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DERIVATIONS OF 2-SUBHOMOGENEOUS C^*-ALGEBRAS

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Abstract A characterization is given of those unital, 2-subhomogeneous, Fell C^*-algebras which have only inner derivations. This proves Sproston and Strauss's conjecture from 1992. Various examples are given of phenomena which cannot occur for separable C^*-algebras. In particular, an example is given of a C^*-algebra with only inner derivations which has a quotient algebra admitting outer derivations. This answers a question of Akemann, Elliott, Pedersen and Tomiyama from 1976.

Keywords: C^*-algebras; derivations; subhomogeneous

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1. Introduction

It is still unknown which C^*-algebras have only inner derivations. For separable C^*-algebras it is known that such a C^*-algebra must be the direct sum of a C^*-algebra with continuous trace and a C^*-algebra with discrete primitive ideal space [**1**, **6**], but for inseparable C^*-algebras the problem remains wide open. It is natural, therefore, to begin by looking at the simplest cases, such as homogeneous and subhomogeneous C^*-algebras.

Every homogeneous C^*-algebra has only inner derivations [**5**, **14**], while for unital, 2-subhomogeneous Fell C^*-algebras, both necessary conditions and sufficient conditions are known for an algebra to have only inner derivations [**15**, **16**], but there is a gap between these. It was conjectured by Sproston and Strauss [**16**] that the known necessary conditions are in fact sufficient. Our main purpose in this paper is to prove this conjecture. Our methods are similar to those in [**16**].

At the same time we are able to provide examples of behaviour which cannot arise in separable C^*-algebras. For instance, we give an example of a C^*-algebra for which the set of inner derivations is a proper, norm-dense subset of the space of all derivations. We also give an example of a C^*-algebra with only inner derivations which has a quotient algebra admitting outer derivations. This answers a question of Akemann *et al.* from 1976 [**2**, Problem 6.4].

2. Results

We begin by recalling the definitions and notation from [**15**]. Let A be a C^*-algebra and let $\hat{A}$ be the spectrum of A, that is, the set of equivalence classes of irreducible *-representations of A, equipped with the Jacobson topology. The C^*-algebra A is said to be *2-subhomogeneous* if all the irreducible *-representations of A are of dimension 1 or 2. A unital 2-subhomogeneous C^*-algebra can be described in the following way [**17**]. For $i = 1, 2$, let $T^i(A)$ be the space of all equivalence classes of non-zero i-dimensional *-representations of A. Let X be the space of all equivalence classes of irreducible two-dimensional *-representations, and let Y be the space of all equivalence classes of irreducible one-dimensional *-representations. It is shown in [**17**] how to topologize $T^2(A)$ with X as an open subset. Let $\bar{X}$ be the closure of X in $T^2(A)$. Set $\partial X = \bar{X} \setminus X$. Then a point of ∂X can be written as an unordered pair $[y_1, y_2]$ of points of Y. Let Z be the open subset of ∂X consisting of all pairs $[y_1, y_2] \in \partial X$ with $y_1 \neq y_2$. We shall be interested in the case when $Z = \partial X$. The description in [**17**] shows that this happens if and only if A satisfies Fell's condition for each $\pi \in \hat{A}$ [**4**, Proposition 4.5.3(iii)]. *Fell's condition* is that there should exist an $a \in A$ such that $\sigma(a)$ is a rank 1 projection for every σ in some neighbourhood of π in $\hat{A}$. Thus, if $Z = \partial X$, A is a *Fell algebra* in the sense of [**3**]. If $\bar{X}$ is the Stone–Čech compactification of X, then A is said to have the *Stone–Čech property.*

Let T denote the space $\bar{X} \cup Y$, topologized so that a subset S is open in T if and only if $S \cap \bar{X}$ is open in $\bar{X}$ and $S \cap Y$ is open in Y. For $t \in T$, let A_t be the quotient of A by the common kernel of the representations in the class t. Each element $a \in A$ defines an operator field on T with values in $\{A_t\}$ by defining $a(t)$ to be the canonical image of a in A_t. In this way we can identify A with a full algebra of operator fields on T. In general, A is not a maximal field. Let B be the maximal C^*-algebra of cross-sections containing A. Then B is generated by A and its centre is $Z(B)$ (which can be identified with the abelian C^*-algebra of continuous functions on T; see [**8**, Lemma 1.7]). We shall use the map $\Phi : \hat{B} \to \hat{A}$ given by $\Phi(t) = t|_A$ $(t \in \hat{B})$ [**8**, Theorem 1.1]. This map is both continuous and surjective [**8**, Lemma 1.10].

