BULL. AUSTRAL. MATH. SOC. 76D30
VOL. 19 (1978), 309-318.

Action of a force near the planar surface between semi-infinite immiscible liquids at very low Reynolds numbers: Addendum

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1. First order approximation to interface shape

In the paper by Aderogba and Blake [1], the first order approximation to the shape of the interface between two immiscible liquids, at which surface tension acted, was obtained in the case of a point force directed parallel to the interface. In the case of the normal force no solution was obtained due to the logarithmic singularity associated with problems of this type. However, the problem is not physically well-posed in the case of the normal force. Surface tension is not suitable because it cannot alone balance the induced stress on the interface due to a point force. We need an additional force to balance the action of the point force on the interface. The obvious solution is to include a density difference $\Delta\rho^*$ between the two fluids

$$\Delta\rho^* = \rho_1^* - \rho_2^* > 0 \, , \tag{1}$$

where ρ_1^* and ρ_2^* are the densities of the lower and upper fluid respectively.

Thus equation (6) in Aderogba and Blake [1] becomes

Received 6 November 1978. The authors acknowledge the useful comments of Dr Nadav Liron of the Weizmann Institute of Science.

https://doi.org/10.1017/S0004972700008819 Published online by Cambridge University Press

$$(2) \qquad T\nabla_2^2\zeta + \left(\rho_2^*-\rho_1^*\right)g\zeta = \left[p^{(1)}-p^{(2)}+2\left\{\mu_2 \frac{\partial u_3^{(2)}}{\partial x_3} - \mu_1 \frac{\partial u_3^{(1)}}{\partial x_3}\right\}\right]_{x_3=0} ,$$

where T is the surface tension, ζ, the interface elevation g, gravitational acceleration, p, pressure, u_3, the normal velocity, and μ_1 and μ_2 the viscosities in the lower and upper fluids respectively. For the case of the point force parallel to the interface the integral representation for the interface shape is:

$$(3) \qquad \zeta\left(x_1, x_2\right) = \frac{F_\alpha}{2\pi T}\frac{\partial}{\partial x_\alpha}\int_0^\infty \frac{\xi J_0(\rho\xi)e^{-\xi h}}{\xi^2+\varepsilon^2}\,d\xi ,$$

where $\varepsilon^2 = \dfrac{\left(\rho_1^*-\rho_2^*\right)g}{T}$ and $\rho^2 = x_1^2 + x_2^2$.

The special cases correspond to

(a) $\rho_1^* = \rho_2^*$ $(\varepsilon = 0)$ and

(b) $T = 0$ $(\varepsilon = \infty)$, no surface tension.

The solution to (a) was given in [1] as

$$(4) \qquad \zeta = \frac{F_\alpha h}{2\pi T}\frac{x_\alpha}{\rho}\left\{1 - \frac{h}{\left(\rho^2+h^2\right)^{\frac{1}{2}}}\right\} , \quad \alpha = 1, 2 ;$$

and for case (b),

$$(5) \qquad \zeta = \frac{3h^2F_\alpha x_\alpha}{2\pi\left(\rho_1^*-\rho_2^*\right)g}\left(\rho^2+h^2\right)^{-5/2} .$$

For the general case $0 < \varepsilon < \infty$, (3) can be evaluated numerically.

In the case of the normal force, the axisymetric case,

$$(6) \qquad \zeta(\rho) = \frac{F_3}{2\pi T}\int_0^\infty \frac{\xi(1+h\xi)e^{-h\xi}J_0(\rho\xi)\,d\xi}{\xi^2+\varepsilon^2} .$$

There is no meaningful solution for the case (a) $\varepsilon = 0$ $\left(\rho_1^* = \rho_2^*\right)$, but case (b) has the solution

(7) $$\zeta(\rho) = \frac{3h^3 F_3}{2\pi(\rho_1^* - \rho_2^*)g} (\rho^2 + h^2)^{-5/2} .$$

Obviously, the force balance

(8) $$F_3 = 2\pi(\rho_1^* - \rho_2^*)g \int_0^\infty \rho\zeta d\rho$$

is obtained.

It appears that one has to resort to numerical techniques to obtain $\zeta(\rho)$ from (6) for general ε . The elevation of the interface at $\rho = 0$ can be obtained in closed form as follows:

(9) $$\zeta(0) = \frac{F_3}{2\pi T} \left[1 + \sin(\varepsilon h)\left(\pi/2 - Si(\varepsilon h) + \varepsilon h Ci(\varepsilon h)\right) + \cos(\varepsilon h)\left(Ci(\varepsilon h) - \pi/2\ \varepsilon h + \varepsilon h Si(\varepsilon h)\right)\right] ,$$

where $Ci(x)$ and $Si(x)$ are the cosine and sine integrals defined as follows:

(10a) $$Ci(x) = \int_x^\infty \frac{\cos t}{t} dt ,$$

(10b) $$Si(x) = \int_0^x \frac{\sin t}{t} dt .$$

The interface elevation is logarithmically singular in εh .

