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ON THE BRANCH POINTS IN THE BRANCHED COVERINGS OF LINKS

BY
SHIN'ICHI KINOSHITA

ABSTRACT. Let l be a polygonal link in a 3-sphere S^3 and $\tilde{M}$ a branched covering of l, which depends on the choice of a monodromy map ϕ. Let $\tilde{l}$ be the link in $\tilde{M}$ over l. In this paper we determine the exact position of $\tilde{l}$ in $\tilde{M}$ for some cases. For instance, if l is a torus link $((n+1)p, n)$ and ϕ is an appropriate monodromy map of the fundamental group of $S^3 - l$ into the symmetric group of degree $n+1$, then $\tilde{M}$ is an S^3 and l is a torus link (np, n^2). The 3-fold irregular branched covering of a doubled knot k is an S^3, if it exists. The position of the link $\tilde{l}$ over k is shown in a figure. The link $\tilde{l}$ over knot 6_1 is obtained by K. A. Perko and the author, independently, and shown without proof in a paper by R. H. Fox [Can. J. Math. **22** (1970), 193–201]. The result mentioned in the above includes this case.

Let l be a polygonal link in a 3-sphere S^3 and $\tilde{M}$ a branched covering of l, which depends on the choice of a monodromy map ϕ. Let $\tilde{l}$ be the link in $\tilde{M}$ over l. Though the branched covering $\tilde{M}$ of l and the position of $\tilde{l}$ in $\tilde{M}$ have been studied, it seems that the exact position of $\tilde{l}$ in $\tilde{M}$ has never been systematically investigated. In this paper we determine the exact position of $\tilde{l}$ in $\tilde{M}$ for some cases. For instance, if l is a torus link $((n+1)p, n)$ and ϕ is an appropriate monodromy map of the fundamental group of $S^3 - l$ into the symmetric group of degree $n+1$, then $\tilde{M}$ is an S^3 and $\tilde{l}$ is a torus link (np, n^2). Other cases are also discussed.

1. Let l be a polygonal link in a 3-sphere S^3 and M a finite covering of $S^3 - l$. Hence, if $O \in S^3 - l$, O is covered by a finite number of points $O_1, O_2, \ldots, O_n$ of M. Let G be the fundamental group $\pi_1(S^3 - l, O)$ of $S^3 - l$ with base point O. For every $g \in G$ and $i (i = 1, 2, \ldots, n)$, there is a path $k_{g(i)}$ in M such that the initial point of $k_{g(i)}$ is O_i and $[p(k_{g(i)})] = g$, where p is the covering map and $[p(k_{g(i)})]$ is the path equivalence class of $p(k_{g(i)})$ with base point O. Let $O_{g(i)}$ be the end point of $k_{g(i)} (i = 1, 2, \ldots, n)$. Then, we have a map ϕ of G into the symmetric group S_n of n letters $1, 2, \ldots, n$ such that

$$\phi(g) = (\, : \; : \; : g(i) : \; : \; : \,),$$

where $g(i)$ is the index of $O_{g(i)}$. The map ϕ is a homomorphism and is called a monodromy map of the covering, and $\phi(G)$ is a transitive subgroup of S_n. Conversely,

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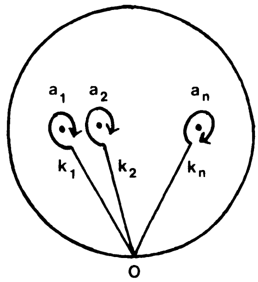

Fig. 1

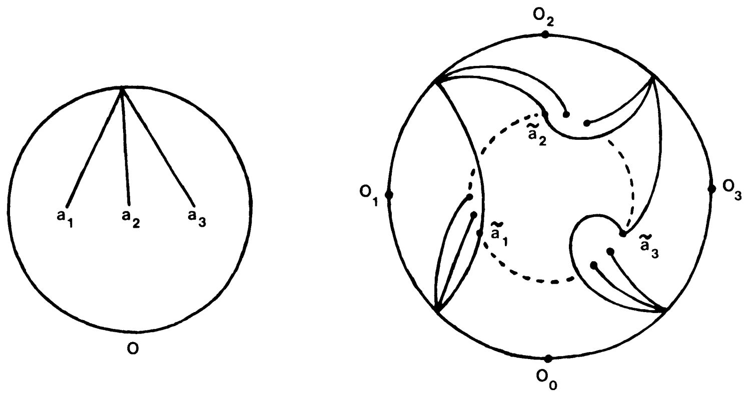

Fig. 2

if there is a homomorphism ϕ of G onto a transitive subgroup of S_n, then there is a finite covering M of $S^3 - l$, whose monodromy map is ϕ.

