THE CLASSIFICATION OF NON-EUCLIDEAN PLANE CRYSTALLOGRAPHIC GROUPS

A. M. MACBEATH

To Professor H. S. M. Coxeter on his sixtieth birthday

1. Introduction. This paper deals with the algebraic classification of non-euclidean plane crystallographic groups (NEC groups, for short) with compact quotient space. The groups considered are the discrete groups of motions of the Lobatschewsky or hyperbolic plane, including those which contain orientation-reversing reflections and glide-reflections. The corresponding problem for Fuchsian groups, which contain only orientable transformations, is essentially solved in the work of Fricke and Klein **(6)**. However, it is difficult to find an explicit formulation and proof of the result in the printed literature, and for this reason an account is given here (§2), partly also to prepare the way for the more complicated discussion which follows. Particular NEC groups with reflections and glide-reflections, such as Dyck's groups, are well known and are dealt with, for instance, in Coxeter and Moser **(4)**, where presentations for the 17 euclidean plane groups can also be found. A fairly complete theory of NEC groups was recently developed by Wilkie **(9)**, who showed that every NEC group has a presentation of a certain type. While Wilkie was able to exhibit a number of isomorphisms between groups defined by formally different presentations in his scheme, he did not succeed in determining necessary and sufficient conditions for two of his standard presentations to define isomorphic groups. The aim of this paper is to supply the missing conditions, and we shall see that the list of isomorphisms given by Wilkie is incomplete, but can be completed by the addition of a fifth class of isomorphisms to the four classes found by him.

I am particularly happy to include this work in an issue celebrating Professor Coxeter's birthday, not only because it adds something to the subject of discrete groups, which his work has so enriched, but also because a key role is played by the lemma that every isomorphism can be realized geometrically. The classic interaction between geometry and algebra, which motivates so much of Coxeter's own work, plays its part here too.

2. Fuchsian groups. As a preparation for what follows, we deal first with the solution of the corresponding problem for Fuchsian groups. The discussion is taken from the author's cyclostyled lecture notes, and has not appeared in print before.

Received May 12, 1966.

Klein associated a "signature"—really an ordered set of non-negative integers—with a Fuchsian group. One of the integers describes the genus of the quotient space. One, which is redundant and will be omitted by us, is the number of periods. The others are the periods written in order. (As we shall see, it does not matter what order.) Klein allowed infinite values for the periods, but we shall restrict ourselves here, as in the rest of the paper, to groups with compact quotient space, in which infinite periods cannot occur. The periods must be integers not less than two, whereas the genus may take any non-negative integral value.

Thus we are led to define an *F-signature* as an ordered set of integers

$$(g; m_1, \ldots, m_r)$$

where $g \geqslant 0$ and the *periods* $m_1, \ldots, m_r \geqslant 2$. The integer r may be zero and the set of periods empty. The same integer may be repeated any number of times in the set of periods. The number of times an integer occurs in the set of periods is called the *multiplicity* of the period.

With the F-signature $(g; m_1, \ldots, m_r)$ we associate the following group presentation:

Generators: $x_1, \ldots, x_r; a_1, b_1, \ldots, a_g, b_g$.

Relations:

$$(1) \qquad \begin{gathered} x_1^{m_1} = 1, \ldots, x_r^{m_r} = 1, \\ x_1 x_2 \ldots x_r a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1. \end{gathered}$$

It is well known that every Fuchsian group has such a presentation, and, conversely, that if

$$2g - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) > 0,$$

then there is a Fuchsian group with the presentation of signature $(g; m_1, \ldots, m_r)$. In the group Γ defined by the presentation (1) it is known that every element of finite order is contained in a maximal finite cyclic subgroup which is conjugate to one of the cyclic groups $\{x_1\}, \ldots, \{x_r\}$, of which no two are conjugate. Thus the numbers $m_1, \ldots, m_r$, and the multiplicities with which they occur, though not their order, correspond to the conjugacy classes of maximal finite cyclic subgroups and are algebraic invariants of the group. The genus g is also an algebraic invariant, being the rank of the commutator factor group of Γ. We deduce:

If the group Γ of F-signature $(g; m_1, \ldots, m_r)$ is isomorphic to the group Γ' of F-signature $(g'; m_1', \ldots, m_r')$, then $g = g'$, $r = r'$, and there is a permutation ϕ of the set $(1, 2, \ldots, r)$ such that $m_i' = m_{\phi(i)}$ $(i = 1, 2, \ldots, r)$.

