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A GENERALIZED FOURIER TRANSFORMATION FOR $L_1(G)$-MODULES

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Abstract

Let G be a compact abelian group with dual $\hat{G}$ and let K be a Banach $L_1(G)$-module. We introduce the notion of character convolution transformation of K which reduces to ordinary Fourier or Fourier-Stieltjes transformation when K is one of the spaces $L_p(G)$, $M(G)$. We show that the question of what maps $\hat{G} \to K$ extend to multipliers of K is a question of asking for descriptions of the character convolution transforms. In this setting some results of Helson-Edward and Schoenberg-Eberlein find generalizations, as do some classical results, including the inversion formula and the Parseval relation. We then apply these results to transformation groups, obtaining a variant of a theorem of Bochner and an extension of a theorem of Ryan.

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Introduction

Let G be a compact abelian group.

As is well-known, $L_1(G)$ is a commutative Banach algebra under convolution. A *Banach* $L_1(G)$-*module* K (see [4; 32.14]) is a Banach space K that is also a module over the ring $L_1(G)$, such that (if $*$ denotes the module multiplication)

$$f * \alpha x = \alpha f * x = \alpha(f * x) \qquad (\alpha \in \mathbf{C};\ f \in L_1(G);\ x \in K)$$

and

$$\|f * x\| \leqslant \|f\|\|x\| \qquad (f \in L_1(G);\ x \in K).$$

Under convolution, $L_p(G)$ $(1 \leqslant p \leqslant \infty)$, $C(G)$ and $M(G)$ are Banach $L_1(G)$-modules. More examples are given in [4; Section 32], [1; Section 4] and in Section 3 of this paper.

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We are going to occupy ourselves with the following two problems, that turn out to be closely related. Let K be a Banach module over $L_1(G)$.

(α) Introduce an analog of the Fourier-Stieltjes transformation that reduces to the ordinary Fourier or Fourier-Stieltjes transformation if K is one of the spaces $L_p(G)$, $M(G)$.

(β) A *multiplier* of K is a continuous module homomorphism $L_1(G) \to K$. (See [5].) Consider the dual group $\hat{G}$ of G as a subset of $L_1(G)$. As $\hat{G}$ spans a dense linear subspace of $L_1(G)$, a map $\hat{G} \to K$ has at most one continuous linear extension $L_1(G) \to K$. What maps $\hat{G} \to K$ extend to multipliers of K?

For $K = L_1(G)$ the relation between (α) and (β) is easy to describe: by Wendel's characterization of the multipliers of $L_1(G)$ [4; 35.5] one sees that a map ϕ: $\hat{G} \to L_1(G)$ extends to a multiplier if and only if there exists a $\mu \in M(G)$ such that $\phi(\gamma) = \hat{\mu}(\gamma)\gamma$ for all $\gamma \in \hat{G}$.

1

Our notations are mostly those used by E. Hewitt and K. A. Ross in [4].

Throughout the paper, G is a compact abelian group whose dual group is denoted $\hat{G}$, and K is a Banach $L_1(G)$-module. Both convolution $L_1(G) \times L_1(G) \to L_1(G)$ and module multiplication $L_1(G) \times K \to K$ are indicated by $*$. The Haar integral of $f \in L_1(G)$ are written $\int f(s)\,ds$.

K is called *order-free* if for every $x \in K$, $x \neq 0$ there exists an $f \in L_1(G)$, $f * x \neq 0$. The *trigonometric polynomials* (that is, the linear combinations of characters) form a dense linear subspace of $L_1(G)$. It follows that K is order-free if and only if for every $x \in K$, $x \neq 0$ there exists a $\gamma \in \hat{G}$ such that $\gamma * x \neq 0$.

By [4; 32.22] the products $f * x$ ($f \in L_1(G)$; $x \in K$) form a closed linear subspace K_{abs} of K. This K_{abs} is a Banach $L_1(G)$-submodule of K. As $L_1(G)$ has an approximate identity K_{abs} is order-free. In particular, if $x, y \in K_{abs}$ and if $\gamma * x = \gamma * y$ for all $\gamma \in \hat{G}$, then $x = y$.

K is said to be *absolutely continuous* if $K = K_{abs}$. Examples: $L_p(G)$ $(1 \leq p < \infty)$ and $C(G)$ are absolutely continuous [4; 32.20 and 32.31]; $L_\infty(G)$ is not [4; 20.16]; neither is $M(G)$ [4; 19.18].

For $f \in L_1(G)$ define $f^* \in L_1(G)$ by $f^*(s) = f(s^{-1})$ $(s \in G)$. We make the conjugate space K^* of K into a Banach $L_1(G)$-module by defining

$$(x, f * h) = (f^* * x, h) \qquad (f \in L_1(G); x \in K; h \in K^*).$$

(We might just as well have taken $f * x$ instead of $f^* * x$. However, $f^* * x$ is more appropriate in the more general situation where one does not confine ones attention to compact abelian group G.)

