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GENERALIZED FRACTIONAL LÉVY PROCESSES WITH FRACTIONAL BROWNIAN MOTION LIMIT

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Abstract

Fractional Lévy processes generalize fractional Brownian motion in a natural way. We go a step further and extend the usual fractional Riemann–Liouville kernel to a regularly varying function. We call the resulting stochastic processes generalized fractional Lévy processes (GFLPs) and show that they may have short or long memory increments and that their sample paths may have jumps or not. Moreover, we define stochastic integrals with respect to a GFLP and investigate their second-order structure and sample path properties. A specific example is the Ornstein–Uhlenbeck process driven by a time-scaled GFLP. We prove a functional central limit theorem for such scaled processes with a fractional Ornstein–Uhlenbeck process as a limit process. This approximation applies to a wide class of stochastic volatility models, which include models where possibly neither the data nor the latent volatility process are semimartingales.

Keywords: Shot-noise process; fractional Brownian motion; fractional Lévy process; generalized fractional Lévy process; fractional Ornstein–Uhlenbeck process; functional central limit theorem; regular variation; stochastic volatility model

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1. Introduction

This paper contributes to current discussions in various areas of applications, where high-frequency and unequally spaced data lead to continuous-time modeling. This applies in particular to financial and computer network traffic data, but also to environmental and climate data, where remote sensing, satellite, and/or radar data have become available.

Practitioners, engineers, and scientists observe different characteristics in such data. In particular, we have to distinguish Gaussian and non-Gaussian distributions (specifically heavy tails), no jumps or jumps, which are triggered by market forces or discontinuities in physical processes, short and long memory of various origin, as well as stochastic variability (volatility) observed in high-frequency measurements. We shall define a new class of models, which allows for flexible modeling of the three essential properties: distributions, memory, and jump behavior.

All stochastic objects used in this paper are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$, which satisfies the usual conditions of completeness and right continuity of the filtration. Recall from Marquardt (2006) that a fractional Lévy process (FLP) on $\mathbb{R}$ has

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the representation

$$S(t)=\int_{\mathbb{R}}\{(t-x)_+^{H-1/2}-(-x)_+^{H-1/2}\}\,\mathrm{d}L(x),\qquad t\in\mathbb{R},\tag{1.1}$$

where $u_+=\max(u,0)$, $H\in(0,1)$, and L is a two-sided Lévy process. For L being Brownian motion (BM) this process defines fractional Brownian motion (FBM) denoted by B^H and has been studied extensively. We extend the class of processes (1.1) to

$$S(t)=\int\{g(t-x)-g(-x)\}\,\mathrm{d}L(x),\qquad t\in\mathbb{R}\tag{1.2}$$

for appropriate functions g and call S a *generalized fractional Lévy process* (GFLP). The class of functions g is determined such that $S(t)$ exists for all $t\in\mathbb{R}$.

As a classic approach, short range dependence models are integrated over a fractional kernel, thus obtaining long memory versions of such processes. This applies in particular for processes driven by BM; see Comte *et al.* (2012) and the references therein.

A different approach modifies the driving BM to an FBM, thus obtaining stochastic differential equations driven by an FBM; see Buchmann and Klüppelberg (2006) and Zähle (1998). It is then a natural step to extend an FBM to FLPs providing more flexible distributions and tail behavior than Gaussian processes, retaining the long memory increments. This implies immediately that Ornstein–Uhlenbeck (OU) processes driven by an FLP constitute a rich distributional class with long memory (cf. Marquardt (2006)). They have been extended to general stochastic differential equations driven by an FLP by, e.g. Fink and Klüppelberg (2011). All these processes are long memory models, and they all have continuous sample paths.

On the other hand, OU processes driven by a Lévy process provide, besides flexible distributions, both continuous sample paths (when driven by BM) and sample paths with jumps (when driven by a Lévy process with jumps). In recent years substantial research focused on Lévy-driven models with mostly short memory, exemplified in Barndorff-Nielsen and Shephard (2001) and in Klüppelberg *et al.* (2004). However, all these processes have exponential autocovariances; hence, short memory.