The following ideal J will be important to us. Let J be the set of elements $a \in A$ such that $a(t)$ is zero for $t \in T \setminus X$. Then J is a closed, two-sided ideal in both A and B. Note that A/J and B/J are abelian C^*-algebras, because all their irreducible *-representations are one dimensional.

The first proposition is a combination of Theorems 1 and 2 of [**15**].

Proposition 2.1 (see [15]). *Let A be a unital 2-subhomogeneous C^*-algebra. Suppose either that $\partial X \setminus Z$ contains a point with a countable base of neighbourhoods, or that A is a Fell algebra but does not have the Stone–Čech property. Then A has outer derivations.*

It seems probable that in fact every unital 2-subhomogeneous C^*-algebra that is not a Fell algebra has outer derivations, but we have not been able to prove this.

Now let A be any C^*-algebra. For $a \in A$, let $\mathrm{ad}(a)$ be the inner derivation of A induced by a. The map $a \mapsto \mathrm{ad}(a)$ defines a bounded, linear map from A to $\Delta(A)$, the Banach space of all derivations of A equipped with the operator norm. The range of this map is the set $\Delta_0(A)$ of inner derivations, and the kernel is $Z(A)$, the centre of A.

Lemma 2.2. *Let A be a unital 2-subhomogeneous Fell C^*-algebra with the Stone–Čech property. Then, with the notation above:*

(i) $\Delta_0(A) = \Delta(A)$ *if and only if* $A/J + (Z(B) + J)/J = B/J$; *and*

(ii) $\Delta_0(A)$ *is dense in* $\Delta(A)$ *if and only if* $A/J + (Z(B) + J)/J$ *is dense in* B/J.

Proof. On the one hand, Sproston showed in [**15**] that, under the assumptions above, every derivation of A is implemented by an element of B. On the other hand, if $b \in B$ then $\mathrm{ad}(b)$ maps B into the ideal J defined above. Thus $\mathrm{ad}(b)|_A$ is a derivation of A. Now suppose that $b_1, b_2 \in B$ and that b_1 and b_2 induce the same derivation on A. Then $b_1 - b_2$ commutes with every element of A, which implies that $(b_1 - b_2)(t)$ is scalar-valued for all $t \in X$. It follows from this that $b_1 - b_2$ belongs to $Z(B)$, the centre of B. Thus, the space $\Delta(A)$ of derivations of A is bicontinuously isomorphic to the Banach space $B/Z(B)$, while the subspace $\Delta_0(A)$ of inner derivations of A is isomorphic to the subspace $(A+Z(B))/Z(B)$ of $B/Z(B)$. In particular, $\Delta_0(A)$ is dense in $\Delta(A)$ if and only if $(A + Z(B))/Z(B)$ is dense in $B/Z(B)$.

Part (i) now follows from the fact that, because $Z(B)$ and J are both subsets of $A + Z(B)$,

$$\frac{A + Z(B)}{Z(B)} = \frac{B}{Z(B)} \iff A + Z(B) = B \iff \frac{A}{J} + \frac{Z(B) + J}{J} = \frac{B}{J}.$$

Part (ii) follows in the same way using elementary approximation arguments. □

Let A_1 and A_2 be the C^*-subalgebras $A_1 = A/J$ and $A_2 = (Z(B)+J)/J$ of the abelian C^*-algebra B/J. Lemma 2.2 shows that we are interested in the question of when A_1+A_2 is dense in B/J, or is equal to B/J.