If by a similar formula one puts a module structure on K^{**}, the natural map $K \to K^{**}$ is a module homomorphism.

For $h \in K^*$ we have $L_1(G) * h = \{0\}$ if and only if h vanishes on K_{abs}. Hence, K^* is order-free if and only if K is absolutely continuous.

The continuous linear module homomorphisms $L_1(G) \to K$ are called the *multipliers* of K; they form a Banach space Mult K. Every $x \in K$ induces a $T_x \in \text{Mult } K$ by

$$T_x f = f * x \qquad (f \in L_1(G)).$$

This T: $K \to$ Multi K is injective if and only if K is order-free. In particular, the restriction of T to K_{abs} is injective (it is also an isometry; see [3; 5.1(iv)]).

1.1 Lemma. *For $\gamma \in \hat{G}$, define $K_\gamma = \gamma * K\,(= \{\gamma * x \mid x \in K\})$. Then*

$$K_\gamma = \{x \in K \mid \gamma * x = x\}$$
$$= \{x \in K \mid f * x = \hat{f}(\gamma)x \text{ for every } f \in L_1(G)\}.$$

K_γ is closed linear subspace of K. The map $x \mapsto \gamma * x$ is a continuous idempotent map of K onto K_γ. If $\beta \in \hat{G}$, $\beta \neq \gamma$, then $\beta * K_\gamma = \{0\}$. Further,

$$K_{\text{abs}} = \text{clo} \sum_{\gamma \in \hat{G}} K_\gamma.$$

Proof. We only prove the last sentence; the proof of the rest is simple. Obviously $\gamma * K \subset K_{\text{abs}}$ for all $\gamma \in \hat{G}$, so $K_{\text{abs}} \supset \text{clo}\,\Sigma\{K_\gamma \mid \gamma \in \hat{G}\}$. Conversely, $\hat{G}$ spans a dense linear subspace of $L_1(G)$; hence if $x \in K$, then $\{\gamma * x \mid \gamma \in \hat{G}\}$ spans a dense linear subspace of $L_1(G) * x$. It follows that $\text{clo}\,\Sigma_\gamma K_\gamma \supset K_{\text{abs}}$.

For Hilbert spaces we have a more detailed knowledge:

1.2 Theorem. *Let K be a Hilbert space; let $\langle\ ,\ \rangle$ be its inner product. Then*

$$\langle f * x, y \rangle = \langle x, \tilde{f}^* * y \rangle \qquad (x, y \in K;\ f \in L_1(G)).$$

If β, γ are distinct, then $K_\beta \perp K_\gamma$. For each γ, the map $x \mapsto \gamma * x$ is the orthogonal projection of K onto K_γ. For every $x \in K$ the sum $\Sigma_{\gamma \in \hat{G}}\, \gamma * x$ converges in the sense of the norm. The map $x \mapsto \Sigma \gamma * x$ is the orthogonal projection of K onto K_{abs}. Its kernel is

$$\{x \in K \mid L_1(G) * x = \{0\}\}.$$

Proof. Take $\gamma \in \hat{G}$, put $Px = x - \gamma * x$ for $x \in K$. Then $P = P^2$ and $\|I - P\| \leqslant 1$. Let $P(K)^\perp = \{x \mid x \perp P(K)\}$. If $x \in P(K)^\perp$, then $x \perp Px$, so

$$\|x\|^2 + \|Px\|^2 = \|x - Px\|^2 \leqslant \|x\|^2.$$

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Hence, $P(K)^{\perp} \subset P^{-1}(0)$. Conversely, every $x \in P^{-1}(0)$ can be written as $x = y + z$ where $y \in P(K)$, $z \in P(K)^{\perp}$. (Notice that $P(K) = (I - P)^{-1}(0)$ is closed.) Then $z \in P^{-1}(0)$, so $y = x - z \in P^{-1}(0)$. But $y \in P(K)$ and $P = P^2$: it follows that $y = 0$ and $x = z \in P(K)^{\perp}$. Therefore, $P(K)^{\perp} = P^{-1}(0)$. Consequently, P is an orthogonal projection. Then so is the map $x \mapsto \gamma * x$. We see that

$$\langle f * x, y \rangle = \langle x, \bar{f}^* * y \rangle \qquad (x, y \in K)$$

if $f \in \hat{G}$. The same formula holds, by linearity, for all trigonometric functions f, and, by continuity, for all $f \in L_1(G)$. The rest is easy.

Note. The formula

$$T_f x = f * x \qquad (f \in L_1(G);\, x \in K)$$

yields a correspondence between the module structures $*$ on K and the representations T of $L_1(G)$ in K for which $\|T_f\| \leqslant \|f\|$ ($f \in L_1(G)$). By the above theorem, every such representation is a $*$-representation.