Now, let M be a finite covering of $S^3 - l$, with monodromy map ϕ. Then, the covering map p has the completion $\tilde{p}$ that maps a uniquely determined set $\tilde{M}$ onto S^3, where $M \subset \tilde{M}$ (see R. H. Fox [2]). The set $\tilde{M}$ is called the covering of S^3 branched along l, or briefly called a branched covering of l. A point $x \in \tilde{M}$ with $\tilde{p}(x) \in l$ is called a branch point of the branched covering of l, and the set of all such x's forms a polygonal link $\tilde{l}$ in $\tilde{M}$. For more details on branched coverings see R. H. Fox [3], L. P. Neuwirth [7] and H. Seifert and W. Threlfall [8].

2. Let D be a 2-cell and A the set of n points $a_1, a_2, \ldots, a_n$ in $\dot{D}$. Let O be a point on ∂D and let $k_1, k_2, \ldots k_n$ be n closed paths as shown in Fig. 1. Let $G = \pi_1(D - A, O)$ and consider a monodromy map ψ of G onto the symmetric group S_{n+1} of $n + 1$ letters $0, 1, \ldots, n$, generated by the correspondence

$$\psi([k_i]) = (0\,i)$$

for $i = 1, 2, \ldots, n$. The $(n + 1)$-fold branched covering $\tilde{D}$, associated with ψ, is a 2-cell, which satisfies the following conditions:

(1) $\partial\tilde{D}$ is the $(n + 1)$-fold cyclic covering of ∂D,

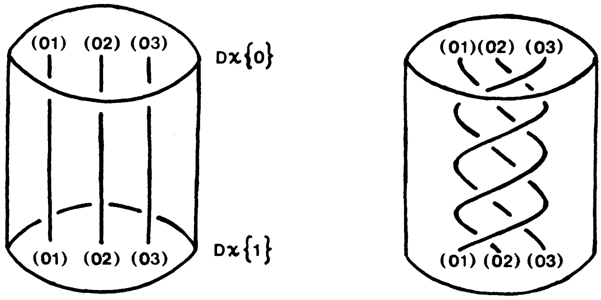

Fig. 3

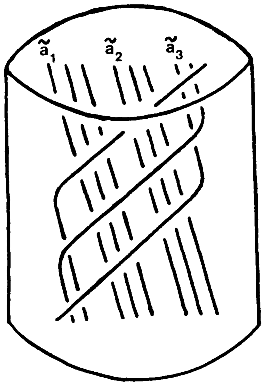

Fig. 4

(2) there are n points in $\tilde{D}$, each of which is over a_i, for $i = 1, 2, \ldots, n$, and

(3) one of these n points has the branching index 2 and others have the branching index 1 (see Fig. 2, where $n = 3$).

Let $C_1 = D \times I$, $\tilde{C}_1 = \tilde{D} \times I$ and $A_1 = A \times I$. Then, $G \approx \pi_1(C_1 - A_1, \{O\} \times \{0\})$ by a natural isomorphism, and $\tilde{C}_1$ is the branched covering of C_1, associated with ψ. We assume that $\{O_0\} \times I$ is over $\{O\} \times I$, where $O_0 \in \partial\tilde{D}$ is a point over O.

Now consider the 3-cell $C_2 = D \times I$, which contains n polygonal simple arcs A_2 obtained through twisting A_1 by $2(n + 1)\pi/n$ radians (see Fig. 3, where $n = 3$). The fundamental group $\pi_1(C_2 - A_2, \{O\} \times \{0\})$ is isomorphic to G and hence, $\tilde{C}_2 = \tilde{D} \times I$ is the branched covering of C_2, associated with ψ. However, our purpose is to identify the branched covering $\tilde{C}_2$ of C_2 such that $\{O_0\} \times I$ is over $\{O\} \times I$.

