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ON BASIC CYCLES OF A_n, B_n, C_n AND D_n

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1. Introduction. In this paper, we investigate a conjecture of Dixmier [**2**] on the structure of basic cycles. Our interest in basic cycles arises primarily from the fact that the irreducible modules of a simple Lie algebra L having a weight space decomposition are completely determined by the irreducible modules of the cycle subalgebra of L. The basic cycles form a generating set for the cycle subalgebra.

First some notation: F denotes an algebraically closed field of characteristic 0, L a finite dimensional simple Lie algebra of rank n over F, H a fixed Cartan subalgebra, $U(L)$ the universal enveloping algebra of L, $C(L)$ the centralizer of H in $U(L)$, Φ the set of nonzero roots in H^*, the dual space of H, $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ a base of Φ, and $\Phi^+ = \{\beta_1, \ldots, \beta_m\}$ the positive roots corresponding to Δ. For nonnegative integers p_i, l_i, q_i, the set of monomials

$$u(\overline{p}, \overline{l}, \overline{q}) = X^{p_m}_{-\beta_m} \ldots X^{p_1}_{-\beta_1} h^{l_1}_{\alpha_1} \ldots h^{l_n}_{\alpha_n} X^{q_1}_{\beta_1} \ldots X^{q_m}_{\beta_m}$$

in $U(L)$ constitutes a Poincaré-Birkhoff-Witt basis for $U(L)$.

Definition 1.1. Any monomial $u(\overline{p}, \overline{l}, \overline{q})$ where

$$\sum_{i=1}^{m} (q_i - p_i)\beta_i = 0$$

is called a *cycle*. It is clear that $C(L)$ can be realized as the linear span of all cycles in $U(L)$.

Definition 1.2. A cycle $c = u(\overline{p}, \overline{l}, \overline{q})$ with $\overline{l} = \overline{0}$ is called a *basic cycle* provided the set $r(c)$ of all roots (with multiplicities) appearing as subscripts contains no proper subset of roots which sums to zero. The cycles h_μ with $\mu \in \Delta$ are called *trivial* basic cycles.

Let $c \in C(L)$ be a basic cycle with $r(c) = \{\mu_1, \ldots, \mu_k\}$, and let

$$\sigma(r(c)) = r(\sigma(c)) = \{\sigma(\mu_1), \ldots, \sigma(\mu_k)\} \quad \text{for } \sigma \in W(L),$$

the Weyl group of L. For convenience we introduce the following notation.

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Definition 1.3. For any basic cycle c we denote by neg $r(c)$ the number of negative roots in the set $r(c)$. We also set $M(c)$ equal to the minimum value of neg $r(\sigma(c))$ for all σ belonging to the Weyl group $W(L)$ and $M(L)$ equal to the maximum value of $M(c)$ for all basic cycles c in $C(L)$.

Dixmier conjectured that $M(L) = 1$ for any finite dimensional simple Lie algebra L over F. Van den Hombergh [**4**] proved that this conjecture is true for A_n, B_2, B_3, D_4, D_5, E_6, and G_2 and that it is not true for the other simple Lie algebras. The validity of this conjecture for A_n played a central role in shortening arguments in our classification of all irreducible modules of A_n having at least one 1-dimensional weight space [**1**]. It seems that a complete determination of irreducible modules of L having a weight space decomposition will involve a consideration of basic cycles in one form or another.

This article centers around the concept of a circle representation for a basic cycle. Once this term is defined and justified, we prove

THEOREM. (i) $M(A_n) = 1$,
(ii) $M(B_n) = [(n + 2)/3]$,
(iii) $M(C_n) = [(n + 1)/2]$, *and*
(iv) $M(D_n) = [n/3]$.

Other information about the structure and properties of basic cycles can be readily obtained from these circle representations. In particular, the fact that $C(L)$ is finitely generated becomes obvious.

Throughout this paper, $\{\epsilon_i | i = 1, \ldots, n\}$ denotes the standard orthonormal the basis of the n-dimensional vector space $\mathbf{R}^n$, and ϵ_{-i} denotes $-\epsilon_i$.

2. Basic cycles of A_{n-1}. The root system of A_{n-1} can be realized in $\mathbf{R}^n$ as

$$\Phi = \{\epsilon_i - \epsilon_j | 1 \leqq i, j \leqq n, i \neq j\}.$$

A base for Φ is given by

$$\Delta = \{\epsilon_i - \epsilon_{i+1} | i = 1, 2, \ldots, n - 1\}.$$

The roots $r(c)$ associated with a basic cycle c can be described using the μ-notation

$$\mu(a_i, a_j) = \epsilon_{a_i} - \epsilon_{a_j}.$$

This is explained by

THEOREM 2.1. (i) *For each sequence* $[a_1, \ldots, a_k]$ *of* $k \geqq 2$ *distinct elements in* $\{1, \ldots, n\}$ *the set of roots*

$$c[a_1, \ldots, a_k] \equiv \{\mu(a_1, a_2), \mu(a_2, a_3), \ldots, \mu(a_n, a_1)\}$$

equals $r(c)$ *for some basic cycle* $c \in C(A_{n-1})$.

(ii) *For a basic cycle* $c \in C(A_{n-1})$, *there exists a sequence* $[a_1, \ldots, a_k]$ *of* $k \geqq 2$ *distinct elements in* $\{1, \ldots, n\}$ *such that* $r(c) = c[a_1, \ldots, a_k]$.

Proof. (i) Let $[a_1, \ldots, a_k]$ be a sequence of two or more distinct elements in $\{1, \ldots, n\}$. Evidently, in order to prove (i) we must prove

(2.2) $\quad \mu(a_1, a_2) + \ldots + \mu(a_{m-1}, a_m) + \mu(a_m, a_1) = 0,\quad$ and

(2.3) $\quad \mu(a_{i_1}, a_{i_1+1}) + \mu(a_{i_2}, a_{i_2+1}) + \ldots + \mu(a_{i_k}, a_{i_k+1}) \neq 0$

for any nonempty proper subset

$$\{\mu(a_{i_1}, a_{i_1+1}), \ldots, \mu(a_{i_k}, a_{i_k+1})\} \subset \{\mu(a_1, a_2), \ldots, \mu(a_{m-1}, a_m), \mu(a_m, a_1)\}.$$

Condition (2.2) follows immediately from the definition of $\mu(a_i, a_j)$. To prove (2.3), let j be the minimal element in $S = \{a_{i_1}, \ldots, a_{i_k}\}$ such that $j + 1$ read modulo n is not in S. Then the left hand side of (2.3), when expanded using the basis $\{\epsilon_i | 1 \leqq i \leqq n\}$, has a -1 appearing as the coefficient of ϵ_{j+1}. Hence, (2.3) is true.