1.3 LEMMA. *Let K be a Banach $L_1(G)$-module. For a map T: $L_1(G) \to K$ the following conditions are equivalent.*

(i) $T \in \operatorname{Mult} K$.

(ii) *T is linear and continuous; $T\gamma \in K_\gamma$ for every $\gamma \in \hat{G}$.*

(iii) *$T(f * g) = f * Tg$ for all $f, g \in L_1(G)$.*

PROOF. (i) $\Rightarrow$ (iii) is obvious.

(ii) $\Rightarrow$ (i). For $\gamma \in \hat{G}$ we have $\gamma * T\gamma = T\gamma = T(\gamma * \gamma)$. If $\beta, \gamma \in \hat{G}$ are distinct, then $\beta * T\gamma = 0 = T(\beta * \gamma)$. Hence, $f * Tg = T(f * g)$ if $f, g \in L_1(G)$ are trigonometric polynomials. These forming a dense subspace of $L_1(G)$ we find $f * Tg = T(f * g)$ for all $f, g \in L_1(G)$.

(iii) $\Rightarrow$ (ii). (See [9].) Clearly T maps $L_1(G)_{\text{abs}}$ into K_{abs}. But $L_1(G)_{\text{abs}} = L_1(G)$ [4; 32.30], so the range of T lies in K_{abs}. For all $f, g \in L_1(G)$, $f * Tg = T(f * g) = T(g * f) = g * Tf$. If $g_1, g_2 \in L_1(G)$ and $c_1, c_2 \in \mathbf{C}$, then for all $f \in L_1(G)$

$$\begin{aligned} f * [c_1 Tg_1 + c_2 Tg_2] &= c_1 f * Tg_1 + c_2 f * Tg_2 \\ &= c_1 g_1 * Tf + c_2 g_2 * Tf = (c_1 g_1 + c_2 g_2) * Tf \\ &= f * T(c_1 g_1 + c_2 g_2). \end{aligned}$$

As $c_1 Tg_1 + c_2 Tg_2 - T(c_1 g_1 + c_2 g_2) \in K_{\text{abs}}$ and K_{abs} is order-free, it follows that $c_1 Tg_1 + c_2 Tg_2 = T(c_1 g_1 + c_2 g_2)$. Thus, T is linear. The continuity of T is proved with the aid of the Closed Graph Theorem. Let $f_1, f_2, \ldots$ be a sequence in $L_1(G)$ such that $\lim f_n = 0$ while $\lim Tf_n$ exists in K. Then $\lim Tf_n \in K_{\text{abs}}$, and for all $g \in L_1(G)$, $g * \lim Tf_n = \lim g * Tf_n = \lim f_n * Tg = 0$. Hence, $\lim Tf_n = 0$ and T is continuous. Finally, for $\gamma \in \hat{G}$ one has $T\gamma = T(\gamma * \gamma) = \gamma * T\gamma \in K_\gamma$.

The implication (ii) $\Rightarrow$ (i) gives the situation a new perspective. Apparently, a map $\phi\colon \hat{G} \to K$ extends to a multiplier if and only if $\phi \in \prod_\gamma K_\gamma$ and ϕ admits a continuous linear extension $L_1(G) \to K$. The question remains: what $\phi \in \prod_\gamma K_\gamma$ do admit such an extension?

2

For $T \in \text{Mult}\, K$ we denote by $\tilde{T}$ the restriction of T to $\hat{G}$. T is determined by $\tilde{T}$, since the characters of G span a dense linear subspace of $L_1(G)$.

Every $x \in K$ determines a multiplier $T_x\colon f \mapsto f * x$. Instead of $\tilde{T}_x$ we write $\tilde{x}$; thus,

$$\tilde{x}_\gamma = \tilde{x}(\gamma) = \gamma * x \qquad (\gamma \in \hat{G};\, x \in K).$$

If $K = L_1(G)$, then $\tilde{x}_\gamma = \hat{x}(\gamma)\gamma$, so $\tilde{x}$ actually is the Fourier "series" of x. For arbitrary modules K we call $\tilde{x}$ the *character convolution transform* of x.

We know by Wendel's theorem [4; 35.5] that Mult $L_1(G)$ may be identified with $M(G)$. If $T \in \text{Mult}\, L_1(G)$ corresponds to $\mu \in M(G)$, then $\tilde{T}(\gamma) = \hat{\mu}(\gamma)\gamma$. Thus, the map $T \mapsto \tilde{T}$ can be viewed as a generalization of the Fourier-Stieltjes transformation.

We see now how our problems (α) and (β) converge: the character convolution transformation is an answer to (α), and (β) asks for descriptions of character convolution transforms.

A few simple observations:

2.1 LEMMA. *For $x \in K$,*

$$\tilde{x} = 0 \text{ if and only if } L_1(G) * x = \{0\}.$$

In particular, if $x, y \in K_{\text{abs}}$ and $\tilde{x} = \tilde{y}$, then $x = y$.