For this purpose, observe that one complete rotation of D causes the rotation of $2\pi/(n + 1)$ radians of $\tilde{D}$, because $\psi([\partial D]) = (0\,1 \ldots n)$. Hence, when we rotate D by $2(n + 1)\pi/n$ radians, $\tilde{D}$ is rotated by $2\pi/n$ radians. Therefore, the branched covering $\tilde{C}_2$ of C_2, which we are looking for, is obtained from C_1 through twisting $\tilde{A}_1$ by $2\pi/n$ radians (see Fig. 4, where $n = 3$).

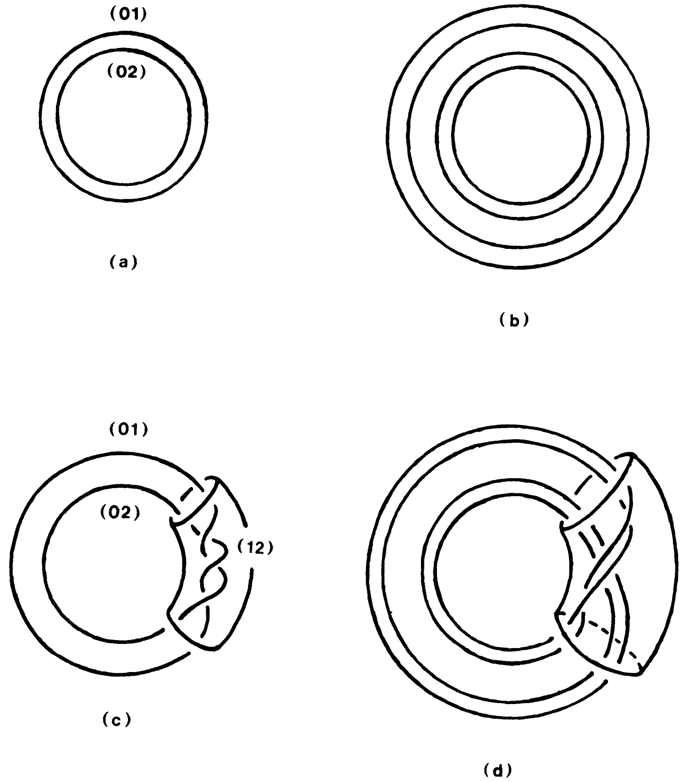

Fig. 5

REMARK. In Fig. 2 the points $\tilde{a}_1, \ldots, \tilde{a}_2, \ldots, \tilde{a}_3$ are on a circle: however, these points are placed on a segment in Fig. 4, so the reader should adjust for the distortion.

3. Now let l be a torus link $((n + 1)p, n)$ in S^3, where p is an integer. Hence, the link l lies in the interior of a trivially embedded solid torus V and there is a meridian disk B of V such that $l \cap B$ consists of n points. Let ϕ be a monodromy map of $\pi_1(S^3 - l)$ onto the symmetric group of $n + 1$ letters $0, 1, \ldots, n$, such that the effect of ϕ on B is the same to that of ψ on D in 2. Now we can apply the result in 2.

THEOREM 1. *Let l be a torus link $((n + 1)p, n)$, and ϕ a monodromy map as above. Then the branched covering $\tilde{M}$ of l, associated with ϕ, is a 3-sphere, and the link $\tilde{l}$ over l is a torus link (np, n^2).*

Proof. The proof of the theorem can easily be seen from the example in Remark 1.

REMARK 1. Fig. $5(a)$ is a trivial link l with two components. The branched covering of l, associated with ϕ, is an S^3 and the link $\tilde{l}$ is described in Fig. $5(b)$. Fig. $5(c)$ is a knot 3_1, obtained from l by applying the operation discussed in 2. The corresponding branched covering is an S^3, and the position of $\tilde{3}_1$ is described in Fig. $5(d)$.