(ii) For $\mu_1 = \epsilon_a - \epsilon_q, \mu_2 = \epsilon_{a'} - \epsilon_{q'} \in \Phi$ with $\mu_1 + \mu_2 = \epsilon_{a_1} - \epsilon_{a_2} \in \Phi$ then either $a = q'$ or $q = a'$, and in either case there is some $1 \leqq a_3 \leqq n$ such that

(2.4) $\quad \{\mu_1, \mu_2\} = \{\mu(a_1, a_3), \mu(a_3, a_2)\}.$

Now let c be a basic cycle in $C(A_{n-1})$ with k = number of elements in $r(c)$. We proceed by induction on $k \geqq 2$ implementing the previous remark.

In the case of $k = 2$, we have $r(c) = \{\mu, -\mu\}$ with $\mu = \epsilon_a - \epsilon_q$ so we take $[a_1, a_k]$ to be $[a, q]$. Assume now that the result is true up to but not including the case of $k = K$. Let c be a basic cycle with

$$r(c) = \{\mu_1, \ldots, \mu_k\}.$$

Since two of the μ_i's add to a root, we may assume $\mu_1 + \mu_2 \in \Phi$ and hence there is a basic cycle $c' \in C(A_{n-1})$ of degree $K - 1$ such that

$$r(c') = \{\mu_1 + \mu_2, \mu_3, \ldots, \mu_k\}.$$

The induction hypothesis implies

(2.5) $\quad \{\mu_1 + \mu_2, \mu_3, \ldots, \mu_k\} = c[a_1, \ldots, a_{k-1}]$

with the a_i's distinct in $\{1, \ldots, n\}$. If $\mu_1 + \mu_2 = \mu(a_i, a_{i+1})$ then by (2.4), there is some $1 \leqq a \leqq n$ such that

$$\{\mu_1, \mu_2\} = c[a_i, a, a_{i+1}] \text{ and hence}$$

(2.6) $\quad \{\mu_1, \mu_2, \ldots, \mu_k\} = c[a_1, \ldots, a_i, a, a_{i+1}, \ldots, a_{k-1}].$

If $a = a_j$ for any $1 \leqq j \leqq k - 1$, then c is not basic. Hence, (ii) is proved.

Since

$$c[a_1, \ldots, a_k] = \{\mu(a_1, a_2), \mu(a_2, a_3), \ldots, \mu(a_k, a_1)\}$$

it is instructive to think of the a_i's as being placed on a circle and ordered from a_1 to a_k in a counterclockwise manner so that a_k and a_1 are adjacent. Hence, we call $[a_1, \ldots, a_k]$ a circle representation for the cycle c such that $r(c) = c[a_1, \ldots, a_k]$ and write

$$CR(c) = [a_1, \ldots, a_k].$$

THEOREM 2.7. [**4**] $M(A_{n-1}) = 1$.

Proof. As noted by [**3**, page 64], the Weyl group $W(A_{n-1})$ is the symmetric group of all permutations on $\{\epsilon_i | 1 \leqq i \leqq n\}$. Hence, there is a $\sigma \in W(A_{n-1})$ such that

(2.8) $$\sigma(r(c)) = \{\sigma(\mu_1), \ldots, \sigma(\mu_k)\} = c[1, \ldots, k].$$

The right hand side of (2.8) has $\epsilon_k - \epsilon_1$ as its only negative root.

3. Basic cycles of C_n. The root system of C_n can be realized in $\mathbf{R}^n$ as

$$\Phi = \{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \leqq i < j \leqq n\} \cup \{\pm 2\epsilon_i | 1 \leqq i \leqq n\}$$

with base

$$\Delta = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}.$$

The terminology and notation used in this section has analogous, but not identical, meaning to the same terms in Section 2. We again have a notion of a circle representation of a basic cycle.

Definition 3.1. A sequence $[a_1, \ldots, a_k]$ of $k \geqq 2$ values in $\{\pm 1, \ldots, \pm n\}$ has Property C provided (i) $a_i \neq a_j$ for $i \neq j$ and (ii) there do not exist indices $1 \leqq i_1 < i_2 < i_3 < i_4 \leqq k$ with

$$a_{i_1} = -a_{i_3} \quad \text{and} \quad a_{i_2} = -a_{i_4}.$$

Such a sequence $[a_1, \ldots, a_k]$ is called a *circle representation* of a basic cycle in $C(C_n)$. This terminology is justified by Lemma 3.6.

LEMMA 3.2. *Let $\mu_1, \mu_2, \mu_1 + \mu_2 \in \Phi$. If $\mu(a_1, a_2) = \mu_1 + \mu_2$ then there is some $a \in \{\pm 1, \ldots, \pm n\}$ such that*

(3.3) $$\{\mu(a_1, a), \mu(a, a_2)\} = \{\mu_1, \mu_2\}.$$

Proof. Assume $r, s, t, u \in \{\pm 1, \ldots, \pm n\}$ are such that $\mu_1 = \mu(r, s)$ and $\mu_2 = \mu(t, u)$. Since

$$\mu(r, s) + \mu(t, u) = \mu(a_1, a_2) \in \Phi,$$

we must have one of the following $r = -t$, $r = u$, $s = t$ or $s = -u$. Since $\mu(q, p) = \mu(-p, -q)$, there is no loss of generality in assuming $s = t$. Now, we have

(3.4) $\mu(r, s) + \mu(s, u) = \mu(r, u) = \mu(a_1, a_2)$.

This implies either $r = a_1$ and $u = a_2$ or $r = -a_2$ and $u = -a_1$. In the first case we have what we want. In the second, we write

(3.5) $$\begin{aligned}\mu(r, s) + \mu(s, u) &= \mu(-u, -s) + \mu(-s, -r)\\ &= \mu(a_1, -s) + \mu(-s, a_2) = \mu(a_1, a_2)\end{aligned}$$

with $\mu(-u, -s) = \mu_1$, and $\mu(-s, -r) = \mu_2$ as required.

LEMMA 3.6. (i) *For each sequence* $[a_1, \ldots, a_k]$ *having Property C, the set of roots*

$$c[a_1, \ldots, a_k] = \{\mu(a_1, a_2), \mu(a_2, a_3), \ldots, \mu(a_k, a_1)\}$$

equals $r(c)$ *for some basic cycle* $c \in C(C_n)$.

(ii) *For a basic cycle* $c \in C(C_n)$, *there is a sequence* $[a_1, \ldots, a_k]$ *having Property C such that*

$$r(c) = c[a_1, \ldots, a_k].$$

Proof. (i) It is obvious that for any sequence $[a_1, \ldots, a_k]$ having Property C, we have

$$\mu(a_1, a_2) + \mu(a_2, a_3) + \ldots + \mu(a_k, a_1) = 0.$$

Hence, we need only show that no proper subset of roots in $c[a_1, \ldots, a_k]$ sum to 0.