2.2 LEMMA. *We have the relations*

$$(Tf)^{\sim} = \hat{f}\tilde{T} \qquad (f \in L_1(G);\, T \in \text{Mult}\, K)$$

and

$$(f * x)^{\sim} = \hat{f}\tilde{x} \qquad (f \in L_1(G);\, x \in K).$$

The following extension of the Helson-Edwards Theorem [7; 3.8.1] holds.

2.3 THEOREM. *$\phi \in \prod_\gamma K_\gamma$ can be extended to a multiplier of K if and only if $\hat{f}\phi \in \tilde{K}$ for every $f \in L_1(G)$. (We put $\tilde{K} = \{\tilde{x} \mid x \in K\}$.)*

PROOF. If $T \in \operatorname{Mult} K$ and $\phi = \tilde{T}$, then for every $f \in L_1(G)$ we have $\hat{f}\phi = \hat{f}\tilde{T} = (Tf)^{\tilde{}} \in \tilde{K}$. Conversely, suppose $\phi \in \prod_\gamma K_\gamma$ and $\hat{f}\phi \in \tilde{K}$ for all $f \in L_1(G)$. Every $f \in L_1(G)$ can be written as $f = f_1 * f_2$ with certain $f_1, f_2 \in L_1(G)$; then $\hat{f}\phi = \hat{f}_1(\hat{f}_2\phi) \in \hat{f}_1\tilde{K} = (f_1 * K)^{\tilde{}} \subset (K_{\text{abs}})^{\tilde{}}$. By Lemma 2.1, for every $f \in L_1(G)$ there is a unique $Tf \in K_{\text{abs}}$ such that $\hat{f}\phi = (Tf)^{\tilde{}}$. If $f, g \in L_1(G)$, then $(f * Tg)^{\tilde{}} = \hat{f}(Tg)^{\tilde{}} = \hat{f}\hat{g}\phi = (f * g)^{\hat{}}\phi = (T(f * g))^{\tilde{}}$, so $f * Tg = T(f * g)$. By Lemma 1.3 T is a multiplier of K. Further, $(T\gamma)^{\tilde{}} = \hat{\gamma}\phi = (\phi_\gamma)^{\tilde{}}$, so $T\gamma = \phi_\gamma$ for every $\gamma \in \hat{G}$.

Another characterization displays a certain analogy with the Schoenberg-Ebelein Theorem [4; 33.20], [7; 1.9.1].

2.4 THEOREM. *$\phi \in \prod_\gamma K_\gamma$ can be extended to a multiplier of K if and only if there exists a constant c such that*

$$(*) \qquad \left\| \sum_{i=1}^{n} c_i \phi_{\gamma_i} \right\| \leq c \left\| \sum_{i=1}^{n} c_i \gamma_i \right\|_1$$

for every trigonometric polynomial $\sum c_i \gamma_i$ on G.

PROOF. If $T \in \operatorname{Mult} K$ and $\phi = \tilde{T}$, then for every trigonometric polynomial $\sum c_i \gamma_i$ we have

$$\left\| \sum c_i \phi_{\gamma_i} \right\| = \left\| \sum c_i T(\gamma_i) \right\| = \left\| T\left(\sum c_i \gamma_i \right) \right\| \leq \|T\| \left\| \sum c_i \gamma_i \right\|_1.$$

Conversely, if $\phi \in \prod_\gamma K_\gamma$ and if there exists a constant c such that $(*)$ holds for every trigonometric polynomial, then (as the trigonometric polynomials are dense in $L_1(G)$) we have a continuous linear $T\colon L_1(G) \to K$ such that $T(\sum c_i \gamma_i) = \sum c_i \phi_{\gamma_i}$ for all $\sum c_i \gamma_i$. In particular, $T\gamma = \phi_\gamma$ for $\gamma \in \hat{G}$. Then $T \in \operatorname{Mult} K$ by the implication (ii) → (i) of Lemma 1.3.

Note. A better analogy with the Schoenberg-Eberlein Theorem would be obtained if in $(*)$ we could replace the L_1-norm by the L_∞-norm. This change, however, would make the theorem false, as one sees from the example $K = C(G)$, $\phi_\gamma = \gamma$.

The following theorem, and also 2.9, are inversion theorems, stating that certain elements of a module are the sums of their character convolution transforms, as many functions of $L_1(G)$ are the sums of their Fourier series. F denotes the directed set of all finite subsets of $\hat{G}$.