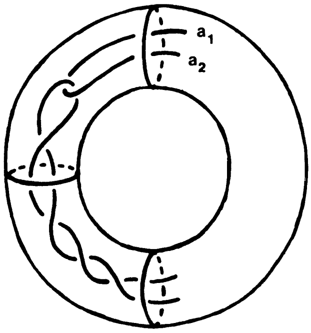

Fig. 6

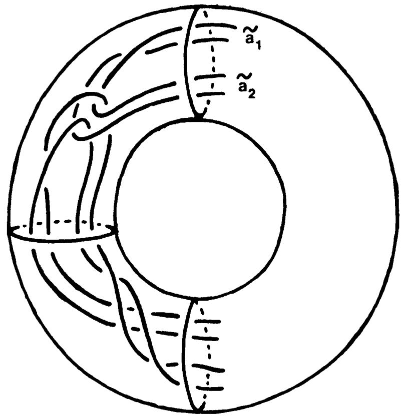

Fig. 7

This example explains Figure (a) on page 215 in the paper by R. H. Fox [4]. See also G. Burde [1], C. McA. Gordon and W. Heil [6], and D. Rolfson [8].

REMARK 2. Let k be a knot in S^3 and l a cable link $((n+1)p, n)$ with core k. Assume that a monodromy map ϕ similar to the above can be defined. Then, the branched covering $\tilde{M}$ of l, associated with ϕ, is homeomorphic to the $(n+1)$-fold cyclic branched covering of k. The link $\tilde{l}$ over l is a cable link (np, n^2) with core $\tilde{k}$, where $\tilde{k}$ is over k in the $(n+1)$-fold cyclic branched covering of k. In this case, the meridian and the longitude of $\tilde{k}$ must be interpreted from k.

REMARK 3. There are many ways to define a monodromy map ψ of $\pi_1(D-A)$ onto S_{n+1} such that the branched covering $\tilde{D}$ of D, associated with ψ, is a 2-cell. For instance, consider the monodromy map ψ_1 defined by

$$\psi_1([k_i]) = (n-i+2 \quad n-i+1),$$

where $i = 1, 2, \ldots, n$ and $n+1$ should be read as 0. When we twist A_1 to obtain A_2 in C_2, the corresponding monodromy map ϕ_1 must have the same effect on $D \times \{0\}$ and $D \times \{1\}$; so, we must twist A_1 by $2(n+1)p\pi$ radians, if $n > 2$. Therefore, we modify our theorem as follows; the branched covering $\tilde{M}$ of a torus link $((n+1)np, n)$, associated with ϕ_1, is a 3-sphere, and the link $\tilde{l}$ is a torus link (n^2p, n^2).

4. Let k be a doubled knot whose carrier is a trivial knot (see Fig. 6). Then, a monodromy map ϕ of the fundamental group of $S^3 - k$ onto the symmetric group S_3 of degree 3 exists if and only if $\Delta_k(-1) = 0$ mod. 3, where $\Delta_k(t)$ is the Alexander polynomial of the knot k (see, for instance, R. H. Fox [5] and L. P. Neuwirth [7]). Further, the monodromy map ϕ is unique up to conjugation, if it exists. The branched covering of k, associated with ϕ, is obviously a 3-fold irregular branched covering of k.