Let $1 \leqq i_1 < i_2 < \ldots < i_j \leqq k$ be such that $\{\mu(a_{i_1}, a_{i_1+1}), \ldots, \mu(a_{i_j}, a_{i_j+1})\}$ is a proper subset of $c[a_1, \ldots, a_k]$. We are making free use of the convention $a_{k+1} = a_1$. Suppose

(3.7) $\mu(a_{i_1}, a_{i_1+1}) + \ldots + \mu(a_{i_j}, a_{i_j+1}) = 0$

or in other words

(3.8) $(\epsilon_{a_{i_1}} - \epsilon_{a_{i_1+1}}) + \ldots + (\epsilon_{a_{i_j}} - \epsilon_{a_{i_j+1}}) = 0.$

Since (3.7) does not involve all roots of $c[a_1, \ldots, a_k]$, there is some $1 \leqq p \leqq j$ such that $i_p + 1 \neq i_{p+1}$. We must have $+\epsilon_{a_{i_p+1}}$ appearing in (3.8). Since the terms in $[a_1, \ldots, a_k]$ are distinct, $+\epsilon_{a_{i_p+1}}$ must appear as

$$-\epsilon_{-a_{i_p+1}} = -\epsilon_{a_{i_q+1}} \quad \text{for some } q.$$

However, the number of summands of (3.8) appearing between $-\epsilon_{a_{i_q+1}}$ and $-\epsilon_{a_{i_p+1}}$ is odd. Therefore for some value a_j of the given sequence its paired value $-a_j = a_k$ must be separated by the pair a_{i_p+1} and $a_{i_q+1} = -a_{i_p+1}$ contrary to Property C.

(ii) We induct on k equal the degree of c. If $k = 2$ the proof is the same as in the case of $L = A_{n-1}$. Let $k \geqq 3$ and let c be a basic cycle in $C(C_n)$ of degree k. Let $r(c) = \{\mu_1, \dots, \mu_k\}$ be ordered so that $\mu_1 + \mu_2 \in \Phi$.

By the induction hypothesis, there is a sequence $[a_1, \dots, a_{k-1}]$ with Property C such that

$$c[a_1, \dots, a_{k-1}] = \{\mu_1 + \mu_2, \mu_3, \dots, \mu_k\}$$

and we may assume

$$\mu_1 + \mu_2 = \mu(a_1, a_2).$$

By Lemma 3.2, there is some $a \in \{\pm 1, \dots, \pm n\}$ such that $a \neq a_1$ or a_2 with

$$\{\mu_1, \mu_2\} = \{\mu(a_1, a), \mu(a, a_2)\}.$$

It suffices to show that the sequence $[a_1, a, a_2, \dots, a_{k-1}]$ has Property C.

If $a = a_i$ for $3 \leqq i \leqq K$, then the sum of the roots in

$$\{\mu(a, a_2), \mu(a_2, a_3), \dots, \mu(a_{i-1}, a_i)\} \subset r(c)$$

is zero, contrary to c being basic.

Relabel the sequence $[a_1, a, a_2, \dots, a_{k-1}]$ as $[a_1, a_2, \dots, a_k]$. Using this notation, we find that if $1 \leqq r < s < t < u \leqq k$ with $a_r = -a_t$ and $a_s = -a_u$ then

$$0 = \mu(a_r, a_{r+1}) + \dots + \mu(a_{s-1}, a_s) + \mu(a_t, a_{t+1}) + \dots + \mu(a_{u-1}, a_u)$$

contrary to c being basic. Thus $[a_1, a, a_2, \dots, a_{k-1}]$ is a circle representation of c as required.

As in Section 2, if c is a basic cycle in $C(C_n)$ such that

$$r(c) = c[a_1, \dots, a_k],$$

we denote the sequence $[a_1, \dots, a_k]$ by $CR(c)$. We refer to a_i as being paired in $CR(c)$ provided both a_i and $-a_i$ appear in the sequence $[a_1, \dots, a_k]$. Otherwise, we say that a_i is singular in $CR(c)$. In determining $M(c)$ for $c \in C(C_n)$, we find that we can ignore the singular values of $CR(c)$. The proof of this requires some knowledge about the action of the Weyl group.

The Weyl group, $W(C_n)$, of C_n consist of all one to one maps

$$\sigma:\{\pm 1, \dots, \pm n\} \to \{\pm 1, \dots, \pm n\}$$

such that $\sigma(-i) = -\sigma(i)$. The action of $\sigma \in W(C_n)$ is given by

$$\sigma(\mu(i, j)) = \sigma(\epsilon_i - \epsilon_j) = \epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} = \mu(\sigma(i), \sigma(j)).$$

Clearly, the set of all basic cycles is invariant under the action of $W(C_n)$. Moreover, the action of $\sigma \in W(C_n)$ on a circle representation $CR(c) = [a_1, \dots, a_k]$ is

$$\sigma(CR(c)) = \sigma[a_1, \dots, a_k] = [\sigma(a_1), \dots, \sigma(a_k)] = CR(\sigma(c)).$$

Remark 3.9. The root $\mu(a_1, a_2)$ is positive if and only if one of the following conditions is true; (i) $0 < a_1 < a_2$, (ii) $0 > a_2 > a_1$, or (iii) $a_1 > 0 > a_2$.

It follows then that if $\sigma \in W(C_n)$ such that $a_1\sigma(a_1)$, $a_2\sigma(a_2) > 0$ and σ preserves the order of the absolute values $|a_1|$ and $|a_2|$, then $\mu(a_1, a_2)$ is positive if and only if $\mu(\sigma(a_1), \sigma(a_2))$ is positive.

LEMMA 3.10. *Let c be a basic cycle in $C(C_n)$ with $CR(c) = [a_1, \dots, a_k]$ having at least one paired value and having a_j as a singular value. Then there is some $\sigma \in W(C_n)$ with*

$$\begin{aligned} &\operatorname{neg}[a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_k] \\ &= \operatorname{neg}[\sigma(a_1), \dots, \sigma(a_k)] = \operatorname{neg} CR(\sigma(c)). \end{aligned}$$

Proof. Assume first that $\mu(a_{j-1}, a_{j+1})$ is a negative root. Let $|a_j| = m$. Define $\sigma \in W(C_n)$ such that σ maps $\{i|1 \leqq i \leqq n, i \neq m\}$ onto $\{i|2 \leqq i \leqq n\}$ preserving order and $\sigma(a_j) = 1$. By remark 3.9 for $i \neq j$ or $j - 1$ the root $\mu(a_i, a_{i+1})$ is negative if and only if $\mu(\sigma(a_i), \sigma(a_{i+1}))$ is negative. Also $\mu(\sigma(a_{j-1}), \sigma(a_j))$ is negative and $\mu(\sigma(a_j), \sigma(a_{j+1}))$ is positive. Thus

$$\operatorname{neg} c[a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_k] = \operatorname{neg} c[\sigma(a_1), \dots, \sigma(a_k)]$$

as required.