For large εh we can express the solution to (6) in differential operator form

(11a) $$\zeta(\rho) = \frac{3}{2\pi T \varepsilon^2} \mathcal{D}\left[h(\rho^2 + h^2)^{-3/2}\right] + O(e^{-\varepsilon h}) ,$$

where $\mathcal{D}$ is the differential operator

(11b) $$\mathcal{D} = \left[\left(1 - h\frac{d}{dh}\right)\left(1 + \frac{1}{\varepsilon^2}\frac{d^2}{dh^2}\right)^{-1}\right] .$$

Not surprisingly, the zeroth order solution is given by (7). The next term in the series is

(12) $$\zeta_1(\rho) = \frac{F_3}{2\pi T(\varepsilon h)^4} \left|75h^7(\rho^2+h^2)^{-7/2}-105h^9(\rho^2+h^2)^{-9/2}\right| .$$

The zeroth order term satisfies the force balance equation, (8), exactly; all other terms in the series produce zero contribution.

2. Streamlines

In the case of the normal force we can illustrate the flow field in terms of a Stokes stream function $\psi(\rho, z)$, defined in terms of the velocities as follows;

(13) $$u_z = \frac{1}{\rho}\frac{\partial\psi}{\partial z} , \quad u_\rho = -\frac{1}{\rho}\frac{\partial\psi}{\partial z} .$$

Here ρ is the radial direction and is defined in (3) and z is the vertical coordinate (equal to x_3 , in [1]).

The streamfunctions in each region (using the notation of [1]), are:

REGION I: $z < 0$.

(14) $$\psi = \frac{F_3}{8\pi\mu_1}\left[\frac{\rho^2}{\left(\rho^2+(z+h)^2\right)^{\frac{1}{2}}} - \frac{\rho^2}{\left(\rho^2+(z-h)^2\right)^{\frac{1}{2}}} + \frac{2\theta}{1+\theta}\,\frac{h\rho^2 z}{\left(\rho^2+(z-h)^2\right)^{3/2}}\right] ;$$

REGION II: $z > 0$.

(15) $$\psi = \frac{F_3}{8\pi\mu_1}\left(\frac{2}{1+\theta}\right)\frac{h\rho^2 z}{\left(\rho^2+(z+h)^2\right)^{3/2}} ,$$

where $\theta = \mu_2|\mu_1$. In Region I the shape and magnitude of the streamlines are dependent on θ , whereas in Region II only the magnitude of ψ is dependent on θ . Several examples of streamlines in Region I for different values of θ are illustrated in Figure 4, while the streamlines for Region II are shown in Figure 5.

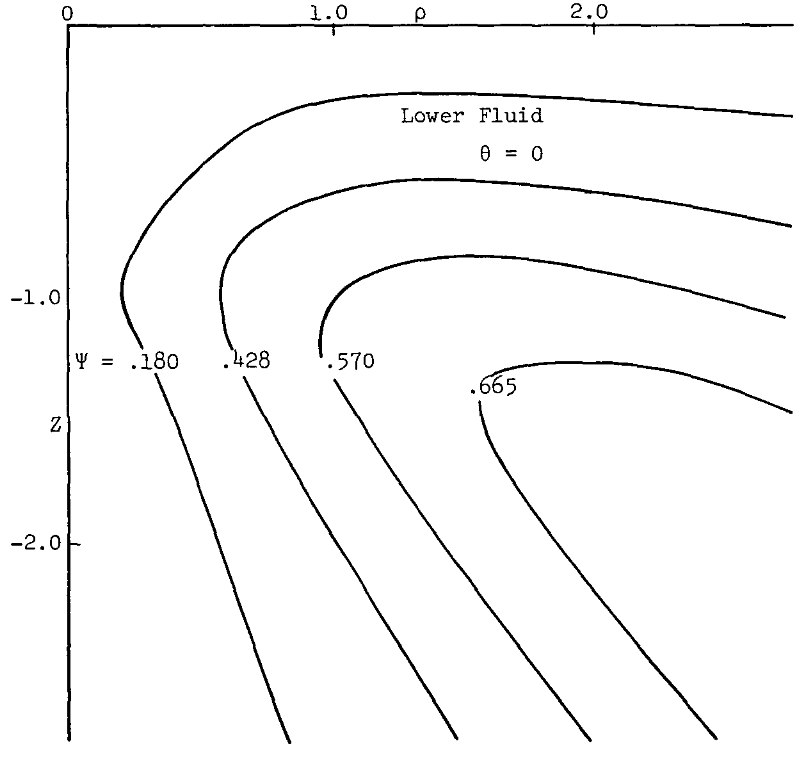

FIGURE 4 (a)