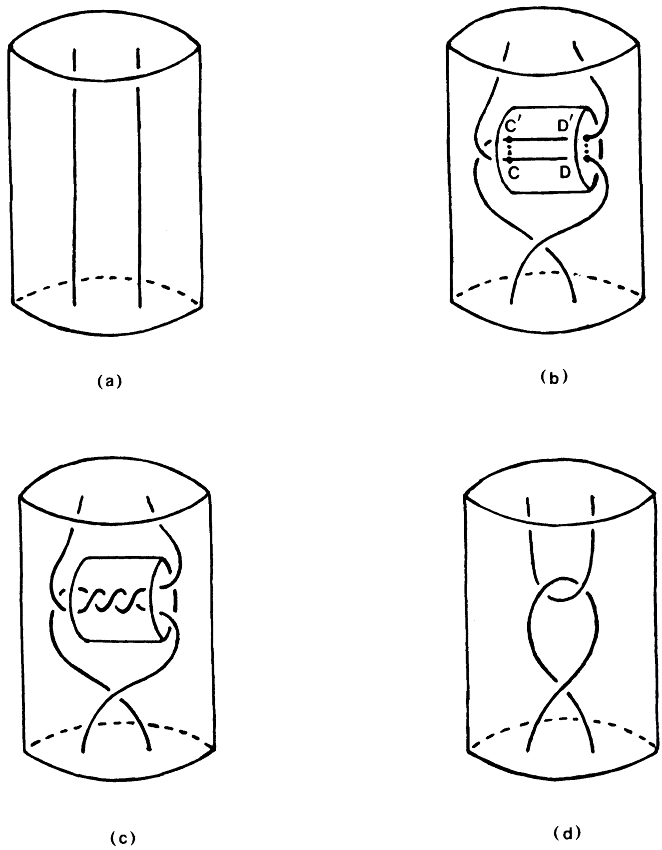

Fig. 8

THEOREM 2. *Let k be a doubled knot whose carrier is a trivial knot. Suppose that* $\Delta_k(-1) = 0$ *mod.* 3. *Then the* 3-*fold irregular branched covering* $\tilde{M}$ *of k is homeomorphic to* S^3, *and the link* $\tilde{l}$ *over k is as described in Fig.* 7.

Proof. The doubled knot k can be obtained from a trivial link with two components by repeatedly applying the operation discussed in 2, where $n = 2$. The operation is applied as described in Fig. 8 for the upper left part of k in Fig. 6, and it is applied as described before for the rest of k. It is easy to see that the branched covering $\tilde{M}$ of k, associated with ϕ, is homeomorphic to the 3-fold cyclic branched covering of the carrier; hence $\tilde{M}$ is homeomorphic to S^3.

We shall identify the position of $\tilde{l}$, which is over the upper left part of k. The rest of $\tilde{l}$ can easily be identified.

Consider the cube that contains the upper left part of k (see Fig. 8(b) and Fig. 9(a)), and its branched covering (see Fig. 9(b)). We shall identify the position of the branched covering of the inner cube in Fig. 9(a) with respect to the branched covering of the

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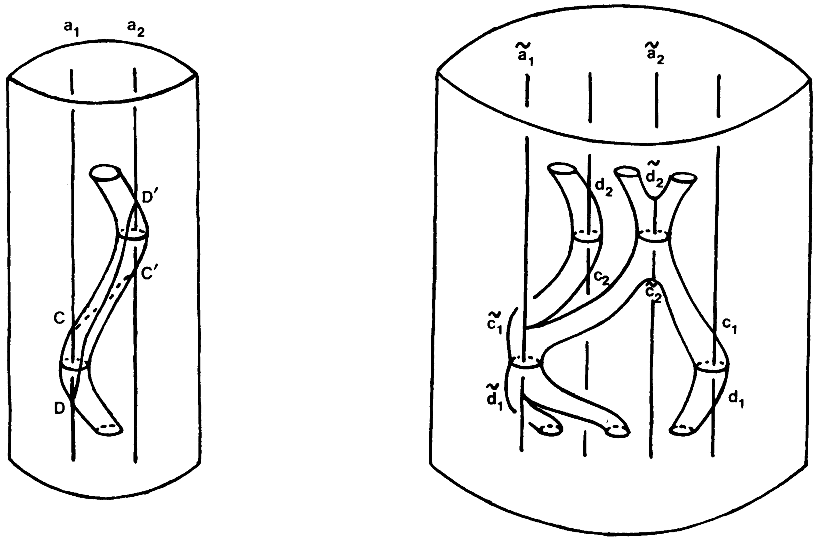


(a) (b)

