int_{s_{k-1}}^{s_k}\int_{0}^{s_{p-l+1}-s_{p-l}}\Psi_2\,\mathrm{d}y\,\mathrm{d}x}\Bigg)\Bigg)\Bigg), \qquad (11)
\end{aligned}
$$

where, in a slight abuse of notation, we define $c'(s) = c(s+1) - c(s)$, and $c''(s) = c(s+2) - 2c(s+1) + c(s)$.

To construct a random field with a varying cross-correlation where the spatial dimension is defined on $\mathbb{R}$ instead of $\{1, 2, \dots, d\}$, one can alter the argument used above to consider the spatial dimension in increments of $1/n$, instead of unit increments, and then let $n \to \infty$. This is essentially how Section 3 and Section 4 differ, with time implicitly considered in increments of $1/n$ in Section 3 as opposed to unit increments in Section 4. In this case the characteristic function of $\mathbb{E}\exp(\mathrm{i}\sum_{h=1}^{d}\sum_{t=1}^{p} a_{h,t} V_{\eta_h}^{n}(s_t))$ is as in (11), but replacing $c(d-1)$ with $c(\eta_d - \eta_1)$, $-c'(d-j)$ with $c(\eta_d - \eta_j) - c(\eta_d - \eta_{j-1})$, $c''(j-i)$ with $c(\eta_j - \eta_i) - c(\eta_{j+1} - \eta_i) - c(\eta_j - \eta_{i-1}) + c(\eta_{j+1} - \eta_{i-1})$, and $-c'(j-1)$ with $c(\eta_j - \eta_1) - c(\eta_{j+1} - \eta_1)$. Alternative cross-correlation structures are possible, with the only limit being the degree of overlap that can be constructed between the $\tilde{Y}_{i,j,h}^{n}$ for varying h.

The constraints outlined above can be relaxed either individually or jointly, and although we have couched this section in terms of the continuous-time process $\{\boldsymbol{V}(t)\}$, similar points hold in the discrete time case for $\{\boldsymbol{X}(t)\}$ also.

6. Conclusion

We have constructed stationary random fields in discrete and continuous time which can have any desired infinitely divisible marginal distribution with finite variance, any autocorrelation function that is positive and convex, and a wide range of cross-correlation functions. This supplements earlier results on Gaussian and related random fields, and makes available non-Gaussian random fields with rich correlation structures which can be used directly in modeling and estimation.

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References

Billingsley, P. (1968). *Convergence of Probability Measures*. John Wiley, New York.

Christakos, G. (1984). On the problem of permissible covariance and variogram models. *Water Resour. Res.* **20,** 251–265.

Chung, K. L. (2001). *A Course in Probability Theory*, 3rd edn. Academic Press, San Diego, CA.

Cramér, H. and Leadbetter, M. R. (1967). *Stationary and Related Stochastic Processes: Sample Function Properties and Their Applications*. John Wiley, New York.

Du, J. and Ma, C. (2013). Vector random fields with compactly supported covariance matrix functions. *J. Statist. Planning Infer.* **143,** 457–467.

Feller, W. (1966). *An Introduction to Probability Theory and Its Applications*, Vol. II. John Wiley, New York.

Finlay, R. and Seneta, E. (2007). A gamma activity time process with noninteger parameter and self-similar limit. *J. Appl. Prob.* **44,** 950–959.

Finlay, R., Fung, T. and Seneta, E. (2011). Autocorrelation functions. *Internat. Statist. Rev.* **79,** 255–271.

Gikhman, I. I. and Skorokhod, A. V. (1969). *Introduction to the Theory of Random Processes*. W. B. Saunders, Philadelphia, PA.

Khoshnevisan, D. (2002). *Multiparameter Processes: An Introduction to Random Fields*. Springer, New York.

Kolmogoroff, A. (1923). Sur l'ordre de grandeur des coefficients de la série de Fourier-Lebesgue. *Bull. Internat. Acad. Polon. Sci. Lett. Cl. Ser. Sci. Math. Nat. A*, **1923,** 83–86.

Lukacs, E. (1960). *Characteristic Functions*. Hafner, New York.

Ma, C. (2009). Construction of non-Gaussian random fields with any given correlation structure. *J. Statist. Planning Inf.* **139,** 780–787.

Ma, C. (2011a). Covariance matrices for second-order vector random fields in space and time. *IEEE Trans. Signal Process.* **59,** 2160–2168.

Ma, C. (2011b). Covariance matrix functions of vector χ^2 random fields in space and time. *IEEE Trans. Commun.* **59,** 2554–2561.

Ma, C. (2011c). Vector random fields with long-range dependence. *Fractals* **19,** 249–258.

Ma, C. (2011d). Vector random fields with second-order moments or second-order increments. *Stoch. Anal. Appl.* **29,** 197–215.

Marfè, R. (2012). A multivariate pure-jump model with multi-factorial dependence structure. *Int. J. Theoret. Appl. Finance* **15,** 1250028.

Marfè, R. (2014). Multivariate Lévy processes with dependent jump intensity. *Quant. Finance* **14,** 1383–1398.

Øksendal, B. (2003). *Stochastic Differential Equations: An Introduction with Applications*, 6th edn. Springer, Berlin.

Pólya, G. (1949). Remarks on characteristic functions. In *Proc. Berkeley Symp. Math. Statist. Prob. 1945, 1946*, University of California Press, Berkeley, CA, pp. 115–123.

Young, W. H. (1913). On the Fourier Series of bounded functions. *Proc. London Math. Soc.* **12,** 41–70.

Zygmund, A. (1968). *Trigonometric Series*, Vol. I, 2nd edn. Cambridge University Press.