Assume now that $\mu(a_{j-1}, a_{j+1})$ is a positive root. Let $|a_j| = m$. Define $\sigma \in W(C_n)$ on $\{i|1 \leqq i \leqq n, i \neq m\}$ as the composite of the map σ_1 which sends $\{i|1 \leqq i \leqq n, i \neq m\}$ onto $\{1, \dots, n - 1\}$ preserving order and the map σ_2 which sends $\{1, \dots, n - 1)$ onto $\{i|1 \leqq i \leqq n, i \neq |a_{j-1} + 1|\}$ preserving order. Also set

$$\sigma(a_j) = \sigma_1(a_{j-1}) + 1.$$

Again by remark 3.9 for $i \neq j$ or $j - 1$ the root $\mu(a_i, a_{i+1})$ is negative if and only if $\mu(\sigma(a_i), \sigma(a_{i+1}))$ is negative and both $\mu(\sigma(a_{j-1}), \sigma(a_j))$ and $\mu(\sigma(a_j), \sigma(a_{j+1}))$ are positive roots. Therefore

$$\operatorname{neg} c[a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_k] = \operatorname{neg} c[\sigma(a_1), \dots, \sigma(a_k)]$$

as required.

LEMMA 3.11. *Let c be a basic cycle in $C(C_n)$ with $CR(c) = [a_1, \dots, a_k]$ having a as a singular value. Then if $c' \in C(C_n)$ with*

$$CR(c') = [a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_k]$$

then $M(c) = M(c')$.

Proof. Let $\sigma' \in W(C_n)$ such that

$$\text{neg}\,[\sigma'(a_1), \ldots, \sigma'(a_{j-1}), \sigma'(a_{j+1}), \ldots, \sigma'(a_k)] = M(c').$$

Then by Lemma 3.10, there is a $\sigma \in W(C_n)$ such that

$$\text{neg}\,[\sigma'(a_1), \ldots, \sigma'(a_{j-1}), \sigma'(a_{j+1}), \ldots, \sigma'(a_k)]$$
$$= \text{neg}\,[\sigma\sigma'(a_1), \ldots, \sigma\sigma'(a_k)].$$

Hence $M(c') \geqq M(c)$.

Now let $\sigma \in W(C_n)$ be such that

$$M(c) = \text{neg}\,[\sigma(a_1), \ldots, \sigma(a_k)].$$

Then since $M(c') \geqq M(c)$, we know

$$\text{neg}\,[\sigma(a_1), \ldots, \sigma(a_{j-1}), \sigma(a_{j+1}), \ldots, \sigma(a_k)] \tag{3.12}$$
$$\geqq \text{neg}\,[\sigma(a_1), \ldots, \sigma(a_k)].$$

Either equality holds in (3.12) and we are done or $\mu(\sigma(a_{j-1}), \sigma(a_{j+1}))$ is negative while both $\mu(\sigma(a_{j-1}), \sigma(a_j))$ and $\mu(\sigma(a_j), \sigma(a_{j+1}))$ are positive but this is impossible since

$$\mu(a_{j-1}, a_{j+1}) = \mu(a_{j-1}, a_j) + \mu(a_j, a_{j+1}).$$

LEMMA 3.13. *If c_k denotes a basic cycle of C_n with*

$$CR(c_k) = [a_1, -a_1, \ldots, a_k, -a_k]$$

then $M(c_k) = [(k+1)/2]$, where $[\ \ ]$ denotes the greatest integer function.

Proof. For $k = 1$ the result is obvious. Assume now that $k \geqq 2$. Observe that regardless of the values of a_i, a_{i+1} the three roots $\mu(a_i, -a_i)$, $\mu(-a_i, a_{i+1})$ and $\mu(a_{i+1}, -a_{i+1})$ must contain at least one negative root. Thus

$$M(c_k) \geqq [(k+1)/2].$$

Now take $\sigma \in W(C_n)$ where

$$\sigma(a_i) = \begin{cases} k - (i-1)/2 & \text{for } i \text{ odd} \\ -i/2 & \text{for } i \text{ even.} \end{cases}$$

By counting, we find that

$$\text{neg}\,CR(\sigma(c_k)) = [(k+1)/2]$$

and hence $M(c_k) = [(k+1)/2]$.

Remark. 3.14. Let $CR(c) = [a_1, \ldots, a_k]$. In view of Remark 3.9, we see that

$$\text{neg}\,r(c) = \text{neg}\,r(\sigma(c))$$

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provided $a_i\sigma(a_i) > 0$ and σ preserves relative size of the absolute values of the a_i's.

Definition 3.15. Let $[a_1, \ldots, a_k] = CR(c)$ and associate k equally spaced points on the circumference of a circle labelled respectively by $a_1, \ldots, a_k$ as one moves in a counter-clockwise direction around the circle. Join all paired labels by a line segment so that the i^{th} and j^{th} points are connected provided $a_i = -a_j$. Our definition of $CR(c)$ excludes the possibility that two of these line segments intersect. The components of $CR(c)$ are the pieces obtained when we cut our circle along each of these line segments. We define a component to be *trivial* provided its boundary contains exactly two values, namely the two paired values we cut along.

THEOREM 3.16. *If* $[a_1, \ldots, a_k] = CR(c)$ *for some basic cycle* $c \in C(C_n)$ *has N non-trivial components after the singular values are removed and the* i^{th} *component contains* k_i *paired values then*

$$M(c) = \sum_{i=1}^{N} [(k_i + 1)/2] - N + 1.$$

Proof. By Lemma 3.11 we may assume that all values a_i for $1 \leqq i \leqq K$ are paired. A pair $a_i = -a_j$ is said to be interior to $[a_1, \ldots, a_k]$ provided $i - j \neq \pm 1 \bmod K$. Since each interior pair partitions the circle into two non-trivial parts, $[a_1, \ldots, a_k]$ contains exactly $N - 1$ interior pairs accounting for the N non-trivial components. If we sum over each of the N non-trivial components we obtain a minimum of

$$\sum_{i=1}^{N} [(k_i + 1)/2]$$

negative roots. Each interior pair $a_i = -a_j$ occurs in exactly two non-trivial components once as $\mu(a_i, a_j)$ and once as $\mu(a_j, a_i)$. Thus each interior pair contributes one negative root to the above sum which does not belong to c. Therefore

$$M(c) \geqq \sum_{i=1}^{N} [(k_i + 1)/2] - (N - 1).$$

To complete the proof we must show that there exists a $\sigma \in W(C_n)$ such that

$$\text{neg } \sigma[a_1, \ldots, a_k] = \sum_{i=1}^{N} [(k_i + 1)/2] - (N - 1).$$

For $N = 1$ the result follows from Lemma 3.13. Assume now that $N \geqq 2$ and that $[a_i, \ldots, a_j = -a_i]$ is the ν^{th} non-trivial component of