For this, we need the following results. Let $C(S)$ be the C^*-algebra of continuous functions on a compact, Hausdorff space S. Suppose that A_1 and A_2 are C^*-subalgebras of $C(S)$. Let $A_1 + A_2$ be the linear span of A_1 and A_2. For $i = 1, 2$, let S_i be the quotient of S obtained by identifying the points a and b whenever $f(a) = f(b)$ for all $f \in A_i$. Let p_i be the natural projection of S onto S_i. We define a *trip* of length $n - 1$ with respect to (A_1, A_2) to be a finite ordered set $\{a_1, \dots, a_n\}$, contained in S, such that $a_i \neq a_{i+1}$ for $i = 1, \dots, n - 1$, and either $p_1(a_1) = p_1(a_2)$, $p_2(a_2) = p_2(a_3)$, $p_1(a_3) = p_1(a_4), \dots$, or $p_2(a_1) = p_2(a_2)$, $p_1(a_2) = p_1(a_3)$, $p_2(a_3) = p_2(a_4), \dots$. A trip is a *round trip* if $n > 2$ and $a_1 = a_n$. Notice that S contains a two-point round trip $\{a_1, a_2, a_1\}$ if and only if $A_1 + A_2$ fails to separate points in S. Notice, too, that if S contains a round trip, then S contains trips of arbitrary length. Let us say that two points of S are equivalent if there is some trip to which they both belong. This defines an equivalence relation on S, and the equivalence classes for this relation are called *orbits*.

The next result is a combination of [**16**, Theorem 1 and Remark 2] and [**10**, Proposition 2]. In [**10**], the proposition is stated in terms of real-valued functions, but the

Stone–Weierstrass Theorem means that the result extends automatically to the complex situation. More recent results in this direction, and further references, can be found in [**16**]; the whole area is related to Hilbert's 13th Problem.

Proposition 2.3 (see [10, 16]). *Let S be a compact, Hausdorff space. Let A_1 and A_2 be C^*-subalgebras of $C(S)$ containing the constant functions.*

(i) *$A_1 + A_2 = C(S)$ if and only if there is a finite upper bound to the length of trips in S with respect to (A_1, A_2).*

(ii) *If all orbits are closed, then $A_1 + A_2$ is dense in $C(S)$ if and only if S contains no round trip with respect to (A_1, A_2).*

We now show the connection between the terminology of Proposition 2.3 and that used in [**12**]. The following definitions are usually made in the context of the primitive ideal space of A, rather than the spectrum of A. For 2-subhomogeneous C^*-algebras, however, these spaces are homeomorphic, and it seems simpler for us to work with the spectrum. For $\pi, \sigma \in \hat{A}$, let $\pi \sim \sigma$ if π and σ cannot be separated by disjoint open sets. Define a graph structure on $\hat{A}$ by saying that π and σ are adjacent if $\pi \sim \sigma$. We shall refer to the connected components of the graph $\hat{A}$ as *orbits*. We are thus using the name 'orbit' in two different contexts, but we shall see in Lemma 2.4 that the two uses harmonize. Let $\mathrm{Orc}(A)$ be the supremum of the diameters of the orbits of $\hat{A}$ in this graph structure (with the convention that a singleton has diameter 1 rather than 0). Thus $\mathrm{Orc}(A) = 1$ if and only if $\sim$ is an equivalence relation on $\hat{A}$. It was shown in [**12**, Corollary 4.6] that $\mathrm{Orc}(A) < \infty$ if and only if the space of inner derivations of A is closed in the space of all derivations of A. Furthermore, it follows from [**12**, Corollary 2.3] that if $\mathrm{Orc}(A) < \infty$, then each orbit is a closed subset of $\hat{A}$ in the Jacobson topology. A cycle in the graph $\hat{A}$ is called a *proper circle* in [**16**]. If the graph $\hat{A}$ is cycle-free, then we say that $\hat{A}$ has *no proper circles.*

