A CHARACTERIZATION OF MACHINE MAPPINGS

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Introduction. A *generalized sequential machine* (abbreviated gsm) is a 6-tuple $(K, \Sigma, \Delta, \delta, \lambda, p_1)$, where K, Σ, Δ are finite non-empty sets (of "states," "inputs," and "outputs" respectively), δ (the "next state" function) is a mapping of $K \times \Sigma$ into K, λ (the "output" function) is a mapping of $K \times \Sigma$ into Δ^*, and p_1 (the "start" state) is a distinguished element of K. (For sets of words X and Y,

$$XY = \{xy/x \in X, y \in Y\} \quad \text{and} \quad X^* = \bigcup_{i=0}^{\infty} X^i,$$

where $X^0 = \{\epsilon\}$, ϵ being the empty word. Thus, for an arbitrary set E of symbols, E^* is the free semi-group generated by E.) The functions δ and λ are extended to $K \times \Sigma^*$ by defining

$$\delta(p, \epsilon) = p, \qquad \delta(p, x_1 \ldots x_k) = \delta[\delta(p, x_1 \ldots x_{k-1}), x_k], \qquad \lambda(p, \epsilon) = \epsilon,$$

and

$$\lambda(p, x_1 \ldots x_k) = \lambda(p, x_1 \ldots x_{k-1})\lambda[\delta(p, x_1 \ldots x_{k-1}), x_k]$$

for each p in K and each sequence $x_1, \ldots, x_k$ of elements of Σ. A function f of Σ^* into Δ^* is said to be a *machine mapping*, or *realized* by a gsm, if $f(x) = \lambda(p_1, x)$ for all x in Σ^* for some gsm S. The machine mapping of a gsm S is also denoted by S.

The purpose of this note is to show that a function f of Σ^* into Δ^* is realized by a gsm if and only if it satisfies each of the following conditions:

(i) $f(\epsilon) = \epsilon$.

(ii) f preserves initial subwords, i.e., if u is an initial subword of v, then $f(u)$ is an initial subword of $f(v)$.

(iii) f has bounded output, i.e., there is a number M such that

$$|f(ua)| - |f(u)| \leqslant M$$

for all u in Σ^* and a in Σ (if u is a word, then $|u|$ denotes its length).

(iv) f^{-1} preserves regular sets, i.e., for each regular set $Y \subseteq \Delta^*$, $f^{-1}(Y) = \{x/f(x) \in Y\}$ is regular. (An *automaton* is a 5-tuple $A = (K, \Sigma, \delta, p_1, F)$, where K and Σ are finite non-empty sets, δ is a mapping of $K \times \Sigma$ into K, p_1 is a distinguished element of K, and F is a (possibly empty) subset of K. δ is extended to $K \times \Sigma^*$ as in a gsm. A set U is *regular* if there exists an automaton A such that $U = \{w \in \Sigma^* \mid \delta(p_1, w) \in F\}$.)

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Since conditions (i)–(iv) imply that f is realized by a gsm, these conditions imply **(5)**:

(v) f preserves regular sets.

There are three sections. A proof of the main result constitutes §1. Counter-examples that show the independence of conditions (i)–(iv) even in the presence of (v), are furnished in §2. Two corollaries are given in §3.

1. Proof. If f is realized by a gsm, then conditions (i), (ii), and (iii) are obvious, while (iv) is proved in **(5)**.

Suppose that f satisfies (i)–(iv). By (iii) there exists an integer M such that

$$|f(wa)| - |f(w)| \leqslant M \tag{1}$$

for each w in Σ^* and a in Σ. Let

$$U(a, w) = \{u/f(ua) = f(u)w\}$$

for each w in $\bigcup_0{}^M \Delta^j$ and each a in Σ. We first show that

$$U(a, w) \text{ is a regular set.} \tag{2}$$

To see this, let $g_a(X) = \{u/ua \in X\}$ for each a in Σ and $X \subseteq \Sigma^*$. By **(5**, Theorem 2.2**)** $g_a(X)$ is regular if X is regular. Let $\Delta_t = (\Delta^{M+1})^*\Delta^t$ for each $0 \leqslant t \leqslant M$ and