$[a_1, \ldots, a_k]$. By induction we may assume that $[a_1, \ldots, a_i, a_j, \ldots, a_k]$ has been labelled with the values $\pm\{1, 2, \ldots, p\}$ such that

$$\operatorname{neg}[a_1, \ldots, a_i, a_j, \ldots, a_k] = \sum_{i \neq \nu} [(k_i + 1)/2] - (N - 2).$$

Assume that $a_i = \pm q$. By Remark 3.9 if $\sigma \in W(C_n)$ such that

$$\sigma(i) = \begin{cases} i & \text{for } 1 \leqq i \leqq q \\ i + k_\nu - 1 & \text{for } i > q \end{cases}$$

then

$$\operatorname{neg} \sigma[a_1, \ldots, a_i, a_j, \ldots, a_k] = \operatorname{neg}[a_1, \ldots, a_i, a_j, \ldots, a_k]$$

Finally using Lemma 3.13 together with Remark 3.14 we use the values $\pm\{q, \ldots, q + k_\nu - 1\}$ to label the component $[a_i, \ldots, a_j]$ such that it contains $[(k_\nu + 1)/2]$ negative roots. Note that in this labelling the value of a_i may change. However, we can maintain its sign and its relative position in $[a_1, \ldots, a_i, a_j, \ldots, a_k]$. Assume that σ has been redefined (if necessary) to accomplish this relabelling. Then

$$\begin{aligned} \operatorname{neg} \sigma[a_1, \ldots, a_k] &= \operatorname{neg} \sigma[a_1, \ldots, a_i, a_j, \ldots, a_k] \\ &\quad + \operatorname{neg}[a_i, \ldots, a_j] - 1 \\ &= \sum_{i=1}^{N} [(K_i + 1)/2] - (N - 1). \end{aligned}$$

From a global point of view we have

THEOREM 3.17. $M(C_n) = [(n + 1)/2]$.

Proof. Since by Lemma 3.13 $M(c_k) = [(k + 1)/2]$ we have that

$$M(C_n) \geqq [(n + 1)/2].$$

Now take any cycle $c \in C(C_n)$ and assume that $CR(c) = [a_1, \ldots, a_k]$ decomposes into N components where the i^{th} component contains K_i pairs of labels. Since there at most n distinct pairs of labels available and each interior pair occurs in exactly two components we have that

$$\sum_{i=1}^{N} K_i - (N - 1) \leqq n.$$

By Theorem 3.16, we have

$$M(c) = \sum_{i=1}^{N} [(K_i + 1)/2] - (N - 1)$$

$$\leqq \left[\left(\sum_{i=1}^{N} K_i + N - 2N + 2\right)/2\right]$$

$$\leqq \left[\left(\sum_{i=1}^{N} K_i - (N - 1) + 1\right)/2\right] \leqq [(n + 1)/2].$$

Therefore $M(C_n) = [(n + 1)/2]$.

In the proof of Theorem 3.16, we have outlined an inductive procedure which associates with each $c \in C(C_n)$ an element $\sigma(c) \in C(C_n)$ with $M(c) = \text{neg } r(\sigma(c))$. Using a modification of this procedure, one can describe a unique $\sigma(c)$ with this property. Hence, the problem of determining the number of basic cycles in $C(C_n)$ amounts to finding the order of the stabilizer of c under the action of $W(C_n)$ and being able to determine if two basic cycles are in the same orbit. We have a rather lengthy algorithm to do this but we do not present it here.

4. Basic cycles in D_n. The root system of D_n can be realized in $\mathbf{R}^n$ as

$$\Phi = \{\pm(\epsilon_j \pm \epsilon_k) | 1 \leqq j < k \leqq n\}.$$

A base for Φ is given by

$$\Delta = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} + \epsilon_n\}.$$

The Weyl Group of D_n, $W(D_n)$, consists of all $\sigma \in W(C_n)$ such that $\{\sigma(i) | i = 1, \dots, n\}$ has an even number of negative values. With these identifications, the root system of D_n is a subset of the root system of C_n and $W(D_n)$ is a proper subgroup of $W(C_n)$. Also, it follows that every basic cycle of D_n is a basic cycle of C_n and every basic cycle of C_n not containing a long root is a basic cycle of D_n. We define the circle representation of a basic cycle in D_n to be the circle representation of the corresponding cycle in C_n, and we keep the notation $CR(c)$. A basic cycle $c \in C(C_n)$ is in $C(D_n)$ exactly when no paired value is next to its negative in

$$CR(c) = [a_1, \dots, a_k].$$

Recall a_1 is next to a_k. Finally, we set

$$M_{C_n}(c) = \min \{\text{neg } CR(\sigma(c)) | \sigma \in W(C_n)\} \quad \text{and}$$

$$M_{D_n}(c) = \min \{\text{neg } CR(\sigma(c)) | \sigma \in W(D_n)\}.$$

LEMMA 4.1. *For each basic cycle $c \in C(D_n)$, we have $M_{C_n}(c) = M_{D_n}(c)$.*

Proof. Let c be a basic cycle in $C(D_n)$ with $CR(c) = [a_1, \dots, a_k]$. If $CR(c)$ contains no paired values then there is a $\sigma \in W(C_n)$ such that σ changes an even number of signs and

$$\sigma[a_1, \dots, a_k] = [1, \dots, k], \text{ or } [-1, 2, \dots, k].$$

In either case, $CR(\sigma(c))$ is the circle representation of a cycle having exactly one negative root, and the lemma is proved in this case.

Now let $\sigma \in W(C_n)$ be such that

$$M_{C_n}(c) = \text{neg } r(CR(\sigma(c))),$$

and

$$CR(\sigma(c)) = [a_1, \dots, a_k]$$

with $a_i = -a_{i+t}$ separated by singular values a_{i+p}, $1 \leqq p \leqq t - 1$. Define $\sigma' \in W(C_n)$ so that it preserves the signs and relative orders of the absolute values of the terms of $CR(\sigma(c))$ on the set $\{a_1, \dots, a_i, a_{i+t}, \dots, a_k\}$, and is defined according to one of the following two cases.

Case I. Assume $a_i > 0$. Let

$$|\sigma'(a_p)| \in \{1, \dots, n - t + 1\} \text{ for } p \notin \{i + 1, \dots, i + t - 1\}$$

and

$$\sigma'(a_{i+p}) = n - t + 1 + p \text{ for } 1 \leqq p \leqq t - 1.$$

If $\sigma'\sigma$ changes an odd number of signs then change the definition of $\sigma'(a_{i+t-1})$ to $-n$.

Case II. Assume $a_i < 0$. Let

$$|\sigma'(a_p)| \in \{t, \dots, n\} \text{ for } p \notin \{i + 1, \dots, i + t - 1\}$$

and

$$\sigma'(a_{i+p}) = p \text{ for } 1 \leqq p \leqq t - 1.$$

If $\sigma'\sigma$ changes an odd number of signs then change the definition of $\sigma'(a_{i+1})$ to -1.