The converse is also true. If ϕ is a permutation of $(1, 2, \ldots, r)$, then the group Γ of signature $(g; m_1, \ldots, m_r)$ is isomorphic to the group Γ' of signature $(g; m_{\phi(1)}, \ldots, m_{\phi(r)}$. To prove this, it is enough to show, since every permutation can be achieved by a succession of transpositions of neighbours, that the group Γ is isomorphic to the group Γ' of signature $(g; m_1, \ldots, m_{i-1}, m_{i+1}, m_i, m_{i+2}, \ldots, m_r)$. Here only the ith and $(i+1)$th periods are interchanged, the rest being unaltered. Then Γ has the presentation (1) while Γ' has the presentation

$$x_k'^{m_k} = 1 \qquad (k \neq i, i+1),$$
$$x'_{i+1}{}^{m_i} = 1, \qquad x'_i{}^{m_{i+1}} = 1,$$
$$x_1' \ldots x_r' a_1' b_1' a_1'^{-1} b_1'^{-1} \ldots a_g' b_g' a_g'^{-1} b_g'^{-1} = 1.$$

The desired isomorphism $f\colon \Gamma \to \Gamma'$ is defined by assigning the following images to the generators of Γ:

$$f(x_k) = x_k' \qquad (k \neq i, i+1),$$
$$f(x_i) = x_i' x'_{i+1} x_i'^{-1},$$
$$f(x_{i+1}) = x_i',$$
$$f(a_i) = a_i', \qquad f(b_i) = b_i'.$$

Thus, while the order in which the periods are written affects the presentation of the group, it makes no difference to the isomorphism class. This depends only on the periods as an unordered set, where, however, the multiplicity is important. This recalls the partition of numbers—a partition of n is an unordered set of integers, with repetitions allowed, whose sum is n. For this reason one talks of the *period partition* of a Fuchsian group and the isomorphism theorem may be expressed:

Two Fuchsian groups with compact quotient space are isomorphic if and only if their quotient spaces have the same genus and their period partitions are the same.

Geometrically the result is plausible when we consider the action of Γ on the hyperbolic plane which, now and later in the paper, we denote by D. Let p denote the quotient map $p\colon D \to D/\Gamma$ which maps every point of D on the point of the quotient space D/Γ represented by the orbit to which it belongs. At most points of D/Γ the map p is a smooth covering, but there are r points in D/Γ over which the covering is branched. At the ith point the sheets come together in groups of m_i. One does not expect the ordering of the branch points to be significant, since there is a homeomorphism of the surface D/Γ which will permute them according to any permutation given in advance.

3. Wilkie's signatures. We now define two kinds of NEC signature. One, denoted by a plus sign and used for NEC groups with orientable quotient space, will be called *orientable*. The other, denoted by a minus sign and used for NEC groups with non-orientable quotient space, will be called *non-orientable*. In addition to having an ordered set of periods, it includes an ordered set of *period-cycles*, each period-cycle being a further ordered set of periods. We shall first give the bare definition, describing the associated geometry and algebra later.

Definition. A NEC signature consists of a sign $\pm$ and a sequence of integers with certain subsequences bracketed together in the following manner:

(0) a sign $\pm$ (plus for orientable, minus for non-orientable),

(1) an integer $g \geqslant 0$,

(2) an ordered set of integers $m_1, \ldots, m_r$ ($m_i \geqslant 2$), called the *proper periods* of the signature,

(3) an ordered set of period-cycles:

$$C_1 = (n_{11}, \ldots, n_{1s_1}), \ldots, \quad C_k = (n_{k1}, \ldots, n_{ks_k}) \qquad (n_{ij} \geqslant 2).$$

The signature just defined will be written

$$(g, \pm, [m_1, \ldots, m_r] \{C_1, C_2, \ldots, C_k\}),$$

the proper periods being enclosed in square brackets and the cycles in curly brackets. More explicitly, the signature can be written in full as follows:

$$(g, \pm, [m_1, \ldots, m_r] \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}),$$

the cycle periods being enclosed in round brackets.

Just as in the case of F-signatures, the set of periods may be empty. Not only this, but also individual period-cycles may be empty of periods, and the whole set of period-cycles may be empty. In such cases, the signature will be written with the brackets inserted, but with no symbols between them. Thus the signature

$$(g, +, [\ ]\ \{\ \})$$

has no proper periods and no period-cycle. As a further example, the signature

$$(g, +, [m]\ \{(\ )\ (\ )\})$$

has one proper period m and two empty period-cycles.

So far we have defined only a numerical-combinatorial structure which, of itself, would seem to have minimal interest. To each NEC signature, however, we shall assign a marked polygon, a marked surface, and a group presentation. The marked polygon and the marked surface derived from it will be described in the next section.