Certain models, which give more flexibility for distributions and memory have been considered; for instance, continuous-time autoregressive moving average (CARMA) models (see Brockwell and Lindner (2009) and the references therein) extend the class of Lévy-driven OU processes. Although they allow for more flexible autocovariance functions than simple exponentials, they are restricted to short range dependence modeling. Long range dependent models like the fractionally integrated continuous-time autoregressive moving average (FICARMA) (Brockwell and Marquardt (2005)) or the infinite factor supOU process by Barndorff-Nielsen (2001) have been suggested. However, FICARMA processes have again continuous sample paths, and the supOU process is a rather complex model.

As a result, we note a lack of stochastic models, which have flexible memory, flexible jump behavior, and interpolate between algebraic and exponential decay of their autocovariance functions. In the light of these facts, we first propose a GFLP S as defined in (1.2), which contributes via its kernel more flexible models to the discussion. We calculate its second-order structure explicitly. Moreover, we show that S can exhibit both short memory increments (with exponentially or fast polynomially decreasing autocovariances) and long memory increments (with slow polynomially decreasing autocovariances). We investigate the sample path behavior, where we show that S has a càdlàg version and can have continuous paths or jumps.

In the next step we investigate models driven by a GFLP S. Here we focus on OU processes driven by S and calculate their second-order structure and finite-dimensional distributions via the characteristic function (ch.f.).

Furthermore, we show that OU processes driven by a time-scaled GFLP converge (in a functional sense) to an FBM-driven OU process, extending previous work (see Klüppelberg and Kühn (2004) and Klüppelberg and Mikosch (1995)) in a nontrivial way.

As a prominent application we consider stochastic volatility models, where the volatility is given by an OU process driven by a time-scaled GFLP, which can cope with the required properties of volatilities (long memory, sample paths, and distributional tail behavior). Since time-scaled versions converge to an FBM-driven OU process, by proper scaling the model adjusts smoothness of the sample paths and closeness to Gaussian distributions, allowing for long memory. Finally, we prove a bivariate functional convergence result for both the data equation and the latent volatility processes.

The paper is organized as follows. In Section 2 we define the GFLP S. In Section 3 we extend the classic Riemann–Liouville fractional integrals by allowing for more general kernel functions. For a fixed kernel function we determine the class $\mathcal{H}$ of integrands such that the integral with respect to S exists. We present some analytic results for this integral. If the kernel function is positive (or negative) on $\mathbb{R}_+ := [0, \infty)$ the isometry between the two inner product spaces $L^2(\Omega)$ and $\mathcal{H}$ is presented, giving the second-order structure of S. As a prominent example we consider the OU process driven by a GFLP and prove functional convergence of scaled versions to a fractional (Gaussian) OU process. In Section 5 we apply our results to stochastic volatility models, proving joint weak convergence of the data process (driven by BM or FBM) and the volatility process in the Skorokhod space $D(\mathbb{R}_+^2)$.

2. Generalized FLPs

Throughout this paper we work with a two-sided Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$ constructed by taking two independent copies $L_1 = \{L_1(t)\}_{t\geq 0}$ and $L_2 = \{L_2(t)\}_{t\geq 0}$ of a Lévy process and setting $L(t) := L_1(t)\mathbf{1}_{[0,\infty)}(t) - L_2((-t)-)\,\mathbf{1}_{(-\infty,0)}(t)$, where $\mathbf{1}$ is the indicator function. Moreover, we assume that L is centered without a Gaussian component and that the Lévy measure ν satisfies $\int_{|x|>1} x^2\nu(\mathrm{d}x) < \infty$, i.e. $\mathbb{E}[(L(t))^2] = t\mathbb{E}[(L(1))^2] = t\int_{\mathbb{R}} x^2\nu(\mathrm{d}x) < \infty$ for all $t \in \mathbb{R}$. The distribution of L is uniquely defined by the ch.f. $\mathbb{E}[\exp\{\mathrm{i}\theta L(t)\}] = \exp\{t\psi(\theta)\}$ for $t \geq 0$, where

$$\psi(\theta) = \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x)\nu(\mathrm{d}x), \qquad \theta \in \mathbb{R}. \tag{2.1}$$

For more details on Lévy processes we refer the reader to the excellent monograph of Sato (1999).