In either case, $\sigma'\sigma$ changes an even number signs, i.e., $\sigma\sigma' \in W(D_n)$, and

$$\text{neg } \sigma'\sigma(c) = \text{neg } \sigma(c).$$

THEOREM 4.2. $M(D_n) = [n/3]$.

Proof. Let $c \in C(D_n)$ be a basic cycle such that $CR(c)$ has $[n/2]$ positive terms which are paired values and each is separated from its negative by exactly one singular value when the circle is traversed in the counterclockwise direction. Apply Theorem 3.16 to find $M(c) = [n/3]$ when $4 \leqq n \leqq 8$. Hence, $M(D_n) \geqq [n/3]$ for these values of n.

Now let $n = 3k + j$ for $k \geqq 3$ and $j = 0$, 1 or 2. Let $c \in C(D_n)$ be a basic cycle whose circle representation can be realized by Figure 4.3 where the line segments are joining paired values and the dots indicate singular values. $M(c)$ is independent of the values that we assign to the nodes of this diagram and is found to be $[n/3]$ by applying Theorem 3.16. Hence,

$$M(D_n) \geqq [n/3] \quad \text{for all } n.$$

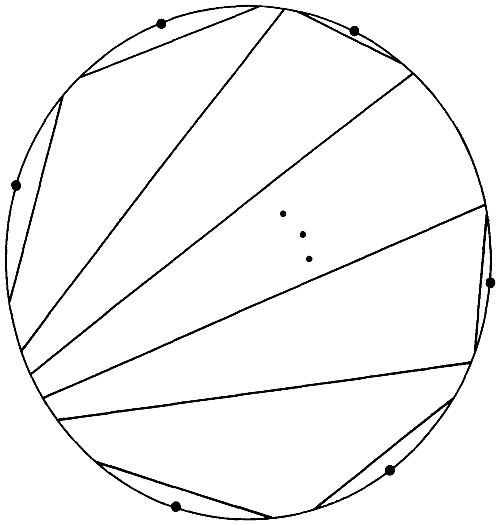

Figure 4.3

To get the inequality in the reverse direction, we use the notion of an interior pair which is a paired value of $CR(c)$ separated from its negative on both sides by more than singular values. Let $CR(c)$ have N nontrivial components after the singular values have been removed. Since each interior pair occurs in exactly two distinct components, there are $N - 1$ interior pairs in $CR(c)$. It follows that there are $(\sum k_i) - (N - 1)$ distinct pairs where we are summing over the numbers k_i of distinct paired values in the i-th component. Evidently, there are $(\sum k_i) - 2(N - 1)$ non-interior paired values in $CR(c)$. Since each non-interior pair must be separated by at least one singular value, we have

$$(4.3) \qquad [\,(\textstyle\sum k_i) - (N - 1)\,] + [\,(\textstyle\sum k_i) - 2(N - 1)\,] \leqq n,$$

$$\text{or } 2\textstyle\sum k_i - 3(N - 1) \leqq n$$

which implies

$$(4.4) \qquad -N \leqq (1/3)[n - 3 - 2\textstyle\sum k_i].$$

Combining Theorem 3.16 with (4.4), we get

$$M(c) \leqq \textstyle\sum [k_i + 1)/2] + (n/3) - (2/3) \textstyle\sum k_i$$

and since $(2/3)k_i \geqq [\,(k_i + 1)/2]$, we have

$$M(c) \leqq [n/3]$$

which completes the proof.

5. Basic cycles on B_n. The root system of B_n can be realized in $\mathbf{R}^n$ as

$$\Phi = \{\pm(\epsilon_i \pm \epsilon_j) \,|1 \leqq i < j \leqq n\} \cup \{\pm\epsilon_i|1 \leqq i \leqq n\}$$

with base

$$\Delta = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \dots, \epsilon_{n-1} - \epsilon_n, \epsilon_n\}.$$

LEMMA 5.1. *Let F denote the set $\{\epsilon_1, \epsilon_1 \pm \epsilon_k|2 \leqq k \leqq n\}$. Then*

(i) *if $\mu_1, \mu_2 \in F$ then $\mu_1 + \mu_2 \notin \Phi$,*

(ii) *if $\mu_1, \mu_2 \in \Phi$ with $\mu_1 + \mu_2 \in F$, then exactly one of μ_1 or μ_2 is in F*

(iii) *if c is a basic cycle in $C(B_n)$ with*

$$r(c) = \{\mu_1, \dots, \mu_k\} \quad \textit{and}$$

$$\sum_{i=1}^{m} \mu_i \in F \quad \textit{for } 1 \leqq m \leqq k - 1$$

then $\mu_1 \in F$ but $\mu_i \notin F$ for $2 \leqq i \leqq k - 1$.

Proof. (i) If $\mu_1 + \mu_2 \in F$ then $\mu_1 + \mu_2 = 2\epsilon_1 + \dots \notin \Phi$.

(ii) If $\mu_1 + \mu_2 \in F$ then $\mu_1 + \mu_2 = \epsilon_1 + \dots$ and hence exactly one of μ_1 or μ_2 has a nonzero coefficient accompanying ϵ_1. This root is in F. Part (iii) follows from (ii).

LEMMA 5.2. *If c is a basic cycle in $C(B_n)$ with $r(c) = \{\mu_1, \dots, \mu_k\}$, then there is some $\sigma \in W(B_n)$ and a permutation π on $\{1, \dots, k\}$ such that*

$$\sum_{i=1}^{m} \sigma(\mu_{\pi(i)}) \in F \quad \textit{for } m = 1, \dots, k - 1.$$

Proof. Our proof is by induction on k. For $k = 2$, we need only map μ_1 to either ϵ_1 or $\epsilon_1 - \epsilon_2$ depending on its length and we are done. Take

$$r(c) = \{\mu_1, \dots, \mu_k\}$$

with $k \geqq 3$. We may assume $\mu_1 + \mu_2 \in \Phi$ and there is a basic cycle c' with

$$r(c') = \{\gamma_1 = \mu_1 + \mu_2, \gamma_2 = \mu_3, \dots, \gamma_{k-1} = \mu_k\}.$$

Hence, there is some $\sigma' \in W(B_n)$ and a permutation π' on $\{1, \dots, k - 1\}$ such that

$$\sum_{i=1}^{m} \sigma'(\gamma_{\pi'(i)}) \in F \text{ for } 1 \leqq m \leqq k - 2.$$

Let

$$p = \pi'^{-1}(i) \text{ and } \mu = \sum_{i=1}^{p-1} \sigma'(\gamma_{\pi'(i)}) \in F.$$

Then

$$\mu + \sigma'(\mu_1) + \sigma'(\mu_2) \in F$$

and exactly one of $\mu + \sigma'(\mu_1)$ or $\mu + \sigma'(\mu_2) \in F$. In the former case let $\pi(p) = 1$ and $\pi(p + 1) = 2$ and in the latter $\pi(p) = 2$ and $\pi(p + 1) = 1$. Complete the definition of π as follows,

$$\pi(i) = \begin{cases} \pi'(i) & \text{for } 1 \leqq i < p \\ \pi'(i - 1) & \text{for } p + 1 < i \leqq K. \end{cases}$$

Now the result follows with $\sigma = \sigma'$.