$$V(a, w) = \bigcup_0^M g_a[f^{-1}(\Delta_t w)] \cap f^{-1}(\Delta_t)$$

for each $a \in \Sigma$ and

$$w \in \bigcup_{j=0}^M \Delta^j.$$

Since $\Delta_t w$ is regular, $f^{-1}(\Delta_t w)$ is regular by (iv). (The family of regular sets is the smallest family of sets containing the finite sets and closed under union, product, and *. Furthermore, the regular sets are closed under subtraction and intersection **(3)**.) Thus $g_a[f^{-1}(\Delta_t w)]$ is regular. Since Δ_t is regular, $f^{-1}(\Delta_t)$ is regular. Therefore $V(a, w)$ is regular. Consider any word $u \in U(a, w)$. Since

$$\bigcup_{t=0}^M \Delta_t = \Delta^*,$$

there exists an integer t such that $f(u) \in \Delta_t$, thus $u \in f^{-1}(\Delta_t)$. Since

$$f(ua) = f(u)w \in \Delta_t w, \qquad ua \in f^{-1}(\Delta_t w)$$

and $u \in g_a[f^{-1}(\Delta_t w)]$. Thus $U(a, w) \subseteq V(a, w)$. Now consider any word $u \in V(a, w)$. Then for some t, u is in both $g_a[f^{-1}(\Delta_t w)]$ and $f^{-1}(\Delta_t)$. Thus $ua \in f^{-1}(\Delta_t w)$, i.e., $f(ua) \in \Delta_t w$, and $f(u) \in \Delta_t$. Then for some integers h and i,

$$|f(ua)| = h(M + 1) + t + |w| \tag{3}$$

and

$$|f(u)| = i(M + 1) + t. \tag{4}$$

By (ii) and (1), $f(ua) = f(u)v$ for some word v, $|v| \leqslant M$. Hence

$$(5) \qquad |f(ua)| = i(M+1) + t + |v|.$$

Since $|v|$, $|w| \leqslant M$, it follows from (3) and (5) that $|w| = |v|$. Therefore $w = v$, so that $f(ua) = f(u)w$. Thus $u \in U(a, w)$. Then $U(a, w) = V(a, w)$ and (2) holds.

Consider the finite set of all homomorphisms τ of Σ^* into Δ^* such that

$$\tau(a) \in \bigcup_0^M \Delta^j$$

for each a in Σ. For each such τ, let

$$U(\tau) = \{u \in \Sigma^* | f(ua) = f(u)\tau(a) \text{ for all } a \in \Sigma\}.$$

Since

$$U(\tau) = \bigcap_{a \in \Sigma} U(a, \tau(a)),$$

$U(\tau)$ is regular by (2). The sets $U(\tau)$ are also pairwise disjoint and have Σ^* as union. For each τ, let A_τ be an automaton $(K_\tau, \Sigma, \delta_\tau, p_\tau, F_\tau)$ such that $U(\tau) = \{u \mid \delta_\tau(p_\tau, u) \in F_\tau\}$. Let $\tau_1, \ldots, \tau_r$ be the distinct mappings τ. Consider the structure $A = (K, \Sigma, \delta_A, p_A, H_1, \ldots, H_r)$, where

$$p_A = (p_{\tau_1}, \ldots, p_{\tau_r}), \qquad K = K_{\tau_1} \times \ldots \times K_{\tau_r},$$

$$\delta_A((p_1, \ldots, p_r), a) = (\delta_{\tau_1}(p_1, a), \ldots, \delta_{\tau_r}(p_r, a))$$

for each $(p_1, \ldots, p_r) \in K$ and $a \in \Sigma$; and for each i,

$$H_{ij} = K_{\tau_j} - F_{\tau_j} \quad \text{or} \quad H_{ij} = F_{\tau_j}$$

according as $j \neq i$ or $j = i$, and $H_i = H_{i1} \times \ldots \times H_{ir}$. Then $\{H_i/1 \leqslant i \leqslant r\}$ is a family of disjoint subsets of K (**3**, p. 109) and a word x is in $U(\tau_i)$ if and only if $\delta_A(p_A, x) \in H_i$. Finally, let S be the gsm $(K, \Sigma, \Delta, \delta_A, \lambda, p_A)$, where for each $p \in K$ and $a \in \Sigma$, $\lambda(p, a) = \tau_i(a)$ if H_i contains p and $\lambda(p, a) = \epsilon$ if

$$p \in K - \bigcup_1^r H_i.$$

To complete the proof, it suffices to show that $\lambda(p_A, x) = f(x)$ for all words x. Now $\lambda(p_A, \epsilon) = f(\epsilon) = \epsilon$. Suppose $\lambda(p_A, x) = f(x)$ for all words x, $|x| \leqslant s$. For $a \in \Sigma$, consider $\lambda(p_A, xa)$. Then

$$\lambda(p_A, xa) = \lambda(p_A, x)\lambda[\delta_A(p_A, x), a] = f(x)\lambda[\delta_A(p_A, x), a].$$