2.5 THEOREM. *Let $\phi \in \prod_\gamma K_\gamma$ be so that the net $(\sum_{\gamma \in \Delta} \phi_\gamma)_{\Delta \in F}$ is bounded. Then ϕ can be extended to a multiplier T of K. For all $f \in L_1(G)$ we have*

$$Tf = \sum_{\gamma \in \hat{G}} f * \phi_\gamma.$$

PROOF. for $\Delta \in F$ put $\phi_\Delta = \Sigma_{\gamma \in \Delta} \phi_\gamma$. Let $c = \sup_{\Delta \in F} \|\phi_\Delta\|$. If $\Sigma c_i \gamma_i$ is a trigonometric polynomial, then for $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ we have

$$\left\| \sum c_i \phi_{\gamma_i} \right\| = \left\| \sum c_i \gamma_i * \phi_\Delta \right\| \leq c \left\| \sum c_i \gamma_i \right\|_1$$

so ϕ is extendable to a multiplier T. Furthermore, if $\Sigma c_i \gamma_i$ is a trigonometric polynomial, then for $\Delta \supset \{\gamma_1, \ldots, \gamma_n\}$ we have

$$T\left(\sum c_i \gamma_i \right) = \sum c_i \phi_{\gamma_i} = \sum c_i \gamma_i * \phi_\Delta.$$

If $g \in L_1(G)$ and $\varepsilon > 0$, there is a trigonometric polynomial $f \in L_1(G)$ such that $\|f - g\|_1 < \varepsilon$; there is a $\Delta_0 \in F$ such that $Tf = f * \phi_\Delta$ for $\Delta \supset \Delta_0$. Then, for $\Delta \supset \Delta_0$,

$$\|Tg - g * \phi_\Delta\| \leq \|T(g - f)\| + \|(f - g) * \phi_\Delta\| \leq \varepsilon(\|T\| + c)$$

Hence, $Tg = \lim_{\Delta \in F} g * \phi_\Delta = \lim_{\Delta \in F} \Sigma_{\gamma \in \Delta} g * \phi_\gamma$.

The following is another variant of the Schoenberg-Eberlein criterion.

2.6 THEOREM. *The following conditions on* $\phi \in \Pi_\gamma (K^*)_\gamma$ *are equivalent*:

(i) $\phi \in (K^*)^{\sim}$.

(ii) ϕ *can be extended to a multiplier of* K^*.

(iii) *There exists a constant c such that for every positive integer n and for all* $\gamma_1, \ldots, \gamma_n \in \hat{G}$ *and* $x_1, \ldots, x_n \in K$,

$$\left| \sum_{i=1}^{n} \left(x_i, \phi_{\bar{\gamma}_i} \right) \right| \leq c \left\| \sum \gamma_i * x_i \right\|.$$

PROOF. (i) $\Rightarrow$ (ii). If $h \in K^*$ and $\phi = \tilde{h}$, then $f \mapsto f * h$ is a multiplier of K^* that is an extension of ϕ.

(ii) $\Rightarrow$ (iii). Let $\phi = \tilde{T}$, $T \in \text{Mult } K^*$. We identify $L_1(G)^*$ with $L_\infty(G)$. It is not difficult to verify that the module operation on $L_1(G)^*$ corresponds to the module operation on $L_\infty(G)$. In particular, for $f \in L_1(G)$ and $h \in L_\infty(G)$,

$$(f, h) = (f^* * h)(e),$$

e denoting the unit element of G. Now T induces a continuous linear S: $K \to L_\infty(G)$ by

$$(f, Sx) = (x, Tf) \qquad (f \in L_1(G);\, x \in K).$$

For $f, g \in L_1(G)$ and $x \in K$.

$$\begin{aligned}(f, g * Sx) &= (g^* * f, Sx) = (x, T(g^* * f)) \\ &= (x, g^* * Tf) = (g * x, Tf) = (f, S(g * x))\end{aligned}$$

so that $g * Sx = S(g * x)$. Now take $\gamma_1, \ldots, \gamma_n \in \hat{G}$ and $x_1, \ldots, x_n \in K$.

$$\begin{aligned}\left|\sum(x_i,\phi_{\bar{\gamma}_i})\right| &= \left|\sum(x_i, T\bar{\gamma}_i)\right| = \left|\sum(\bar{\gamma}_i, Sx_i)\right|\\ &= \left|\sum(\gamma_i * Sx_i)(e)\right| \leqslant \left\|\sum\gamma_i * Sx_i\right\|\\ &= \left\|S\left(\sum\gamma_i * x_i\right)\right\| \leqslant \|S\|\,\left\|\sum\gamma_i * x_i\right\|.\end{aligned}$$

(iii) $\Rightarrow$ (i). By the Hahn-Banach Theorem there exists an $h \in K^*$ such that

$$\left(\sum\gamma_i * x_i, h\right) = \sum(x_i, \phi_{\bar{\gamma}_i})$$

for all γ_i and x_i. In particular, for every $\gamma \in \hat{G}$ and $x \in K$, $(x, \bar{\gamma} * h) = (\gamma * x, h) = (x, \phi_{\bar{\gamma}})$. Hence $\bar{\gamma} * h = \phi_{\bar{\gamma}}$ for all γ, and $\tilde{h} = \phi$.

2.7 Corollary. *For every* $T \in \operatorname{Mult} K^*$ *there exists an* $h \in K^*$ *such that* $Tf = f * h$ $(f \in L_1(G))$.