Definition 5.3. A sequence $[a_1, \ldots, a_k]$ with $a_i \in \{\pm 1, \ldots, \pm n\}$ has Property B provided: (i) $a_i \neq a_j$ if $i \neq j$, (ii) $a_i \neq -a_{i+1}$ for $2 \leqq i \leqq k - 1$, and (iii) there do not exist indices $1 < i < u < j < w$ such that $a_i = -a_j$ and $a_u = -a_w$.

For any sequence $[a_1, \ldots, a_k]$ satisfying Property B we introduce the following notation. Let $v(a_1) = 0$, $v(-a_1) = \epsilon_{a_1}$, $v(a_i) = \epsilon_{a_i} - \epsilon_{a_i}$ for $a_i \neq \pm a_1$ and define

$$\eta(a_i, a_j) = v(a_j) - v(a_i).$$

We note that in this section the $\eta(\, . \, , \, . \,)$ notation will play the same role as the $\mu(\, . \, , \, . \,)$ notation from the previous sections. In particular we associate with the sequence $[a_1, \ldots, a_k]$ the set of roots $c[a_1, \ldots, a_k]$ equal to

$$\{\eta(a_1, a_2) = v(a_2) - v(a_1), \ldots, \eta(a_{k-1}, a_k) = v(a_k) - v(a_{k-1}),$$
$$\eta(a_k, a_1) = v(a_1) - v(a_k)\}.$$

It is clear then that if $[a_1, \ldots, a_k]$ is a sequence such that $c[a_1, \ldots, a_k]$ is a basic cycle of $C(B_n)$ then this sequence satisfies Property B.

A consequence of Lemma 5.2 and Definition 5.3 is

THEOREM 5.4. *If c is a basic cycle in $C(B_n)$ then there exists a sequence $[a_1, \ldots, a_k]$ having Property B such that $r(c) = c[a_1, \ldots, a_k]$.*

Proof. Let σ and π be as in Lemma 5.2. Hence for $l = 1, \ldots, k - 1$ we have

$$\sum_{j=1}^{l} \sigma(\mu_{\pi(j)}) = \epsilon_1 - \epsilon_{b_{l+1}} \in F.$$

Define

$$a_1 = \sigma^{-1}(b_i) \text{ for } i = 2, \ldots, k.$$

Using the ν-notation introduced above it follows that

$$c[a_1, \ldots, a_k] = \{\nu(a_2) - \nu(a_1), \ldots, \nu(a_1) - \nu(a_k)\} = r(c).$$

LEMMA 5.5. *If* $[a_1, \ldots, a_k]$ *is a sequence having Property B then* $c[a_1, \ldots, a_k]$ *equals* $r(c)$ *for some basic cycle* $c \in C(B_n)$.

Proof. Obviously,

$$\eta(a_1, a_2) + \ldots + \eta(a_k, a_1) = 0.$$

Assume that $1 \leqq i_1 \leqq i_2 \leqq \ldots \leqq i_j \leqq k$ is such that

$$S = \{\eta(a_{i_1}, a_{i_1+1}), \ldots, \eta(a_{i_j}, a_{i_j+1})\}$$

is a proper subset of $c[a_1, \ldots, a_k]$ having property

$$\eta(a_{i_1}, a_{i_1+1}) + \ldots + \eta(a_{i_j}, a_{i_j+1}) = 0. \tag{5.6}$$

Since (5.6) does not involve all roots of $c[a_1, \ldots, a_k]$, there is some $1 \leqq p \leqq j$ such that $i_p + 1 \neq i_{p+1}$. If no a_{i_t} or a_{i_t+1}, $1 \leqq t \leqq j$, equals $-a_1$ then (5.6) is equivalent to

$$(\epsilon_{a_{i_1}} - \epsilon_{a_{i_1+1}}) + \ldots + (\epsilon_{a_{i_j}} - \epsilon_{a_{i_j+1}}) = 0$$

and the proof continues in exactly the same way as in the case of C_n (see (3.8)).

Now, assume that $-a_1$ appears in $\{a_{i_1}, a_{i_1+1}, \ldots, a_{i_j}, a_{i_j+1}\}$. If

$$a_{i_t} = -a_1 = a_{i_t+1} \quad \text{for some } 1 \leqq t \leqq j,$$

then the set of roots $c[a_1, \ldots, a_k] - S$ sums to zero and does not involve $-a_1$. Hence, we arrive at the same contradiction as above.

Therefore, we assume $-a_1 = a_{i_p+1} \neq a_{i_{p+1}}$, and hence equation (5.6) becomes

$$\begin{aligned}(\epsilon_{a_{i_1}} - \epsilon_{a_{i_1+1}}) + \ldots + (\epsilon_{a_{i_{p-1}}} - \epsilon_{a_{i_{p-1}+1}}) + \epsilon_{a_{i_p}} \\ + (\epsilon_{a_{i_{p+1}}} - \epsilon_{a_{i_{p+1}+1}}) + \ldots + (\epsilon_{a_{i_j}} - \epsilon_{a_{i_j+1}}) = 0.\end{aligned} \tag{5.7}$$

The negative of $\epsilon_{a_{i_p}}$ must appear in (5.7) in the form of $+\epsilon_{-a_{i_p}} = +\epsilon_{a_{i_t}}$ with $-a_{i_p} = a_{i_t}$ or else $a_{i_{p-1}+1} = a_{i_p}$. In the latter case, add $-\epsilon_{a_{i_{p-1}+1}}$ to $\epsilon_{a_{i_p}}$ to reduce (5.7) to an equation of the same form. Continue in this manner until an equation of the form of (5.7) is arrived at with $a_{i_{p-1}} \neq a_{i_p}$. Remember at each stage $a_{i_0+1} = a_{i_j+1}$ since we are considering the subscripts of i modulo the appropriate j. Finally, we arrive at an equation of the form of (5.7) with the negative of $\epsilon_{a_{i_p}}$ appearing as $+\epsilon_{-a_{i_p}} = +\epsilon_{a_{i_t}}$. By condition (iii) of Property B, the values a_{i_p} and a_{i_t} cannot split paired values. However, between $\epsilon_{a_{i_t}}$ and $\epsilon_{a_{i_p}}$ there are an odd number of terms in the sum

$$(\epsilon_{a_{i_t}} - \epsilon_{a_{i_t+1}}) + \ldots + \epsilon_{a_{i_p}},$$

contrary to Property B.