Now $x \in U(\tau_i)$ for some i. Then $\delta_A(p_A, x) \in H_i$. By the definition of λ, $\tau_i(a) = \lambda[\delta_A(p_A, x), a]$. Since $x \in U(\tau_i)$,

$$f(xa) = f(x)\tau_i(a) = f(x)\lambda[\delta_A(p_A, x), a] = \lambda(p_A, xa),$$

completing the proof.

Remark. The proof shows that condition (iv) may be weakened to require only that $f^{-1}(Y)$ be regular for all sets $Y = (\Delta^{M+1})\Delta^t w$, where $0 \leqslant t \leqslant M$ and

$$w \in \bigcup_{j=0}^{M} \Delta^j.$$

2. Counter-examples. We now show that none of the conditions (i)–(iv) can be relaxed, even in the presence of (v). Although we do not prove it, both f and f^{-1} preserve context-free languages for each of the functions f presented.

Example 1. Let $\Sigma = \Delta = \{a, b\}$. For each word $w \in \Sigma^*$ let $f(w) = aw$. Then f clearly satisfies (ii)–(v) but not (i).

Example 2. Let $\Sigma = \Delta = \{a, b\}$. Let $f(a) = b$, $f(b) = a$, and $f(x) = x$ otherwise. Obviously f satisfies (i), (iii)–(v), but not (ii).

Example 3. Let $\Sigma = \{a, b\}$ and $\Delta = \{a\}$. Let $f(a^i) = a^i$ and $f(a^i bu) = a^{2i}$ for $i \geqslant 0$ and $u \in \Sigma^*$. Obviously f satisfies (i) and (ii), but not (iii). Consider (iv). Let S be the gsm $(\{p_1, p_2\}, \{a\}, \{a\}, \delta_S, \lambda_S, p_1)$ where

$$\delta_S(p_1, a) = p_2, \qquad \delta_S(p_2, a) = p_1, \qquad \lambda_S(p_1, a) = \epsilon, \qquad \text{and } \lambda_S(p_2, a) = a.$$

For each $Y \subseteq a^*$,

$$f^{-1}(Y) = Y \cup [S(Y \cap (a^2)^*)]b\Sigma^*.$$

Thus f^{-1} preserves regular sets. Consider (v). Let T be the gsm

$$(\{q_1, q_2\}, \Sigma, \Delta, \delta_T, \lambda_T, q_1),$$

where

$$\delta_T(q_1, a) = q_1, \qquad \delta_T = q_2 \text{ otherwise,}$$

$$\lambda_T(q_1, a) = a^2, \qquad \text{and } \lambda_T = \epsilon \text{ otherwise.}$$

For each $X \subseteq \Sigma^*$,

$$f(X) = (X \cap a^*) \cup T(X \cap a^* b\Sigma^*).$$

Thus f preserves regular sets.

Example 4. Let $\Sigma = \Delta = \{a, b\}$. Let $f(a^i bvw) = a^{2i+1}b^{|w|}$ if $|v| = i \geqslant 0$ and $f(x) = a^{|x|}$ otherwise. Clearly f is a length-preserving function which satisfies (i), (ii), and (iii). To see that f does not satisfy (iv), in view of the main theorem it suffices to show that f is not realized by a gsm.