Proof. For every T there is an h for which $\tilde{T} = \tilde{h}$. Then $Tf = f * h$ if f is any trigonometric polynomial; hence, if $f \in L_1(G)$.

For absolutely continuous K this result was proved in [3; 5.2].

For Hilbert spaces we obtain from 2.7 and 1.2:

2.8 Corollary. *If K is a Hilbert space, then $\phi \in \prod_\gamma K_\gamma$ can be extended to a multiplier of K if and only if $\sum\|\phi_\gamma\|^2 < \infty$.*

2.9 Theorem. *Let K be absolutely continuous. Let $\phi \in \prod_\gamma (K^*)_\gamma$ be so that the net $(\sum_{\gamma\in\Delta}\phi_\gamma)_{\Delta\in F}$ is bounded. Then this net is w^*-convergent. If h is its w^*-limit then $\phi = \tilde{h}$ and*

$$(x, h) = \sum_{\gamma\in\hat{G}} \left(\tilde{x}_\gamma, \tilde{h}_{\bar{\gamma}}\right) \qquad (x \in K).$$

Note. Apparently, here we have analogs of the inversion formula and the Parseval relation from the theory of Fourier transformation.

Proof. ϕ can be extended to a multiplier T of K^*, and $Tf = \sum f * \phi_\gamma$ for all $f \in L_1(G)$. By 2.7 there is an $h \in K^*$ such that $Tf = f * h$ for all f. Now every $x \in K$ can be written as $f * y$ for certain $f \in L_1(G)$ and $y \in K$. Then

$$\begin{aligned}(x, h) &= (y, f^* * h) = (y, Tf^*) = \left(y, \sum f^* * \phi_\gamma\right)\\ &= \sum\left(y, f^* * \phi_\gamma\right) = \sum\left(x, \phi_\gamma\right).\end{aligned}$$

Hence,

$$h = w^*\text{-}\lim_{\Delta \in F} \sum_{\gamma \in \Delta} \phi_\gamma.$$

For $\gamma \in \hat{G}$, $\phi_\gamma = T\gamma = \gamma * h$; so $\phi = \tilde{h}$. Further, for $x \in K$,

$$\begin{aligned}(x, h) &= \sum (x, \phi_{\bar{\gamma}}) = \sum (x, \bar{\gamma} * \phi_{\bar{\gamma}}) \\ &= \sum (\gamma * x, \phi_{\bar{\gamma}}) = \sum (\tilde{x}_\gamma, \tilde{h}_{\bar{\gamma}}).\end{aligned}$$

3

A linear module homomorphism is simply called a *homomorphism*.

In this section G is a compact abelian group of homeomorphisms of a locally compact Hausdorff space X, such that the mapping $(s, x) \mapsto sx$ $(s \in G; x \in X)$ is jointly continuous. We denote by $C(G)$, $C_0(X)$, $C_{00}(X)$ the spaces of all continuous functions on G, all continuous fuctions on X vanishing at infinity, and all continuous functions on X with compact supports, respectively. The formula

$$(f * k)(x) = \int f(s) k(s^{-1}x)\, ds \qquad (k \in C_0(X); x \in X)$$

turns $C_0(X)$ into an absolutely continuous Banach $L_1(G)$-module. (For details, see [6].) We identify $C_0(X)^*$ with the Banach space $M(X)$ of all bounded Radon measures on X, writing (k, μ) instead of $\int k\, d\mu$ $(k \in C_0(X), \mu \in M(X))$. The induced module composition on $M(X)$ is given by

$$(f * \mu)(Y) = \int f(s) \mu(s^{-1}Y)\, ds$$

where $f \in L_1(G)$, $\mu \in M(X)$, $Y \subset X$, Y a Borel set.

3.1 THEOREM. *Let T: $C(G) \to M(X)$ be a homomorphism. Assume that $Tf \geqslant 0$ whenever $f \in C(G)$ and $f \geqslant 0$. (Such a homomorphism T is called positive.) Then there exists a $\mu \in M(X)$, $\mu \geqslant 0$ such that*

$$Tf = f * \mu \qquad (f \in C(G)).$$

Thus, T can be extended to an element of Mult $M(X)$.