Now let A be a unital 2-subhomogeneous Fell C^*-algebra. Then the description in [**17**] shows that for distinct $\pi_1, \pi_2 \in \hat{A}$, $\pi_1 \sim \pi_2$ if and only if $\pi_1, \pi_2 \in Y$ with $[\pi_1, \pi_2] \in \partial X$. Thus, a path $\pi_1 \sim \pi_2 \sim \cdots \sim \pi_n$ in $\hat{A}$, with the π_is distinct, corresponds to a sequence $[\pi_1, \pi_2], [\pi_2, \pi_3], \ldots, [\pi_{n-1}, \pi_n]$ in ∂X. Recall that the restriction map $\Phi : \hat{B} \to \hat{A}$ is continuous and surjective.

Lemma 2.4. *Let A be a unital 2-subhomogeneous Fell C^*-algebra. Let $A_1 = A/J$ and $A_2 = (Z(B) + J)/J$, and let S be the character space of B/J. Let $s, t \in S$. Then s and t lie in the same orbit in S with respect to (A_1, A_2) if and only if $\Phi(s)$ and $\Phi(t)$ lie in the same orbit in $\hat{A}$.*

Proof. We may identify S with the set of one-dimensional irreducible *-representations of B. Let p_1 and p_2 be the projection maps from S onto the character spaces S_1 and S_2 of A_1 and A_2, respectively. Note that for s and t in $S \subseteq \hat{B}$, $p_1(s) = p_1(t)$ if and only if $\Phi(s) = \Phi(t)$. Thus, p_1 is simply the map $\Phi : \hat{B} \to \hat{A}$, with its domain restricted to the closed subset S of $\hat{B}$ and its range restricted to the closed subset S_1 of $\hat{A}$ (identifying S_1 with the set of one-dimensional irreducible *-representations of A). Note also that for distinct $s, t \in S$, $p_2(s) = p_2(t)$ if and only if $[\Phi(s), \Phi(t)] \in \partial X$. By the remarks just before this lemma, this occurs if and only if $\Phi(s) \sim \Phi(t)$ in $\hat{A}$.

Now let $s, t \in S$. Suppose that s and t lie in the same orbit in S. Then there is a trip $\{s_1, s_2, \ldots, s_n\}$ in S with respect to (A_1, A_2) such that $s_1 = s$ and $s_n = t$. The description in the previous paragraph shows that $\Phi(s)$ and $\Phi(t)$ belong to the same orbit in $\hat{A}$. Suppose on the other hand that $\Phi(s)$ and $\Phi(t)$ belong to the same orbit in $\hat{A}$. Then the previous paragraph of this lemma shows that it is possible to find $\{s_1, s_2, \ldots, s_{2n}\}$ in S with $s_1 = s$ and $s_{2n} = t$, such that $[\Phi(s_{2i-1}), \Phi(s_{2i})] \in \partial X$ for $i \in \{1, \ldots, n\}$ and $\Phi(s_{2i}) = \Phi(s_{2i+1})$ for $i \in \{1, \ldots, n-1\}$. Hence, there is a trip from s to t in S, so s and t lie in the same orbit in S. □

Lemma 2.5. *Let A be a unital 2-subhomogeneous Fell C^*-algebra. Set $A_1 = A/J$ and $A_2 = (Z(B) + J)/J$.*

(i) *Orbits are closed in $\hat{A}$ if and only if orbits are closed in the character space S of B/J with respect to (A_1, A_2).*

(ii) *$\hat{A}$ has proper circles if and only if S has round trips with respect to (A_1, A_2).*

Proof. (i) Suppose that orbits are closed in $\hat{A}$. Let F be an orbit in S. Then $E := p_1(F)$ is an orbit in $\hat{A}$, by Lemma 2.4, so E is closed. Hence, $F = P_1^{-1}(E)$ is closed in S, since p_1 is continuous. On the other hand, suppose that orbits are closed in S. Let $E \subseteq S_1$ be an orbit in $\hat{A}$. Then $F := p_1^{-1}(E)$ is an orbit in S, so F is closed. Thus F is compact, so $E = p_1(F)$ is compact in the Hausdorff space S_1, and hence is closed in S_1, which itself is closed in $\hat{A}$. This establishes (i).