Suppose that there exists a gsm $S = (K, \Sigma, \Delta, \delta, \lambda, p_1)$ such that $f(x) = S(x)$ for all x in Σ^*. Let $p_1, \ldots, p_t$ be the distinct elements of K. Then there exist integers i and j, $1 \leqslant i < i + j \leqslant t + 1$, such that $\delta(p_1, a^i) = \delta(p_1, a^{i+j})$. Now

$$\begin{aligned} a^{2t+3} &= f(a^{t+1}b^{t+2}) = \lambda(p_1, a^{t+1}b^{t+2}) \\ &= \lambda(p_1, a^{i+j})\lambda[\delta(p_1, a^{i+j}), a^{t+1-i-j}b^{t+2})] \\ &= a^{i+j}\lambda[\delta(p_1, a^{i+j}), a^{t+1-i-j}b^{t+2}] \end{aligned}$$

so that

$$\lambda[\delta(p_1, a^{i+j}), a^{t+1-i-j}b^{t+2}] = a^{2t+3-i-j}.$$

Then

$$\lambda(p_1, a^{t+1-j}b^{t+2}) = \lambda(p_1, a^i)\lambda[\delta(p_1, a^i), a^{t+1-i-j}b^{t+2}]$$
$$= a^i\lambda[\delta(p_1, a^{i+j}), a^{t+1-i-j}b^{t+2}] = a^i a^{2t+3-i-j} = a^{2t+3-j} \neq f(a^{t+1-j}b^{t+2}),$$

a contradiction.

Consider (v). Let g be the function of Δ^* into Δ^* defined by $g(a^i) = a^i$ and $g(a^ibw) = a^ib^{|w|+1}$, $i \geqslant 0$. Then g is realized by the gsm

$$U = (\{r_1, r_2\}, \{a, b\}, \{a, b\}, \delta_u, \lambda_u, r_1),$$

where $\delta_u(r_1, a) = r_1$, $\delta_T = r_2$ otherwise, $\lambda_u(p_1, a) = a$, and $\lambda_u = b$ otherwise. Now $f(x) = fg(x)$ for all x in Σ^*. For if $x = a^i$, then $f(a^i) = fg(a^i)$. If $x = a^ibw$, with $|w| \leqslant i$, then

$$f(x) = a^{|x|} = f(a^ibb^{|w|}) = fg(x).$$

If $x = a^ibw$, with $|w| > i$, then

$$f(x) = a^{2i+1}b^{|w|-i} = f(a^ibb^{|w|}) = fg(x).$$

To see that f preserves regular sets, let R be an arbitrary regular set. Since g is realized by a gsm, g preserves regular sets. Thus $g(R) \subseteq a^*b^*$ is regular. Now $\{a^ib^j/j \leqslant i+1\}$ and $\{a^ib^j/j > i+1\}$ are context-free languages. (A *grammar* is a 4-tuple (V, Σ, P, σ), where V is a finite set, Σ is a subset of V, σ is an element of $V - \Sigma$, and P is a finite set of productions $\xi \rightarrow w$ with $\xi \in V - \Sigma$ and $w \in V^*$. For $x, y \in V^*$, write $x \overset{*}{\Rightarrow} y$ if either $x = y$ or there exists an integer k and words

$$x = x_1, \ldots, x_k = y, u_1, \ldots, u_{k-1}, v_1, \ldots, v_{k-1}, y_1, \ldots, y_{k-1}, z_1, \ldots, z_{k-1}$$

such that $x_i = u_i y_i v_i$, $x_{i+1} = u_i z_i v_i$, and $y_i \rightarrow z_i$ is in P for $1 \leqslant i \leqslant k-1$. A subset L of Σ^* is said to be a *context-free language* if there exists a grammar (V, Σ, P, σ) such that $L = \{w \in \Sigma^*/\sigma \overset{*}{\Rightarrow} w\}$.) Since the intersection of a context-free language and a regular set is a context-free language **(1)**,

$$R_1 = g(R) \cap \{a^ib^j/j \leqslant i+1\} \quad \text{and} \quad R_2 = g(R) \cap \{a^ib^j/j > i+1\}$$

are context-free languages. Let τ be the homomorphism of Σ^* into a^* defined by $\tau(a) = \tau(b) = a$. As a homomorphism, τ preserves regular sets and context-free languages **(1)**. Thus $f(R_1) = \tau(R_1) \subseteq a^*$ is a context-free language. By **(4**, Theorem 4. Corollary **2)**, each context-free language in a^* is regular. Thus $f(R_1)$ is regular. Since $g(R) = R_1 \cup R_2$ and

$$f(R) = fg(R) = f(R_1 \cup R_2) = f(R_1) \cup f(R_2),$$

it suffices to show that $f(R_2)$ is regular. To do this, we need

LEMMA. *For each regular set B, there exist regular sets $U_1, \ldots, U_r, V_1, \ldots, V_r$ with the following properties:*

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(1) $\bigcup_1^r U_i V_i = B$.