PROOF. If $\nu \in M(X)$, $\nu \geqslant 0$, then

$$\|\nu\| = \nu(X) = \int \nu(s^{-1}X)\, ds = (1 * \nu)(X) = \|1 * \nu\|.$$

Thus, if $f \in C(X)$, $f \geqslant 0$, then $\|Tf\| = \|1 * Tf\| = \|T(f) * 1\| = \|f * T1\| \leqslant \|f\|_1 \|T1\|$. For an arbitrary $f \in C(G)$ we can write $f = f_1 - f_2 + if_3 - if_4$ where

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$f_j \in C(G)$ and $0 \leqslant f_j \leqslant |f|$ for each j. It follows that $\|Tf\| \leqslant 4\|f\|_1 \|T1\|$. Therefore T has a unique continuous linear extension T_1: $L_1(G) \to M(X)$. By continuity, $T_1 \in \text{Mult } M(X)$. According to Corollary 2.7 there exists a $\mu \in M(X)$ such that $T_1 f = f * \mu$ ($f \in L^1(G)$). To prove $\mu \geqslant 0$ take $j \in C_0(X)$, $j \geqslant 0$ and let $\{u_\iota\}$ be an approximate identity of $L^1(G)$ such that $u_\iota \in C(G)$ and $u_\iota \geqslant 0$ for each ι. By the absolute continuity of $C_0(X)$, j can be written as $f * j'$ where $f \in L_1(G)$, $j' \in C_0(X)$. Then $(j, \mu) = (f * j', \mu) = \lim(u_\iota * f * j', \mu) = \lim(f * j', u_\iota^* * \mu) = \lim(j, T(u_\iota^*)) \geqslant 0$. Thus $\mu \geqslant 0$.

For multipliers of $M(X)$ we can extend Bochner's Theorem [4; 33.3], [7; 1.4.3]. A function ϕ: $\hat{G} \to M(X)$ is said to be *positive definite* if

$$\sum_{i,j=1}^{n} c_i \bar{c}_j \phi\left(\gamma_i \gamma_j^{-1}\right) \geqslant 0$$

for all positive integers n, all complex numbers $c_1, \ldots, c_n$ and $\gamma_1, \ldots, \gamma_n \in \hat{G}$.

3.2 THEOREM. *Let $\phi \in \prod_\gamma M(X)_\gamma$. Then ϕ is positive definite if and only if there exists $\mu \in M(X)$, $\mu \geqslant 0$ such that $\phi = \tilde{\mu}$.*

PROOF. Let $\mu \in M(X)$, $\mu \geqslant 0$; $c_1, \ldots, c_n \in \mathbf{C}$; $\gamma_1, \ldots, \gamma_n \in \hat{G}$. Take $k \in C_0(X)$, $k \geqslant 0$. For every $x \in X$,

$$0 \leqslant \int \left| \sum_i c_i \bar{\gamma}_i(s) \right|^2 k(s^{-1}x)\, dx = \sum_{i,j} c_i \bar{c}_j \int \overline{\gamma_i(s)}\, \gamma_j(s) k(s^{-1}x)\, ds$$
$$= \sum_{i,j} c_i \bar{c}_j \left(\bar{\gamma}_i \gamma_j * k\right)(x).$$

Hence

$$0 \leqslant \sum_{i,j} c_i \bar{c}_j \left(\bar{\gamma}_i \gamma_j * k, \mu\right) = \sum_{i,j} c_i \bar{c}_j \left(k, \left(\bar{\gamma}_i \gamma_j\right)^* * \mu\right)$$
$$= \left(k, \sum_{i,j} c_i \bar{c}_j \tilde{\mu}\left(\gamma_i \gamma_j^{-1}\right)\right)$$

and $\sum_{i,j} c_i \bar{c}_j \tilde{\mu}(\gamma_i \gamma_j^{-1}) \geqslant 0$.

Conversely assume ϕ to be positive definite. For every $k \in C_0(X)$, $k \geqslant 0$, the scalar valued function $\gamma \mapsto (k, \phi(\gamma^*))$ is positive definite. By Bochner's Theorem [4; 33.3] for such k there exists a unique $\mu_k \in M(G)$ such that $(k, \phi(\gamma^*)) = \hat{\mu}_k(\gamma)$, ($\gamma \in \hat{G}$), and we have $\mu_k \geqslant 0$. The map $k \mapsto \mu_k$ can be extended to a linear positive, hence continuous, U: $C_0(X) \to M(G)$. Then

$$(k, \phi(\gamma^*)) = (Uk)\hat{}\,(\gamma) \qquad \left(k \in C_0(X), \gamma \in \hat{G}\right).$$

It is easy to see that $(U(f*k))\hat{} = \hat{f}(Uk)\hat{} = (f*Uk)\hat{}$ for all $f \in L^1(G)$, $k \in C_0(X)$. Thus U is a homomorphism. U in turn induces a positive homomorphism T: $C(G) \to M(X)$ by

$$(k, Tf) = (f, Uk) \qquad (f \in C(G), k \in C_0(X)).$$

Applying Theorem (3.1) we obtain a $\mu \in M(X)$, $\mu \geqslant 0$ such that $Tf = f * \mu$ for all $f \in C(G)$. In particular, $(k, \gamma * \mu) = (k, T\gamma) = (\gamma, Uk) = (Uk)\hat{}(\gamma^*) = (k, \phi(\gamma))$ for all $k \in C_0(X)$ and $\gamma \in \hat{G}$. It follows that $\phi = \tilde{\mu}$.