4. The marked polygon and marked surface of a NEC signature.

By a marked polygon we mean a plane polygon in which certain pairs of sides are related by homeomorphisms. If the first side has vertices P, Q in order as we describe the perimeter of the polygon anticlockwise, and the second has vertices R, S, also in order anticlockwise, then the pairing homeomorphism can pair the edges orientably, mapping P on S and Q on R, or non-orientably, mapping P on R and Q on S. Two sides paired orientably will be indicated by the same letter and a prime, say ξ, ξ'. Two sides paired non-orientably will be indicated by the same letter and an asterisk, as α, α^*. If we label all the edges of a marked polygon according to this scheme, and then write them in order as they occur round the polygon anticlockwise, we obtain the *surface symbol* of the marked polygon, which determines it apart from homeomorphism. As we are dealing with a surface with boundary, some of our edges will be unpaired, and the letters corresponding to these will occur only once, and without a prime or an asterisk.

The marked polygon of the signature

$$(2) \qquad (g, +, [m_1, \ldots, m_r] \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\})$$

is defined by the surface symbol

$$\xi_1 \xi_1' \xi_2 \xi_2' \ldots \xi_r \xi_r' \epsilon_1 \gamma_{10} \gamma_{11} \ldots \gamma_{1s_1} \epsilon_1' \epsilon_2 \gamma_{20} \ldots \gamma_{2s_2} \epsilon_2' \ldots \epsilon_k \gamma_{k0} \ldots \gamma_{ks_k} \epsilon_k' \alpha_1 \beta_1' \alpha_1' \beta_1 \ldots \alpha_g \beta_g' \alpha_g' \beta_g.$$

The marked polygon is determined merely by the number of periods and the number and length of the period-cycles, the actual values of the periods playing no part. However, it is desirable to think of the periods as being associated with certain vertices of the marked polygon. The proper period m_i is attached to the vertex M_i common to the two sides ξ_i, ξ_i'. The cycle period n_{ij} is associated with the vertex N_{ij} common to the edges $\gamma_{i,j-1}$ and γ_{ij}.

The marked polygon associated with the signature

$$(3) \qquad (g, -, [m_1, \ldots, m_r] \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\})$$

is given by the surface symbol

$$\xi_1 \ldots \xi_r' \epsilon_1 \gamma_{10} \ldots \epsilon_k' \alpha_1 \alpha_1^* \alpha_2 \alpha_2^* \ldots \alpha_g \alpha_g^*,$$

which differs from the preceding one only in the last part. This last part represents the surface symbol of a closed non-orientable surface in case (3) while it represents the surface symbol of a closed orientable surface in case (2). In case (3) the proper and improper periods are assigned to vertices of the marked polygon, just as in case (2).

If we identify corresponding points on the related edges of the marked polygon, we obtain from it a surface with boundary. In the orientable case the surface will be a sphere with k disks removed and g handles added. In the non-orientable case it will be a sphere with k disks removed and g cross-caps added. On this surface, the edges α (in the non-orientable case) and the edges α, β (in the orientable case) determine a canonical system of cross-cuts meeting at a base-point Q, say. There are certain distinguished points M_i in the interior of the surface and certain distinguished points $N_{i1}, \ldots, N_{is_i}$ on the ith boundary component. The lines ξ join the base-point to the points M and the line ϵ_i joins the base-point to a point on the ith boundary component between N_{is_i} and N_{i1}. Thus we have a marked surface in a sense very similar to that used in the theory of Teichmuller spaces **(1, 2)**.

5. The presentation associated with a NEC signature. Suppose that P is a fundamental polygon for a NEC group Γ. Then certain pairs of its edges correspond under transformations of Γ. Some edges, corresponding to boundary components of the quotient space, are segments of axes of reflections, and are not paired with other edges. Some of the paired edges are paried orientably and some non-orientably. Thus, with this pairing, the fundamental region is a marked polygon.

In order to derive presentations for NEC groups, Wilkie proved the following result.

Let Γ be a NEC *group. Then there is a fundamental polygon P for Γ which is the marked polygon of a NEC signature, such that the stabilizer of the vertex M_i is a cyclic group of rotations of order m_i, the stabilizer of a vertex N_{ij} is dihedral of order $2n_{ij}$ (containing a cyclic rotation subgroup of index 2), and the stabilizer of a point, other than N_{ij}, $N_{i,j+1}$ of γ_{ij} is a reflection group $\mathbf{Z}_2$. No other points of P are fixed points for Γ.*

From this result, Wilkie deduces a presentation for Γ by standard methods. This presentation is completely determined by the integers m_i, n_{ij} and will be called the *presentation of the signature.* The generators, and to some extent also the relations between them, divide naturally into subsets associated with the different parts of the signature. Thus with each period is associated a generator of finite order and with a cycle of length s is associated a "connecting generator" and a number of generators of order two of which successive pairs define dihedral groups. Finally, there are generators given, essentially, by the system of cross-cuts on the surface. For this reason we give the presentation in the form of a table, the first column listing the appropriate part of the signature, the second column giving the generators, and the third column the relations. First, the orientable signature is

$$(g, +, [m_1, \ldots, m_r] \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}). \tag{2}$$