(2) *For all words u, v such that $uv \in B$, there exists an integer k such that $u \in U_k$ and $V \in V_k$.*

Proof. Let $A = (K, \Sigma, \delta, p_1, F)$ be an automaton such that

$$B = \{w \mid \delta(w, p_1) \in F\}.$$

Let $p_1, \ldots, p_r$ be the elements of K. For each i let

$$U_i = \{u \in \Sigma^* \mid \delta(p_1, u) = p_i\}$$

and

$$V_i = \{u \in \Sigma^* \mid \delta(p_i, u) \in F\}.$$

Obviously $U_1, \ldots, U_r, V_1, \ldots, V_r$ satisfy the conclusion of the lemma.

Now consider $f(R_2)$. Since $g(R)$ is regular, by the lemma there exist regular sets $U_1, \ldots, U_r, V_1, \ldots, V_r$ such that

$$\text{(a)} \bigcup_{i=1}^{r} U_i V_i = g(R)$$

and (b) for all u, v such that $uv \in g(R)$, there exists an integer i such that $u \in U_i$ and $v \in V_i$. For each k let

$$U'_k b = U_k \cap \{a^i b^{i+2} / i \geqslant 0\}.$$

Clearly

$$R_2 = \bigcup_{k=1}^{r} U'_k b V_k.$$

Since U_k is regular, $U'_k b$ is a context-free language **(1)**; thus U'_k is a context-free language **(5)**. For each k, $1 \leqslant k \leqslant r, f(U'_k b V_k) = \tau(U'_k)\mu(b V_k)$, where μ is the homomorphism defined by $\mu(a) = \mu(b) = b$. Then $\tau(U'_k) \subseteq a^*$ is a context-free language and thus regular **(4)**. Since $\mu(b V_k)$ is regular, $f(U'_k b V_k)$, hence $f(R_2) = \bigcup_1^r f(U'_k b V_k)$ is regular.

3. Corollaries. If f is a length-preserving function, then we obtain the following result, first proved (unpublished) by J. Rhodes and E. Shamir:

COROLLARY 1. *Let f be a length and initial subword preserving function of Σ^* into Δ^* such that f^{-1} preserves regular sets. Then f is realized by a complete sequential machine, i.e. by a* gsm *in which λ maps $K \times \Sigma$ into Δ.*

Remarks. (1) Other characterizations of functions that are realized by complete sequential machines are known; cf. **(2; 7)**.

(2) We know of no way to use Corollary 1 to prove the main result.

The question arises as to what conditions on a partial function f allow f to be extended to a function that is realized by a gsm. One set of conditions is now given. (A set of conditions for f to be extended to a complete sequential machine is given in **(6)**.)

COROLLARY 2. *Let $X \subseteq \Sigma^*$ and f be a mapping from X into Δ^* satisfying the following conditions:*

(i) *If $\epsilon \in X$, then $f(\epsilon) = \epsilon$.*

(ii) *If $u \leqslant v$, u and v in X, then $f(u) \leqslant f(v)$.* (The relation $\leqslant$ on Σ^* is the partial order defined by $u \leqslant v$ if and only if u is an initial subword of v.)

(iii) *There exists an integer M with the property that if u and v are in X, $u < v$, and $u < x < v$ for no $x \in X$, then $|f(v)| - |f(u)| \leqslant M$.*

(iv) *f^{-1} preserves regular sets.*

Then f can be extended to a function g over Σ^* that is realized by a gsm.

Proof. Let g be defined as follows. For $x \in X$ let $g(x) = f(x)$. If $u \geqslant x$ for no element $x \in X$, let $g(u) = \epsilon$. If $x \in X$ and $u > x$, with no element $y \in X$ such that $x < y \leqslant u$, let $g(u) = g(x)$. Clearly g is an extension of f over Σ^* which satisfies conditions (i), (ii), and (iii) of the Introduction. To see that g is realized by a gsm, it suffices to show that g^{-1} preserves regular sets.