We specialize further and assume the existence of a positive Radon measure m on X that is invariant under the action of G. Then every $L_p(m)$ $(1 \leqslant p \leqslant \infty)$ can be made into a group algebra module by

$$(f*k)(x) = \int f(s)k(s^{-1}x)\,dx \qquad \big(f \in L_1(G), k \in L_p(m)\big)$$

for locally almost all $x \in X$ (see [1]). For $p < \infty$, $L_p(m)$ is absolutely continuous. The natural linear maps $L_1(m) \to M(X)$, $L_p(m) \to L_q(m)^*$ $(p^{-1} + q^{-1} = 1)$ are isometric homomorphisms. We identify $L_p(m)$ and $L_q(m)^*$ $(p > 1, p^{-1} + q^{-1} = 1)$.

R. Ryan [8] characterizes those Fourier-Stieltjes transforms of measures on G that actually are Fourier transforms of elements of $L_1(G) \cap L_p(G)$. His theorem can be extended in the following way.

3.3 Theorem. *Let* $1 < p \leqslant \infty$, $p^{-1} + q^{-1} = 1$. *Let* $E = \{k \in C_{00}(X) \mid \tilde{k}_\gamma \neq 0$ *for only finitely many* $\gamma \in \hat{G}\}$. *Let* $\mu \in M(X)$ *and assume that there exists a number c such that*

$$\Big|\sum_{\gamma \in \hat{G}} \big(\tilde{k}_\gamma, \tilde{\mu}_{\bar{\gamma}}\big)\Big| \leqslant c\|k\|_q$$

for all $k \in E$. *Then there exists a* $g \in L_1(m) \cap L_p(m)$ *such that* $\mu = gm$ (*that is,* $\mu(A) = \int_A g\,dm$ *for all Borel sets* $A \subset X$).

Proof. If $k \in C_{00}(X)$ and $\beta \in \hat{G}$, then

$$(\beta * k, \mu) = \sum_{\gamma \in \hat{G}} (\beta * k, \bar{\gamma} * \mu) = \sum_{\gamma \in \hat{G}} (\beta * k, \tilde{\mu}_{\bar{\gamma}}).$$

The elements of E are finite sums $\Sigma \beta_i * k_i$. Hence

$$(k, \mu) = \sum_{\gamma \in \hat{G}} (k, \tilde{\mu}_{\bar{\gamma}}) \qquad (k \in E).$$

By the isomorphism between $L_q(m)^*$ and $L_p(m)$ there is a $g \in L_p(m)$ such that

$$(k, \mu) = (k, g) \qquad (k \in E).$$

If we can prove that $g \in L_1(m)$, then μ and gm are bounded regular measures and $(k, \mu) = (k, gm)$ for all k in a dense subspace of $C_0(X)$; then $(k, \mu) = (k, gm)$ for all $k \in C_0(X)$ and $\mu = gm$.

Take $k \in C_{00}(X)$; let S be the support of k. For $A \subset X$ let ξ_A be the characteristic function of A. For every positive integer n let f_n be a trigonometric polynomial on G such that $\|f_n\|_1 \leqslant 1$ and $\|f_n * k - k\|_1 \leqslant 2^{-n}$. Then $\lim f_n * k = k$, a.e. and $f_n * k \in E$. Further, $\|f_n * k\|_\infty \leqslant \|f_n\|_1 \|k\|_\infty \leqslant \|k\|_\infty$, and $f_n * k = 0$ outside the compact set GS. Thus, $\lim (f_n * k)g = kg$ a.e., and $|(f_n * k)g| \leqslant \|k\|_\infty |g|\xi_{GS}$. As $g\xi_{GS} \in L_1(m)$ it follows by the Lebesgue Dominated Convergence Theorem that

$$\left|\int kg\, dm\right| = \lim\left|\int (f_n * k)g\, dm\right|$$
$$= \lim |(f_n * k, \mu)| \leqslant \sup_n \|f_n * k\|_\infty \|\mu\|.$$

Thus,

$$\left|\int kg\, dm\right| \leqslant \|\mu\| \|k\|_\infty \qquad (k \in C_{00}(X)).$$

Now let $C \subset X$ be compact. Let U be an open set containing C and of finite m-measure. Let h be a measurable function, $|h(x)| \leqslant 1$ for all x, such that $hg = |g|\xi_C$ and $h = 0$ off C. For each positive integer n choose $k_n \in C_{00}(X)$, $\|k_n - h\|_1 \leqslant 2^{-n}$, $\|k_n\|_\infty \leqslant 1$, $k_n = 0$ outside U. By another application of the Lebesgue theorem (note that $g\xi_U \in L_1(m)$) we get

$$\int_C |g|\, dm = \int |g|\xi_C\, dm = \int hg\, dm$$
$$= \lim_n \int k_n g\, dm \leqslant \|\mu\| \sup_n \|k_n\| = \|\mu\|.$$

As this is true for all compact C, it follows that $g \in L_1(m)$.

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