The following result is known and we recall it for later reference. It can be found in Proposition 2.1 and Theorem 3.5 of Marquardt (2006) and, in a more general version, in Rajput and Rosinski (1989).

Proposition 2.1. *Let L be a Lévy process. Assume that $\mathbb{E}[L(1)] = 0$ and $\mathbb{E}[(L(1))^2] < \infty$. For $t \in \mathbb{R}$ let $f_t \in L^2(\mathbb{R})$. Then the integral $S(t) := \int_{\mathbb{R}} f_t(u)\,\mathrm{d}L(u)$ exists in the $L^2(\Omega)$ sense. Furthermore, for $s, t \in \mathbb{R}$ we obtain $\mathbb{E}[S(t)] = 0$, and the isometry*

$$\mathbb{E}[(S(t))^2] = \mathbb{E}[(L(1))^2]\|f_t(\cdot)\|^2_{L^2(\mathbb{R})} \tag{2.2}$$

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holds, and

$$\widetilde{\Gamma}(s,t) = \operatorname{cov}(S(s), S(t)) = \mathbb{E}[(L(1))^2] \int_{\mathbb{R}} f_s(u) f_t(u)\,\mathrm{d}u. \tag{2.3}$$

Moreover, the ch.f. of $S(t_1), \ldots, S(t_m)$ *for* $t_1 < \cdots < t_m$ *is given by*

$$\mathbb{E}\bigg[\exp\bigg\{\sum_{j=1}^{m} \mathrm{i}\theta_j S(t_j)\bigg\}\bigg] = \exp\bigg\{\int_{\mathbb{R}} \psi\bigg(\sum_{j=1}^{m} \theta_j f_{t_j}(s)\bigg)\,\mathrm{d}s\bigg\} \quad \text{for } \theta_j \in \mathbb{R},\ j = 1, \ldots, m,$$

where ψ *is given in (2.1).*

We now define a generalized fractional Lévy process.

Definition 2.1. Let L be a Lévy process with $\mathbb{E}[L(1)] = 0$ and $\mathbb{E}[(L(1))^2] < \infty$. Let $g\colon \mathbb{R} \to \mathbb{R}$ with $g(t) = 0$ for $t < 0$ and such that $\int_{\mathbb{R}} (g(t-s) - g(-s))^2\,\mathrm{d}s < \infty$ for all $t \in \mathbb{R}$. The stochastic process $S = \{S(t)\}_{t\in\mathbb{R}}$ defined by

$$S(t) = \int_{\mathbb{R}} \{g(t-u) - g(-u)\}\,\mathrm{d}L(u), \qquad t \in \mathbb{R} \tag{2.4}$$

is called the *generalized fractional Lévy process (GFLP)*.

The process S has stationary increments and is symmetric with $S(0) = 0$, i.e. $S(-t) \stackrel{\mathrm{D}}{=} -S(t),\ t \geq 0$, where '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution. By taking $g(u) = u_+^{H-1/2}$ we obtain an FLP.

The integral (2.4) obviously exists in the $L^2(\Omega)$ sense. In what follows we formulate assumptions on g needed for the existence of a stochastic integral with respect to S considered in Section 3 or for the existence of a functional limit of a scaled family of such processes in Section 4.

Assumption 2.1. *The function* $g\colon \mathbb{R} \to \mathbb{R}$ *satisfies* $g(t) = 0$ *for* $t < 0$ *and is continuously twice differentiable on* $(0, \infty)$*, the limit* $\lim_{u\downarrow 0} |g'(u)|$ *exists and is finite, and* $g''(u) = O(u^{-3/2-\varepsilon})$ *as* $u \to \infty$ *for sufficiently small* $\varepsilon > 0$.