Signature element	Generator(s)	Relation(s)
Period m_i	x_i	$x_i^{m_i} = 1$
Cycle $C_i = (n_{i1}, \ldots, n_{is_i})$	e_i (connecting gen.) $c_{i0}, c_{i1}, \ldots, c_{is_i}$	$c_{is_i} = e_i^{-1} c_{i0} e_i$; $c_{i,j-1}^2 = c_{ij}^2 = (c_{i,j-1} c_{ij})^{n_{ij}} = 1$
$g+$	$a_1, b_1, \ldots, a_g, b_g$	$x_1 x_2 \ldots x_r e_1 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1$

The table for the non-orientable signature (3) obtained from (2) by changing only the sign is exactly the same as the above except for the last row, which now reads

$g-$	$a_1, a_2, \ldots, a_g$	$x_1 x_2 \ldots x_r e_1 e_2 \ldots e_k a_1^2 a_2^2 \ldots a_g^2 = 1$

6. Geometrical properties. The transformations $a, b, \ldots, x, \ldots$ in the presentation are those which carry the fundamental polygon across the sides

$\alpha, \beta, \ldots, \xi, \ldots$ denoted by the corresponding Greek letters. The cyclic group $\{x_i\}$ is the stabilizer of the vertex we have called M_i and the dihedral group of order $2n_{ij}$ generated by the two reflections $c_{i,j-1}$ and c_{ij} is the stabilizer of N_{ij}. We now show that:

An element of finite order in Γ *is conjugate to one of the following:*

(i) *a power of some* x_i $(1 \leqslant i \leqslant r)$,

(ii) *a power of some* $c_{i,j-1}\, c_{ij}$ $(1 \leqslant j \leqslant s_i;\ 1 \leqslant i \leqslant k)$,

(iii) *some* c_{ij} $(0 \leqslant j \leqslant s_i;\ 1 \leqslant i \leqslant k)$.

Proof. An element t of finite order is either a rotation or a reflection. In either case it has a fixed point in D which we may call x. Since P is a fundamental polygon, it contains a point which is a Γ-image of x, say $y = gx \in P$, $g \in \Gamma$. Then $gtg^{-1}y = y$, and the conjugate transformation gtg^{-1} is in the stabilizer of y. However, the only points of P with non-trivial stabilizer are M_i, N_{ij}, and the points of γ_{ij}. Thus gtg^{-1} belongs to the stabilizer of one of these points. These stabilizers constitute the transformations listed under (i), (ii), (iii) above, so our assertion is proved.

7. Geometrical isomorphism. Two NEC groups Γ, Γ' are called *geometrically isomorphic* if there is a homeomorphism $x \rightarrow x'$ of D and a group isomorphism $g \rightarrow g'$ of Γ on Γ' such that $y = gx$ if and only if $y' = g'x'$. If we denote the homeomorphism by t, the condition becomes $tgx = g'tx$, for all x, so that

$$g' = tgt^{-1}.$$

Thus the groups Γ, Γ' are geometrically isomorphic if and only if they are conjugate in the group of all homeomorphisms of D. If an isomorphism (in the algebraic sense) of Γ onto Γ' can be derived from a geometrical isomorphism—in other words, if there is a homeomorphism t of D such that the image of g under the isomorphism is tgt^{-1} for all $g \in \Gamma$, then we say that the isomorphism can be *realized geometrically* and we refer to the homeomorphism t as a *geometrical realization* of the given isomorphism. We do not require t to be an isometry, or, indeed, to have any particular relationship to the geometry of the hyperbolic plane.

A geometrical isomorphism maps rotations on rotations, since these are geometrically characterized by the property of having precisely one fixed point in D. Similarly it maps reflections, which have an infinite fixed point set, into reflections, and it maps translations or glide-reflections, which are without fixed points, into translations or glide-reflections. Moreover, the translations are distinguished from the glide-reflections by the topological property of being orientable mappings, so a geometrical isomorphism cannot map one on the other.

A geometrical isomorphism t maps the Γ-orbit of x on the Γ'-orbit of tx, since

$$t(\Gamma x) = t\Gamma t^{-1}(tx) = \Gamma'(tx).$$

Thus it induces a homeomorphism between the quotient spaces D/Γ, D/Γ'.

https://doi.org/10.4153/CJM-1967-108-5 Published online by Cambridge University Press

We denote this homeomorphism by t^*. Since all points of the same Γ-orbit have geometrically isomorphic stabilizers, we may talk, slightly illogically, of the stabilizer of a point of the quotient space, meaning the isomorphism class to which the stabilizer of any point in the orbit belongs.