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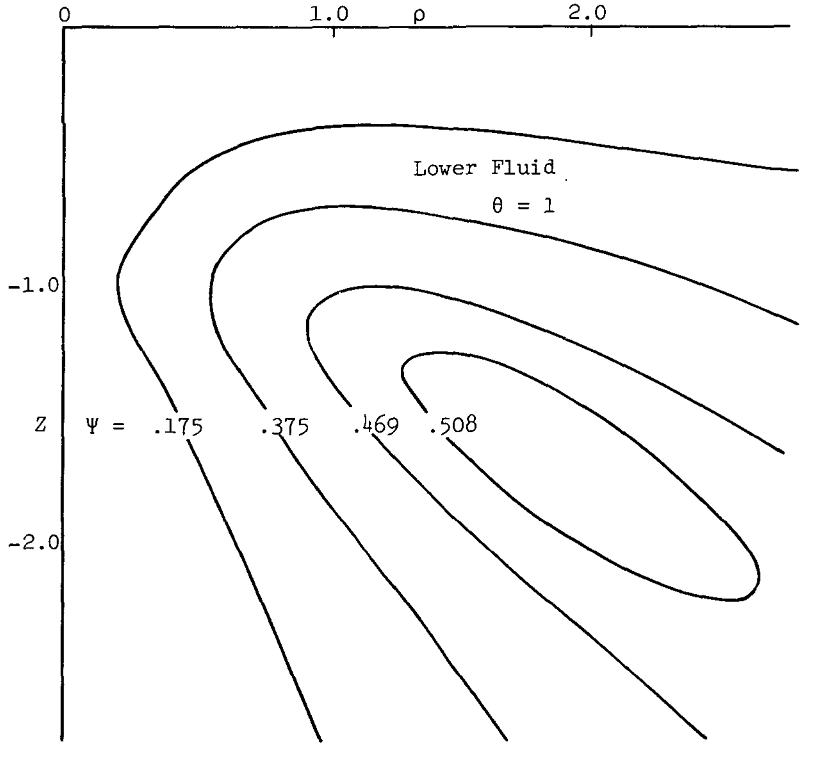

FIGURE 4 (b)

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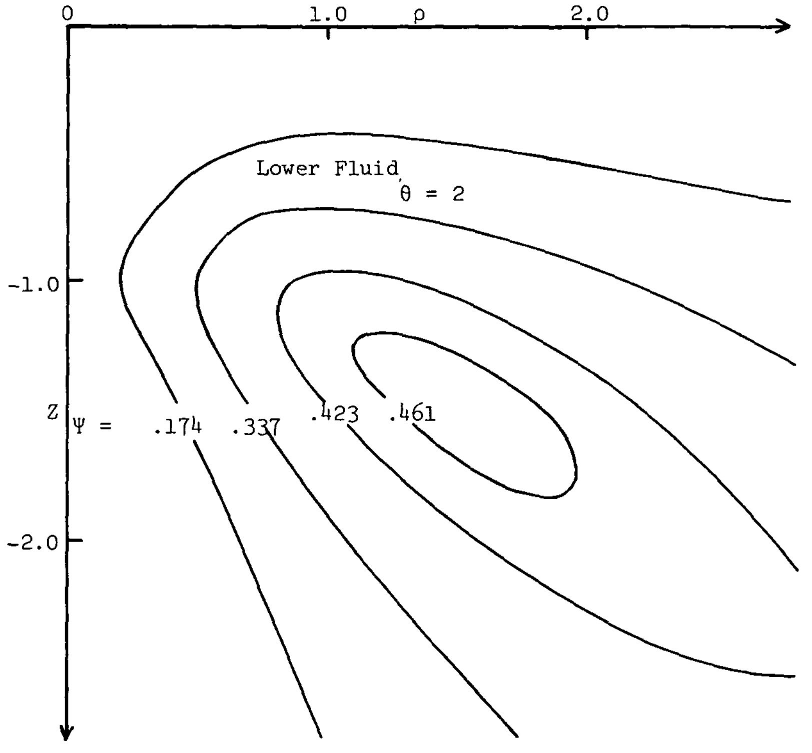

FIGURE 4 (c)