We assume that Assumption 2.1 holds throughout this paper. We start with some sample path properties of a GFLP.

Lemma 2.1. *Let* L *be a Lévy process with* $\mathbb{E}[L(1)] = 0$ *and* $\mathbb{E}[(L(1))^2] < \infty$. *Under Assumption 2.1 on* g*, the GFLP* S *has a càdlàg version. Moreover,* S *has jumps if and only if* $g(0) \neq 0$.

Proof. We let $t > 0$ without loss of generality (w.l.o.g.) since the proof is analogous for $t \leq 0$. Write

$$S(t) = \int_0^t g(t-u)\,\mathrm{d}L(u) + \int_{-\infty}^0 \{g(t-u) - g(-u)\}\,\mathrm{d}L(u) =: S_1(t) + S_2(t).$$

Our assumption on the Lévy process implies the laws of the iterated logarithms (LILs) (see Sato (1999, Propositions 47.11 and 48.9)), giving almost surely (a.s.)

$$\limsup_{t\downarrow 0} \frac{|L(t)|}{(2t \log\log(1/t))^{1/2}} = 0, \qquad \limsup_{t\to\infty} \frac{|L(t)|}{(2t \log\log t)^{1/2}} = (\mathbb{E}[(L(1))^2])^{1/2}.$$

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We use the fact that if f is continuously differentiable,

$$\int_a^b f(s)\,\mathrm{d}L(s) = f(b)L(b) - f(a)L(a) - \int_a^b L(s)\,\mathrm{d}f(s)$$

holds (see Lemma 2.1 of Eberlein and Raible (1999): for fixed ω the sets of jumps of L are an at most countable Lebesgue null set). This, together with the LIL at the origin and the assumptions on g, yields

$$\begin{aligned} S_1(t) &= \int_0^t g(t-u)\,\mathrm{d}L(u) \\ &= g(0)L(t) - \lim_{s\downarrow 0} g(t-s)L(s) + \int_0^t L(u)g'(t-u)\,\mathrm{d}u \\ &= g(0)L(t) + \int_0^t L(u)g'(t-u)\,\mathrm{d}u, \end{aligned}$$

whereas this, together with the LIL at ∞, yields

$$\begin{aligned} S_2(t) &= \lim_{s\downarrow -\infty} \{g(t-s) - g(-s)\}L(s) + \lim_{s\downarrow -\infty} \int_s^0 \{g'(t-u) - g'(-u)\}L(u)\,\mathrm{d}u \\ &= \int_{-\infty}^0 \{g'(t-u) - g'(-u)\}L(u)\,\mathrm{d}u. \end{aligned}$$

As for the expression of S_2, we apply the dominated convergence theorem to

$$S_2(t) - S_2(s) = \int_{-\infty}^0 \{g'(t-u) - g'(s-u)\}L(u)\,\mathrm{d}u$$

to observe $\lim_{t\to s} |S_2(t) - S_2(s)| = 0$. Hence, S_2 is a.s. continuous. Similarly, the integral term of S_1 is continuous. Since L is càdlàg without drift and Gaussian components S_1 and, hence, S have jumps if and only if $g(0) \neq 0$.

Generalized fractional Lévy processes can exhibit both short and long memory increments. By Proposition 2.1, when w.l.o.g. $\mathbb{E}[(L(1))^2] = 1$, the covariance function of the increments has the form for $t, s, h > 0$,

$$\begin{aligned} \gamma(t,h) &= \mathbb{E}[\{S(t+s+h) - S(t+s)\}\{S(s+h) - S(s)\}] \\ &= \int_{-\infty}^h \{g(t+h-u) - g(t-u)\}\{g(h-u) - g(-u)\}\,\mathrm{d}u. \end{aligned} \tag{2.5}$$

Definition 2.2. Assume that a GFLP S has covariance function $\gamma(\cdot, h)$ for fixed lag $h > 0$. If $\int_0^\infty |\gamma(t,h)|\,\mathrm{d}t < \infty$ then S is said to have *short memory increments*. If $\int_0^\infty |\gamma(t,h)|\,\mathrm{d}t = \infty$ then S is said to have *long memory increments*.