Let $Z \subseteq \Delta^*$ be a regular set. By hypothesis, $f^{-1}(Z)$ and $X = f^{-1}(\Delta^*)$ are regular. Then $H_1 = f^{-1}(Z)$, $H_2 = X - f^{-1}(Z)$, and $H_3 = \Sigma^* - X$ are disjoint regular sets whose union is Σ^*. Thus there exists a structure

$$(K, \Sigma, \delta, p_1, F_1, F_2, F_3),$$

the F_i being disjoint with union K, such that

$$H_i = \{w \in \Sigma^* \mid \delta(p_1, w) \in F_i\} \qquad \text{for } i = 1, 2, 3 \text{ } \mathbf{(3)}.$$

For each $p \in K$ let p' and p'' be abstract symbols. For each $a \in \Sigma$ let a' and a'' be abstract symbols. Let $\Sigma' = \{a' \mid a \in \Sigma\}$, $\Sigma_1 = \{a', a'' \mid a \in \Sigma\}$, and let s_0 be a symbol not in $\{p', p'' \mid p \in K\}$. Let τ be the homomorphism of $\Sigma_1{}^*$ into Σ^* defined by $\tau(a') = \tau(a'') = a$ for each a' and a'' in Σ_1. We shall construct a gsm $S = (K_S, \Sigma, \Sigma_1, \delta_S, \lambda_S, s_0)$, where

$$K_S = \{s_0\} \cup \{p', p'' \mid p \in K\},$$

such that for $D = \tau[S(\Sigma^*) \cap \Sigma_1{}^*\Sigma']$ either

$$D \cup \{\epsilon\} = g^{-1}(Z) \tag{*}$$

or

$$D = g^{-1}(Z), \tag{**}$$

thereby proving that $g^{-1}(Z)$ is regular.

(α) $\epsilon \in Z$. Let a be in Σ. If $\delta(p_1, a) \in F_2$, let $\delta_S(s_0, a) = \delta(p_1, a)''$ and $\lambda_S(s_0, a) = a''$. Otherwise let $\delta_S(s_0, a) = \delta(p_1, a)'$ and $\lambda_S(s_0, a) = a'$. If $\delta(p, a) \in F_2$, let $\delta_S(p', a) = \delta(p, a)''$ and $\lambda_S(p', a) = a''$. Otherwise let

$$\delta_S(p', a) = \delta(p, a)'$$

and $\lambda_S(p', a) = a'$. If $\delta(p, a) \in F_1$, let $\delta_S(p'', a) = \delta(p, a)'$ and $\lambda_S(p'', a) = a'$. Otherwise let $\delta_S(p'', a) = \delta(p, a)''$ and $\lambda_S(p'', a) = a''$.

Now $\epsilon \in D \cup \{\epsilon\}$ and $\epsilon \in g^{-1}(Z)$. If $u \geqslant x$, $u \neq \epsilon$, for no $x \in X$, then $u \in D$ and, since $g(u) = \epsilon$, $u \in g^{-1}(Z)$. If $u \neq \epsilon$ is in $f^{-1}(Z)$, then $u \in g^{-1}(Z) \cup D$. If $x \in f^{-1}(Z)$ and $u > x$, with no $y \in X$ such that $x < y \leqslant u$, then

$$u \in g^{-1}(Z) \cap D.$$

If $u \in X - f^{-1}(Z)$, then u is in $\Sigma^* - [D \cup \{\epsilon\}]$ and in $\Sigma^* - g^{-1}(Z)$. If $x \in X - f^{-1}(Z)$ and $u > x$, with no $y \in X$ such that $x < y \leqslant u$, then u is in $\Sigma^* - g^{-1}(Z)$ and $\Sigma^* - [D \cup \{\epsilon\}]$. Thus (*) holds.

(β) $\epsilon \notin Z$. Let $a \in \Sigma$. If $\delta(p_1, a) \in F_1$, let $\delta_S(s_0, a) = \delta(p_1, a)'$ and

$$\lambda_S(s_0, a) = a'.$$

Otherwise let $\delta_S(s_0, a) = \delta(p_1, a)''$ and $\lambda_S(s_0, a) = a''$. Let $\delta_S(p', a)$, $\delta_S(p'', a)$, $\lambda_S(p', a)$, and $\lambda_S(p'', a)$ be defined as in (α). By an argument as in (α), we may show that (**) holds.

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