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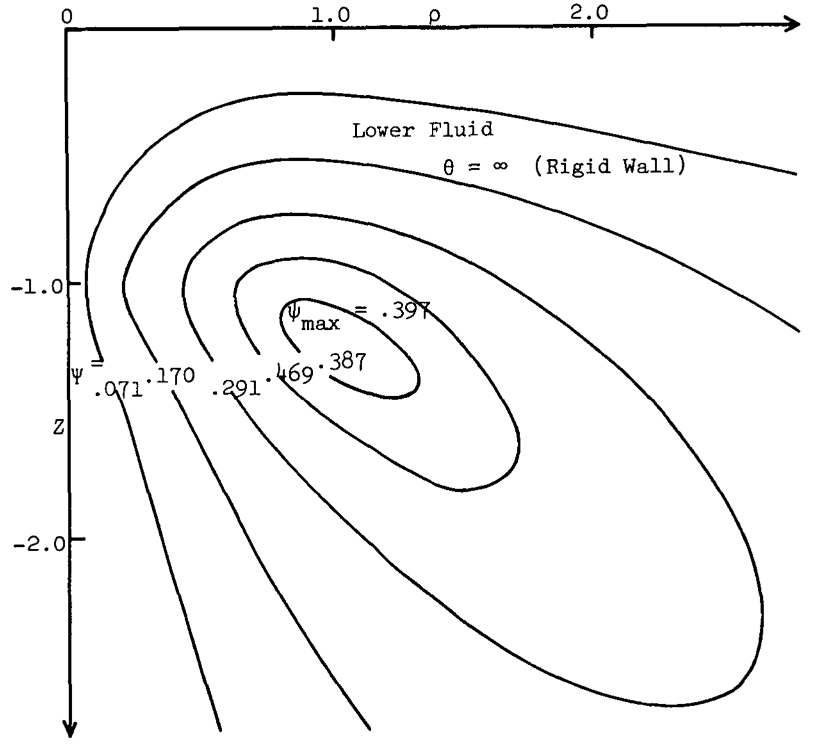

FIGURE 4 (d)

FIGURE 4. Streamlines in Region I for different values of θ with $F_3 = 8\pi\mu_1$ and $h = 1$. Closed streamlines are obtained in all cases except for $\theta = 0$.

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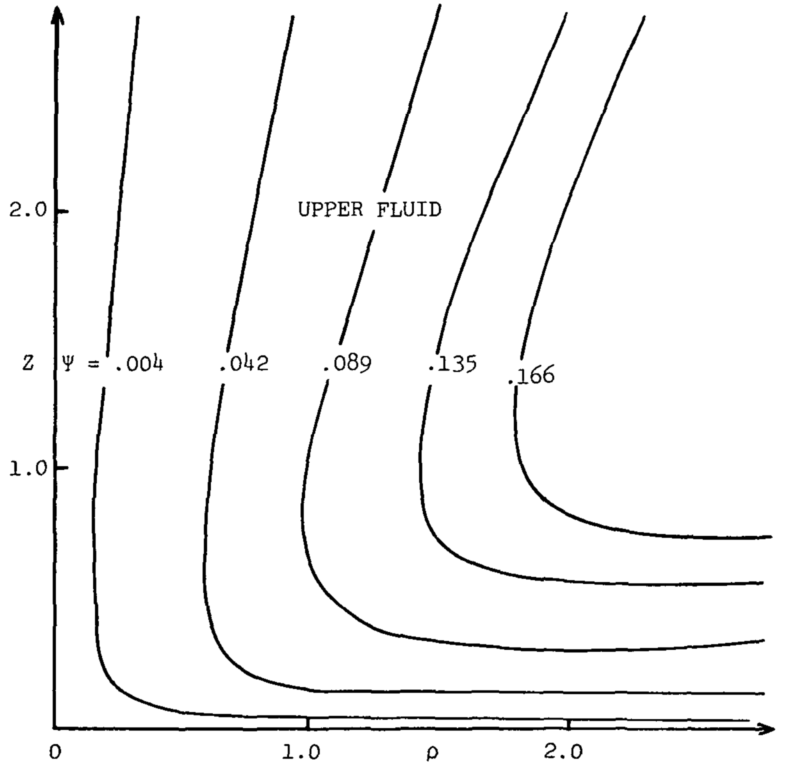

FIGURE 5. Streamlines in Region II for $F_3 = 4\pi\mu_1(1+\theta)$ and $h = 1$. The streamlines are not closed in the upper fluid.

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Reference

[1] K. Aderogba and J.R. Blake, "Action of a force near the planar surface between two semi-infinite immiscible liquids at very low Reynolds numbers", *Bull. Austral. Math. Soc.* 18 (1978), 345-356.

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https://doi.org/10.1017/S0004972700008819 Published online by Cambridge University Press