Now we prove (ii). Suppose first that there is a cycle $\pi_1 \sim \pi_2 \sim \cdots \sim \pi_n = \pi_1$ in $\hat{A}$ (with $n \geqslant 4$). As in Lemma 2.4, there exist $\{s_1, s_2, \ldots, s_{2n}\}$ in S with $[\Phi(s_{2i-1}), \Phi(s_{2i})] \in \partial X$ for $i \in \{1, \ldots, n\}$, and $\Phi(s_{2i}) = \Phi(s_{2i+1})$ for $i \in \{1, \ldots, n-1\}$, such that $\Phi(s_1) = \pi_1 = \pi_n = \Phi(s_{2n})$. Hence, $\{s_1, s_2, \ldots, s_{2n}, s_1\}$ is a round trip in S.

Conversely, suppose that there is a round trip $\{s_1, s_2, \ldots, s_n\}$ in S with respect to (A_1, A_2). Since $A_1 + A_2$ separates the points of S, $n \geqslant 4$. We have that $s_1 = s_n$, and, without loss of generality, we may assume that all the other s_i are distinct. Since distinct points of Z correspond to different unordered pairs $[\pi, \sigma]$ of points of Y, it is easy to check that in fact either $n \geqslant 7$, or $n = 6$ and $p_2(s_1) = p_2(s_2)$. In either case, the first paragraph of Lemma 2.4 shows that there is a proper circle in $\hat{A}$. This establishes (ii). □

Theorem 2.6. *Let A be a unital 2-subhomogeneous Fell C^*-algebra. Suppose that A has the Stone–Čech property, and that orbits are closed in $\hat{A}$. Then the set of inner derivations of A is dense in the norm topology in the set of all derivations of A if and only if there are no proper circles in $\hat{A}$.*

Proof. This follows from Lemma 2.2, Proposition 2.3 (ii), and Lemma 2.5. □

It is known that if A is a separable, unital C^*-algebra, then $\Delta_0(A)$ is dense in $\Delta(A)$ if and only if $\Delta_0(A) = \Delta(A)$ (see [**6**, Theorem 3]). Here we show that this fails for inseparable C^*-algebras.

Example 2.7. A C^*-algebra for which the set of inner derivations is a dense, proper subset of the space of all derivations.

Let B be the C^*-algebra of all continuous functions f from $\beta\mathbb{N}$ into $M_2(\mathbb{C})$ with the property that $f(x)$ is diagonal for $x \in \beta\mathbb{N} \setminus \mathbb{N}$. For $x \in \beta\mathbb{N} \setminus \mathbb{N}$, let $\lambda_x(f)$ be the entry in the top-left corner of $f(x)$ and let $\mu_x(f)$ be the entry in the bottom-right corner. Let $W = \{x_{i,j} : 1 \leqslant i < \infty,\ 1 \leqslant j \leqslant i+1\}$ be a countable discrete subset of $\beta\mathbb{N} \setminus \mathbb{N}$. Let A be the C^*-subalgebra of B consisting of those elements f of B that satisfy

$$\lambda_{x_{i,j}}(f) = \lambda_{x_{i,j+1}}(f) \quad \text{for } j \text{ odd},\ 1 \leqslant j \leqslant i,$$

and

$$\mu_{x_{i,j}}(f) = \mu_{x_{i,j+1}}(f) \quad \text{for } j \text{ even}, 1 \leqslant j \leqslant i,$$

for $1 \leqslant i < \infty$. Thus, in $\hat{A}$ we have, for each i,

$$\ker \mu_{x_{i,1}} \sim \ker \lambda_{x_{i,1}} = \ker \lambda_{x_{i,2}} \sim \ker \mu_{x_{i,2}} = \ker \mu_{x_{i,3}} \sim \ker \lambda_{x_{i,3}} \cdots,$$

and so on, up to $j = i+1$.