The map $t^*: S \to S'$ will map points of S on points of S' with geometrically isomorphic stabilizer. Identifying S and S' with the marked surfaces of their appropriate signatures, we see, in the notation of §4, that $t^*(M_i)$, like M_i, has a stabilizer which is a cyclic rotation group of order m_i. Similarly, the stabilizer of $t^*(N_{ij})$ will be dihedral of order $2n_{ij}$. Thus the map t^* induces a (1–1) mapping of the proper periods of Γ on the proper periods of Γ', and also a (1–1) mapping of the cycle periods of Γ on those of Γ'.

For a fixed value of i, the points N_{ij} are all on the same boundary component of S. Thus their t^*-images all lie on a single boundary component of S'. Thus their images, which are points $N'_{i'j'}$ on S', must all have the same initial suffix i', and each cycle of the signature of Γ corresponds to a cycle of the same length in the signature of Γ', and with the same periods. Within the cycle, the order in which the N_{ij} have been taken round the ith boundary component of S may be the same (cyclically) as the $N'_{i'j'}$, or it may be the reverse. The choice of initial point N_{i1} has no geometrical significance, so cyclic interchanges are possible. To be quite precise, we require the following definitions:

Definitions. Let C, C' be two period-cycles, $C = (n_1, \ldots, n_s)$, $C' = (n'_1, \ldots, n'_{s'})$. Then C and C' are called *directly equivalent* if one is a cyclic permutation of the other, that is, if $s = s'$ and there is an integer k such that

$$n_i = n'_{i+k},$$

suffixes being read modulo s. C, C' are called *inversely equivalent* if one is a cyclic permutation of the other reversed, that is, if $s = s'$ and there is an integer k such that

$$n_i = n'_{k-i},$$

where the suffixes are again reduced modulo s.

We can now state

THEOREM 1a. *If the* NEC *group* Γ *of signature*

$$(g, +, [m_1, \ldots, m_r] \{C_1, C_2, \ldots, C_k\})$$

is geometrically isomorphic to the NEC *group* Γ' *of signature*

$$(g, +, [m'_1, \ldots, m'_{r'}] \{C'_1, \ldots C'_{k'}\}),$$

then $r = r'$, $k = k'$, *the proper periods* $[m']$ *are a permutation of the proper periods* $[m]$, *and there is a permutation* ϕ *of* $(1, 2, \ldots, k)$ *such that either* (i) *for each* i, C'_i *is directly equivalent to* $C_{\phi(i)}$, *or* (ii) *for each* i, C'_i *is inversely equivalent to* $C_{\phi(i)}$.

THEOREM 2a. *If the* NEC *group* Γ *of signature*

$$(g, -, [m_1, \ldots, m_r] \{C_1, \ldots, C_k\})$$

is geometrically isomorphic to the NEC *group* Γ' *of signature*

$$(g, -, [m'_1, \ldots, m'_{r'}] \{C'_1, \ldots C'_{k'}\}),$$

then $r = r'$, $k = k'$, *the proper periods* $[m']$ *are a permutation of the proper periods* $[m]$, *and there is a permutation* ϕ *of* $1, \ldots, k$ *such that, for each* i, C'_i *is either directly or inversely equivalent to* $C_{\phi(i)}$.

Note. In the orientable case, corresponding pairs of cycles are *all* paired in the same way—either all directly or all inversely. *In the non-orientable case,* some may be paired directly and some inversely.

Proof of the theorems. Theorem 2a has practically been proved in the discussion preceding the definitions. If we use $[C_i]$ to denote the component of ∂S which contains the points N_{ij} corresponding to periods in the cycle C_i, then ϕ is defined by the relation

$$[C'_i] = t^*[C_{\phi(i)}]).$$

The cycles C'_i, $C_{\phi(i)}$ are directly or inversely equivalent according as t^* maps the oriented 1-sphere $[C_{\phi(i)}]$ on the oriented 1-sphere $[C'_i]$ with degree $+1$ or -1.

To prove Theorem 1a, we remark that in this case S, S' are orientable and the 1-spheres $[C_i]$, $[C'_j]$ are oriented with the induced orientation of the boundary. If t^* maps S on S' with degree $+1$, then it maps each $[C_{\phi(i)}]$ on the corresponding $[C'_i]$ with degree $+1$, and all pairs of corresponding period-cycles are directly equivalent. The only other possibility is that t^* should map S on S' with degree -1, in which case all pairs of cycles are inversely equivalent. This completes the proof.

8. Geometrical realization of isomorphisms. It is by no means necessary that an isomorphism of a group of transformations be realized geometrically. Examples could be multiplied, but, as one particularly simple example, the map

$$t \longrightarrow t^2$$

yields an automorphism of the cyclic rotation group of order 5 which cannot be realized geometrically.