We denote $[a_1, \ldots, a_k]$ by $CR(c)$ and call it a circle representation of c provided Property B is satisfied and $c[a_1, \ldots, a_k] = r(c)$. We now work towards determining

$$M(c) = \min \{\text{neg } CR(\sigma(c)) \mid \sigma \in W(B_n)\}.$$

Let $[a_1, \ldots, a_k] = CR(c)$ for a basic cycle $c \in C(B_n)$ in which $-a_1$ does not occur. Then $c[a_1, \ldots, a_k]$ is also a basic cycle in C_n with the same circle representation. Since the Weyl groups $W(C_n)$ and $W(B_n)$ are isomorphic and act on circle representations in the same way, this case reverts to our conclusions in $C(C_n)$. It then remains to consider those basic cycles of B_n whose circle representations $[a_1, \ldots, a_k]$ contain $-a_1$. Again we need the concept of a component of $[a_1, \ldots, a_k]$.

Definition 5.8. Let $[a_1, \ldots, a_k] = CR(c)$. Then a component of $CR(c)$ is defined as in 3.14 except that we do not cut along the line segment joining a_1 to $-a_1$ and when counting nontrivial components $-a_1$ is not considered as singular value but a_1 is. Note that if we compute $v(a_{i_{j+1}}) - v(a_{i_j})$ for two consecutive values in a component, then the resultant vector need not be a root of B_n.

LEMMA 5.9. *Let* $[a_1, \ldots, a_k] = CR(c)$ *for a basic cycle* $c \in C(B_n)$ *and let* $[a_i, \ldots, a_j = -a_i]$ *be a non-trivial component containing* $a_q = -a_1$ *and having* m *pairs of values. Then*

$$M[a_i, \ldots, a_j] = [(m + 2)/2].$$

Proof. Consider $CR(\overline{c}) = [a_i, \ldots, a_{q-1}, a_{q+1}, \ldots, a_j]$. By the above remark and Theorem 3.17, we have

$$M(\overline{c}) = [(m + 1)/2].$$

If m is odd then using Lemma 3.13 we may assume that this component has been labelled so that

$$\text{neg } [a_i, \ldots, a_{q-1}, a_{q+1}, \ldots, a_j] = [(m + 1)/2]$$

and moreover $a_{q-1} > 0$, $a_{q+1} < 0$. Since in this case the roots $\eta(a_{q-1}, a_q)$, $\eta(a_q, a_{q+1})$ and $\eta(a_{q-1}, a_{q+1})$ are all positive, it follows that

$$\text{neg } [a_i, \ldots, a_j] = [(m + 1)/2] = [(m + 2)/2].$$

If m is even then again referring to Lemma 3.13 any labelling of $\overline{c} = [a_i, \ldots, a_{q-1}, a_{q-1}, \ldots, a_j]$ such that

$$\text{neg } (\overline{c}) = [(m + 1)/2]$$

must have a_{q-1} and a_{q+1} being of the same sign. In this case, the set

$$\{\eta(a_{q-1}, a_q), \eta(a_q, a_{q+1})\}$$

contains exactly one negative root whereas $\eta(a_{q-1}, a_{q+1})$, is positive. Therefore,

$$\text{neg}\,[a_i, \dots, a_j] = \lfloor (m + 1)/2 \rfloor + 1 = \lfloor (m + 2)/2 \rfloor.$$

THEOREM 5.10. *Let $[a_1, \dots, a_k] = CR(c)$ for a basic cycle $c \in C(B_n)$ and assume that $a_q = -a_1$ for some q. If $CR(c)$ has N non-trivial components and the i^{th} component contains a minimum of Q_i negative roots then*

$$M(c) = \sum_{i=1}^{N} Q_i - (N - 1).$$

Proof. It is clear that

$$M(c) \geqq \sum_{i=1}^{N} Q_i - (N - 1).$$

Thus it suffices to show that there exists $\sigma \in W(B_n)$ such that

$$\text{neg}\,\sigma(c) = \sum_{i=1}^{N} Q_i - (N - 1).$$

If $N = 1$ then the result follows from the previous lemma. Assume now that $N \geqq 2$ and let $[a_i, \dots, a_j = -a_i]$ be the ν^{th} non-trivial component which does not contain $-a_1$. By induction we may assume that $[a_1, \dots, a_i, a_j, \dots, a_k]$ has been labelled so that

$$\text{neg}\,[a_1, \dots, a_i, a_j, \dots, a_k] = \sum_{i \neq \nu} Q_i - (N - 2).$$

Using the same techniques as in Theorem 3.14 we can label $[a_1, \dots, a_k]$ such that

$$\text{neg}\,[a_1, \dots, a_k] = \sum_{i=1}^{N} Q_i - (N - 1).$$

THEOREM 5.11. $M(B_n) = \lfloor (n + 2)/3 \rfloor$.

Proof. Clearly the result is true for $2 \leqq n \leqq 6$. To see that

$$M(B_n) \geqq \lfloor (n + 2)/3 \rfloor \quad \text{for } n \geqq 7$$

construct a circle representation corresponding to the following diagram

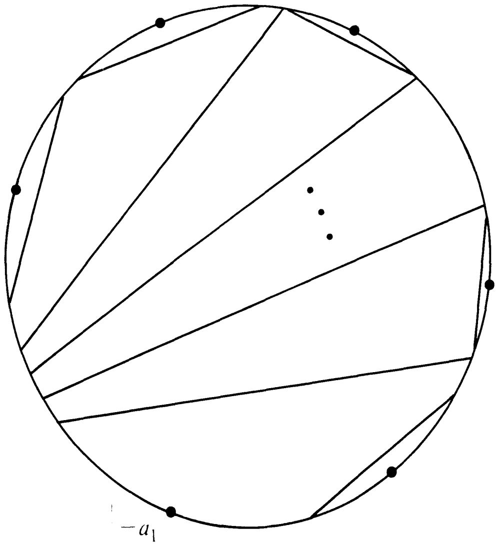

Figure 5.12

where the line segments are joining paired values and the dots indicate singular values. By Theorem 3.16, the minimum number of roots associated with this diagram is exactly the same as the minimum number associated with Figure 4.1, $[(n + 2)/3]$. This means

$$M(B_n) \geqq [(n + 2)/3].$$

Assume $M(B_n) = M > [(n + 2)/3]$ and c is a basic cycle in $C(B_n)$ with $M(c) = M$. Then $CR(c)$ must have $-a_1$ appearing, otherwise c is equivalent to a basic cycle in $C(D_n)$, contrary to Theorem 4.2. Let $[a_j, \ldots, a_i = -a_1, \ldots, a_k]$ be the component of $CR(c)$ containing $-a_1$ and suppose it has m paired values. Then this component contributes $[(m + 2)/2]$ to the sum equaling M. We now replace this component by a new sequence $[a_j, \ldots, a_{i-1}, n + 1, n + 2, -(n + 1), a_{i+1}, \ldots, a_k]$. This substitution produces a basic cycle c' in $C(D_{n+2})$ such that $M(c') = M$, contrary to Theorem 4.2.

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