Evidently, A is a unital, 2-subhomogeneous C^*-algebra, with the Stone–Čech property. The constant operator fields

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

belong to A, showing that A is a Fell algebra. All the orbits in $\hat{A}$ are finite, hence closed, but there are arbitrarily long orbits, so $\mathrm{Orc}(A) = \infty$. Thus, Theorem 2.6 together with [**12**, Corollary 4.6] shows that the set of inner derivations of A is a dense, proper subset of the space of all derivations.

We now prove our main theorem.

Theorem 2.8. *Let A be a unital 2-subhomogeneous Fell C^*-algebra. Then the following are equivalent:*

(i) *A has only inner derivations; and*

(ii) *$\mathrm{Orc}(A)$ is finite, A has the Stone–Čech property, and there are no proper circles in $\hat{A}$.*

Proof. The implication (i) $\Rightarrow$ (ii) is established in [**16**, Appendix, Proposition 2]. The implication (ii) $\Rightarrow$ (i) follows from Theorem 2.6, together with the fact that when $\mathrm{Orc}(A)$ is finite, orbits are closed in $\hat{A}$ [**12**, Corollary 2.3] and the space of inner derivations of A is closed in the space of all derivations of A [**12**, Corollary 4.6]. □

If A is a separable C^*-algebra with only inner derivations, then the characterization theorem [**1**, **6**] shows that $\mathrm{Orc}(A) = 1$. It was observed in [**13**] that there are inseparable C^*-algebras with only inner derivations for which $\mathrm{Orc}(A) \neq 1$. Here we give a specific example.

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Example 2.9. Fix a natural number $n > 1$. Choose n distinct points, $x_1, \dots, x_n$, in $\beta\mathbb{N} \setminus \mathbb{N}$. Let A_n be the C^*-algebra of continuous functions from $\beta\mathbb{N}$ into $M_2(\mathbb{C})$ with the property that $f(x_i)$ is diagonal for $1 \leqslant i \leqslant n$, and the lower diagonal entry of $f(x_i)$ is equal to the upper diagonal entry of $f(x_{i+1})$ for $1 \leqslant i \leqslant n-1$. Then $\mathrm{Orc}(A_n) = n$, but A_n satisfies the conditions of Theorem 2.8, so A_n has only inner derivations.

The question was raised in [**2**, Problem 6.4] of whether the property of every derivation being inner passes to quotients. This is true for separable C^*-algebras by the characterization theorem [**1**, **6**]. Here we show that the general question has a negative answer.

Example 2.10. A C^*-algebra having only inner derivations, but with a quotient algebra having outer derivations.

The space $\beta\mathbb{N} \setminus \mathbb{N}$ is not extremally disconnected [**9**, Problem 6R]. Thus there is an open subset S of $\beta\mathbb{N} \setminus \mathbb{N}$ with the property that $\bar{S}$, the closure of S in $\beta\mathbb{N} \setminus \mathbb{N}$, is not homeomorphic to βS [**9**, Problem 1H6]. On the other hand, [**9**, Theorem 6.7] shows that $\beta(\mathbb{N} \cup S) = \beta\mathbb{N}$. Let A be the C^*-algebra of all continuous functions from $\beta\mathbb{N}$ into $M_2(\mathbb{C})$ which are diagonal on $\beta\mathbb{N} \setminus (\mathbb{N} \cup S)$. Then A is a 2-subhomogeneous Fell C^*-algebra with the Stone–Čech property, and $\mathrm{Orc}(A) = 1$, so A has only inner derivations, by Theorem 2.8.

On the other hand, let D be the C^*-algebra of continuous functions from $\bar{S}$ into $M_2(\mathbb{C})$ which are diagonal on $\bar{S} \setminus S$. Then D is a quotient of A, and D is a 2-subhomogeneous Fell algebra which does not have the Stone–Čech property. Thus D admits outer derivations by Proposition 2.1.

It is known that von Neumann algebras have only inner derivations. One of the main open questions in this area is whether quotients of von Neumann algebras ever admit outer derivations (see [**11**] and [**7**]).

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