However, in the case of plane groups with compact quotient space, every isomorphism of the algebraic group structure can be realized geometrically. For automorphisms of Fuchsian groups without periods, this is a celebrated theorem of Nielsen **(8)**, and for automorphisms of NEC groups without reflections it was proved by Zieschang **(10)**. For automorphisms of general NEC groups, it was proved by the author, being deduced from Nielsen's theorem by an application of the theory of quasi-conformal mappings. Practically the same proof applies to isomorphisms of one group on another, but as the proof is short, we repeat it here with appropriate modifications. Efforts to find a proof independent of quasi-conformal theory, as are Nielsen's and Zieschang's proofs, have not yet met with success.

THEOREM 3. *Let* $\phi\colon \Gamma \to \Gamma'$ *be an isomorphism (of the groups structure only) between two* NEC *groups. Then* ϕ *can be realized geometrically, that is, there is a homeomorphism* $t\colon D \to D$ *such that, for all* $g \in \Gamma$, $\phi(g) = tgt^{-1}$.

Proof. The set of all orientable mappings in Γ is a Fuchsian group of index 1 or 2, which, by a theorem of Fox **(3; 5)**, contains a subgroup of finite index with no periods, which in turn contains a subgroup N of finite index which is normal in Γ. Since N has no periods, it is the fundamental group of a compact orientable surface. Then $N' = \phi(N)$ is also the fundamental group of a compact orientable surface, and, since two compact surfaces with isomorphic fundamental group are homeomorphic, there is a geometrical isomorphism of N with N' (not necessarily inducing ϕ). If $\tau\colon D \to D$ is the homeomorphism which realizes this geometrical isomorphism, let us define $\psi\colon N \to N'$ by $\psi(n) = \tau n \tau^{-1}$. Then $\phi\psi^{-1}$ is an automorphism of N. By Nielsen's theorem, $\phi\psi^{-1}$ can be realized geometrically, and hence $(\phi\psi^{-1})\psi = \phi$ can be realized geometrically. Among the set of homeomorphisms $t\colon D \to D$ such that

$$\phi(n) = tnt^{-1} \qquad \text{for all } n \in N, \tag{4}$$

which we now know to be non-empty, there is, by the theory of Teichmuller spaces **(1; 2)** a unique extremal quasi-conformal homeomorphism. Suppose t to be this unique map. Let now g be any element of Γ. Consider the map $t'\colon D \to D$ defined by

$$t' = \phi(g)tg^{-1}. \tag{5}$$

Since g^{-1}, $\phi(g) \in \Gamma'$ are conformal, the maximal dilation of t' is the same as that of t. Further, t' satisfies the relation (4), since

$$\begin{aligned} t'nt'^{-1} &= \phi(g)t(g^{-1}ng)t^{-1}\phi(g^{-1}) \\ &= \phi(g)\phi(g^{-1}ng)\phi(g^{-1}), \qquad \text{by (4) since } g^{-1}ng \in N, \\ &= \phi(gg^{-1}ngg^{-1}) = \phi(n). \end{aligned}$$

By the uniqueness theorem, $t = t'$, that is, from (5)

$$tgt^{-1} = \phi(g).$$

This being true for all $g \in \Gamma$, t is a geometrical realization of ϕ, not only on the normal subgroup N, but on the whole of Γ. This proves Theorem 3.

9. Main results. We can now state and complete the proof of the two main theorems.

THEOREM 1. *The conditions given in Theorem* 1a *as necessary for geometrical isomorphism are necessary and sufficient for (not necessarily geometrical) isomorphism.*

https://doi.org/10.4153/CJM-1967-108-5 Published online by Cambridge University Press

THEOREM 2. *The conditions given in Theorem* 2a *for geometrical isomorphism are necessary and sufficient for (not necessarily geometrical) isomorphism.*

By Theorem 3, every isomorphism can be taken as geometrical, so the necessity of the conditions has already been proved. The converse part asserts that two groups with the same genus and orientability of quotient space are isomorphic if the signature of one can be derived from the other by a succession of operations of certain types—permuting the periods, permuting the cycles, cyclically permuting the periods within a cycle, or reversing the order of the periods in some or all of the cycles. To prove this it is naturally better to deal with each type of operation separately. Our result thus follows from the succession of lemmas below. The proofs of all are similar, and Lemmas 1, 2, 3, 4 are proved in Wilkie's paper **(9)**. Only Lemma 5, which escaped Wilkie's notice, is proved here.

LEMMA 1. *Groups* Γ, Γ' *which are defined by signatures which are the same except for a permutation of the proper periods are isomorphic.* (See also §2.)