Fig. 9

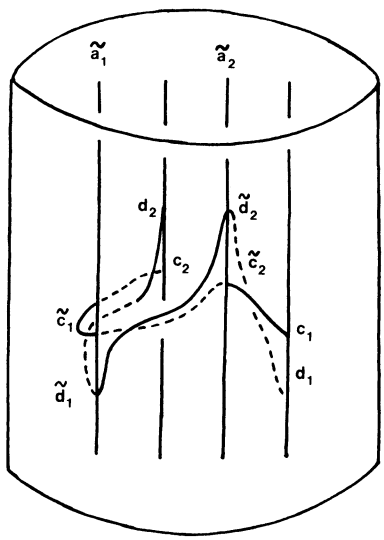
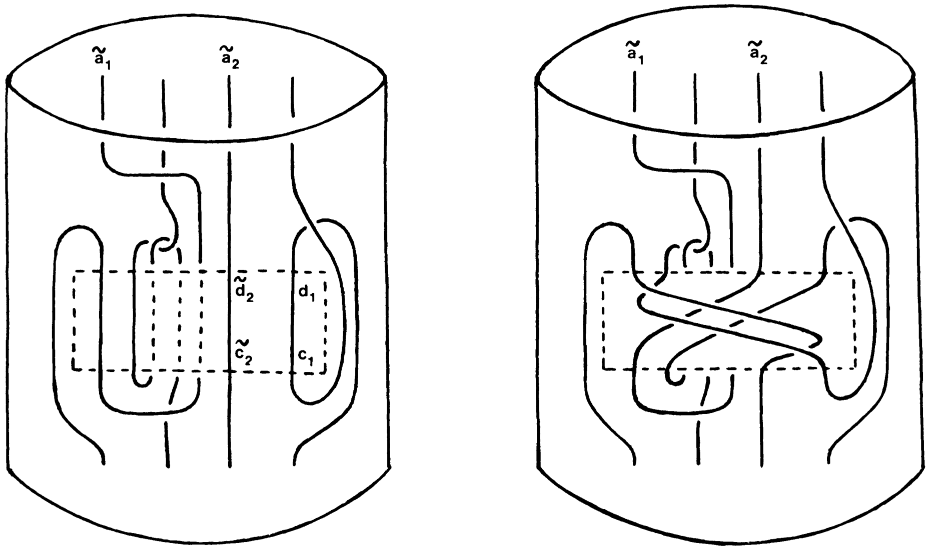


Fig. 10

outer cube, by considering the inner cube as a family of disks, from top to bottom, then tracing the positions of the branched coverings of these disks by referring to Fig. 2. The result is described in Fig. $9(b)$. Note that c_1 and $\tilde{c}_1$ are points over C, and d_1 and $\tilde{d}_1$ are points over D. Similarly d_2 and $\tilde{d}_2$ are points over D', and c_2 and $\tilde{c}_2$ are points over C'.



(a) (b)

Fig. 11

Next, we consider the segment CC' and DD' in Fig. 8(b) (see, also, Fig. 9(a)). The positions of the arcs over CC' and DD' are described in Fig. 10. By rotating the appropriate sections, we obtain Fig. 11(a), where the rectangle with dotted segments indicates the cube that is the branched covering of the inner cube in Fig. 8(b). Applying the operation discussed in 2, we obtain the branched covering of the cube in Fig. 8(c) (see Fig. 11(b)). The last result can be simplified as the upper left part of Fig. 7.

REMARK 1. Knots considered here have Alexander polynomial of the form $\Delta(t) = nt^2 + (1 - 2n)t + n$. Thus Theorem 2 applies to one third of the cases: $n = 1$ mod. 3. The trefoil corresponds to $n = 1$ and the knot 6_1 to $n = -2$. Hence, Theorem 2 explains the figure for the 3-fold irregular branched covering of knot 6_1 obtained by K. A. Perko and by the author, independently, on page 199 in the paper of R. H. Fox [5]. See also, G. Burde [1].

REMARK 2. The result for a doubled knot k_1 with carrier k_1' is similar to the result for a cable link 1_2 with core k_2 in 3. Note that if the irregular 3-fold branched covering of k_1 exists, it is homeomorphic to the 3-fold cyclic branched covering of k_1'.

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DEPARMENT OF MATHEMATICS AND COMPUTER SCIENCE
FLORIDA STATE UNIVERSITY
TALLAHASSEE, FLORIDA 32306