LEMMA 2. *If* ϕ *is a permutation of* $1, 2, \ldots, k$, *the group* Γ *of signature*

$$(g, \pm, [m_1, \ldots, m_r] \{C_1, \ldots, C_k\})$$

is isomorphic to the group Γ' *of signature*

$$(g, \pm, [m_1, \ldots, m_r] \{C_{\phi(1)}, \ldots, C_{\phi(k)}\})$$

LEMMA 3. *The group* Γ *of signature*

$$(g, \pm, [m_1, \ldots, m_r] \{C_1, \ldots, C_k\})$$

is isomorphic to the group Γ' *of signature*

$$(g, \pm, [m_1, \ldots, m_r] \{C_1, \ldots, C_{j-1}, C'_j, C_{j+1}, \ldots, C_k\})$$

where C'_j *is a cyclic permutation of* C_j, *the signatures being otherwise identical.*

LEMMA 4. *If, for any period-cycle* C, *the same set of periods written in reverse order is denoted by* C^*, *then the group* Γ *of signature*

$$(g, \pm, [m_1, \ldots, m_r] \{C_1, \ldots, C_k\})$$

is isomorphic to the group Γ' *of signature*

$$(g, \pm, [m_1, \ldots, m_r] \{C_1^*, \ldots, C_k^*\}).$$

These lemmas are all proved in Wilkie **(9)**, as the last part of his Theorem 3. Finally we state and prove the only new lemma, which is also the only one differentiating between groups with orientable quotient space and those with non-orientable quotient space. It gives algebraic expression to the geometrical fact that one reverses the orientation of a boundary component of a non-orientable surface by continuous variation round a suitable path back to its original position.

LEMMA 5. *The group* Γ *of signature*

$$(g, -, [m_1, \ldots, m_r] \{C_1, \ldots, C_i, \ldots, C_k\})$$

is isomorphic to the group Γ' *of signature*

$$(g, -, [m_1, \ldots, m_r] \{C_1, \ldots, C^*_i, \ldots, C_k\})$$

only the ith cycle being reversed, the signatures being otherwise identical.

Proof. By Lemma 2 we can permute the cycles arbitrarily in both groups without altering the isomorphism class, so there is no loss in assuming that $i = k$. Then Γ is defined by generators x, e, c, a with appropriate suffixes and relations which include:

$$c_{ks_k} = e_k^{-1} c_{k0}\, e_k,$$
$$c_{k,j-1}^2 = c_{kj}^2 = (c_{k,j-1}\, c_{kj})^{n_{kj}} = 1,$$
$$x_1\, x_2 \ldots x_r\, e_1\, e_2 \ldots e_k\, a_1^2 a_2^2 \ldots a_g^2 = 1,$$

while Γ' is defined by generators x', e', c', a' with appropriate suffixes and relations which include

$$c'_{ks_k} = e_k'^{-1} c_{k0}' e_k'$$
$$c'_{k,j-1}{}^2 = c_{kj}'^2 = (c'_{k,j-1}\, c_{kj}')^{n_{k,s_k+1-j}} = 1,$$
$$x_1' \ldots e_k' a_1'^2 a_2'^2 \ldots a_g'^2 = 1.$$

Apart from the relations we have written, all the relations for Γ correspond in an exact manner to those for Γ'.

An isomorphism between Γ, Γ' is given by the following assignment of images for the generators. First, the generators not associated with the kth cycle and the surface generators apart from a_1 are all mapped on the corresponding generators in Γ':

$$\phi(x_i) = x_i', \quad \phi(e_i) = e_i', \quad \phi(c_{ij}) = c_{ij}' \qquad (i = 1, \ldots, k-1),$$
$$\phi(a_i) = a_i' \qquad (i = 2, \ldots, n).$$

Only for e_k, c_{kj} ($j = 1, \ldots, s_k$), and for a_1 are the images different. For these we put, writing d for $e_k' a_1'$:

$$\phi(e_k) = d e_k'^{-1} d^{-1}, \qquad \phi(c_{kj}) = d c'_{k,s_k+1-j}\, d^{-1},$$
$$\phi(a_1) = e_k' a_1'.$$

It is easily checked that the images assigned to these generators satisfy all the appropriate relations, so that ϕ defines an isomorphism of Γ on Γ' and our proof of Theorems 1 and 2 is complete.

Finally we remark that the genus and the orientability of the signature, being geometrical invariants of the quotient space, are also algebraic invariants of the groups. Thus two isomorphic groups must have the same g and the same orientability character. It is interesting to notice that here, too, it is necessary to invoke the geometrical realization Theorem 3, since for general NEC groups the simple characterization of g as the rank of the commutator factor group is not valid.

10. The 17 plane groups of euclidean crystallography. Finally one must confess that the name "NEC signature," emphasizing the non-euclidean, is not really appropriate. Perhaps "plane crystallographic signature" would be a better name. Just as there are certain F-signatures which do not define Fuchsian groups, but are realized as groups of isometries of the 2-sphere or the euclidean plane, so there are NEC signatures which define groups of isometries of the sphere and plane crystallographic groups. One well-known example is the icosahedral group defined by the F-presentation (0; 2, 3, 5), or, what is the same thing, the NEC presentation $(0, +, [2, 3, 5]\{\ \})$.

So much has been written about 17 euclidean space groups that one hesitates to add another notation to those already proposed. As a matter of interest, though, and perhaps also to illustrate and motivate what has gone before, we list the NEC signatures for the euclidean groups(!). We do not give presentations, because, in all cases, these follow automatically as explained in §5, and nearly always contain redundant generators and sometimes redundant relations too. In fact, it is the device of redundant generators preserving symmetry that enabled Wilkie to bring all the NEC groups into a unified scheme. However, an obvious elimination of redundant generators leads, in each case, to the presentation listed in Coxeter and Moser's book. As one example, let us take the group $p3m1$. The NEC presentation of the signature listed in Table I is:

Generators: x_1, e_1, c_0, c_1.

TABLE I

The 17 euclidean groups and their signatures

Symbol (Coxeter–Moser)	Signature	Quotient space
$p1$	$(1, +, [\ ]\ \{\ \})$	Torus
$p2$	$(0, +, [2, 2, 2, 2]\ \{\ \})$	Sphere
pm	$(0, +, [\ ]\ \{(\ )\ (\ )\})$	Annulus = sphere with two holes
pg	$(2, -, [\ ]\ \{\ \})$	Klein bottle
cm	$(1, -, [\ ]\ \{(\ )\})$	Möbius strip
pmm	$(0, +, [\ ]\ \{(2, 2, 2, 2)\})$	Disk = sphere with one hole
pmg	$(0, +, [2, 2]\ \{(\ )\})$	Disk
pgg	$(1, -, [2, 2]\ \{\ \})$	Projective plane
cmm	$(0, +, [2]\ \{(2, 2)\})$	Disk
$p4$	$(0, +, [2, 4, 4]\ \{\ \})$	Sphere
$p4m$	$(0, +, [\ ]\ \{(2, 4, 4)\})$	Disk
$p4g$	$(0, +, [4]\ \{(2)\})$	Disk
$p3$	$(0, +, [3, 3, 3]\ \{\ \})$	Sphere
$p3m1$	$(0 +, [3]\ \{(3)\})$	Disk
$p31m$	$(0, +, [\ ]\ \{(3, 3, 3)\})$	Disk
$p6$	$(0, +, [2, 3, 6]\ \{\ \})$	Sphere
$p6m$	$(0, +, [\ ]\ \{(2, 3, 6)\})$	Disk

https://doi.org/10.4153/CJM-1967-108-5 Published online by Cambridge University Press

Relations: $x_1^3 = 1, \quad c_1 = e_1^{-1}c_0\, e_1,$

$$c_0^2 = c_1^2 = (c_0\, c_1)^3 = 1, \quad x_1\, e_1 = 1.$$

Eliminating c_1 and e_1 by means of the second and last relations and dropping the redundant relation $c_1^2 = 1$, we derive

$$x_1^3 = c_0^2 = (c_0\, x_1\, c_0\, x_1^{-1})^2 = 1$$

as in Coxeter and Moser (**4**, p. 514).

References

1. L. V. Ahlfors, *On quasiconformal mappings*, J. Anal. Math. *4* (1954), 1–58.

2. L. Bers, *Quasiconformal mappings and Teichmüller's theorem*, Princeton Conference on Analytic Functions (1960).

3. S. Bundgaard and J. Nielsen, *On normal subgroups with finite index in F-groups*, Mat. Tidsskr., B (1951), 56–58.

4. H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups* (1957).

5. R. H. Fox, *On Fenchel's conjecture about F-groups*, Mat. Tidsskr., B (1952), 177–195.

6. R. Fricke and F. Klein, *Vorlesungen über die Theorie der automorphen Funktionen* (Leipzig, 1926).

7. A. M. Macbeath, *Geometrical realisation of isomorphisms between plane groups*, Bull. Amer. Math. Soc., *71* (1965), 629–630.

8. J. Nielsen, *Untersuchungen zur Topologie der geschlossenen zweiseitigen Flachen*, Acta Math., *50* (1927), 189–358.

9. H. C. Wilkie, *On non-euclidean crystallographic groups*, Math. Z., *91* (1966), 87–102.

10. H. Zieschang, *Ueber Automorphismen ebener diskontinuierlichen Gruppen*, Math. Ann., *166* (1966), 148–167.

University of Birmingham,
Birmingham, England