Example 2.1. Assume that $\gamma(t,h)$ is continuous and that $\gamma(t,h) \sim Ct^{-\beta}$ as $t \to \infty$ with $C, \beta > 0$ for all $h > 0$. If $\beta \le 1$ then S has long memory increments, whereas if $\beta > 1$ then S has short memory increments.

Whether S has long or short memory increments depends on the asymptotic behavior of g.

Lemma 2.2. *Let $0 < \alpha < \frac{1}{2}$ and $c > 0$. Assume that $g(x) = cx^\alpha$ for $x \ge M > 0$. Then S has long memory increments.*

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Proof. Set w.l.o.g. $c = 1$. Write $\gamma(t, h) = \gamma_1(t, h) + \gamma_2(t, h)$ with

$$\gamma_1(t, h) := \int_{-M}^{h} \{g(t + h - u) - g(t - u)\}\{g(h - u) - g(-u)\}\,\mathrm{d}u,$$
$$\gamma_2(t, h) := \int_{-\infty}^{-M/t} \{g(t + h - tv) - g(t - tv)\}\{g(h - tv) - g(-tv)\}t\,\mathrm{d}v.$$

By the mean value theorem for $x \geq M$ and $y > 0$, we have

$$g(y + x) - g(x) = (y + x)^{\alpha} - x^{\alpha} = \alpha(x + \theta y)^{\alpha-1} y,$$

where the parameter $0 < \theta < 1$ depends on both x and y. We apply this mean value theorem to both γ_1 and γ_2 and observe that, for $\theta = \theta(t, h, u) \in (0, 1)$,

$$\begin{aligned}\gamma_1(t, h) &= \alpha t^{\alpha-1} h \int_{-M}^{h} \left(1 - \frac{u}{t} + \frac{\theta h}{t}\right)^{\alpha-1} \{g(h - u) - g(-u)\}\,\mathrm{d}u \\ &\sim \alpha t^{\alpha-1} h \int_{-M}^{h} \{g(h - u) - g(-u)\}\,\mathrm{d}u, \qquad t \to \infty\end{aligned}$$

and, similarly,

$$\gamma_2(t, h) \sim \alpha^2 h^2 t^{2\alpha-1} \int_{-\infty}^{0} (1 - v)^{\alpha-1}(-v)^{\alpha-1}\,\mathrm{d}v = \alpha^2 h^2 t^{2\alpha-1} B(\alpha, 1 - 2\alpha), \qquad t \to \infty,$$

where we have used the dominated convergence theorem. Hence, $\gamma(t, h) \sim C t^{2\alpha-1}$ as $t \to \infty$ with $0 < \alpha < \frac{1}{2}$.

Example 2.2. (i) Let $g(x) = \mathrm{e}^{-\lambda x} \mathbf{1}_{\{x \geq 0\}}$. Then S is a Lévy OU process whose properties are well known. The process has short memory increments, since (2.5) gives $\gamma(t, h) = \mathrm{e}^{-\lambda t} \int_{-\infty}^{h} (\mathrm{e}^{-\lambda(h-u)} - \mathrm{e}^{-\lambda(-u)})^2\,\mathrm{d}u$ for $t, h > 0$. Moreover, the sample paths of S exhibit jumps, since $g(0) \neq 0$.

(ii) Let $g(x) = x^{\alpha}$ with $0 < \alpha < \frac{1}{2}$ for $x \geq M$ and some $M > 0$. Then the sample paths of S can have jumps or not, depending on the behavior of g in 0, while S has long memory increments by Lemma 2.2.

(iii) Consider $g(x) = 1/(\alpha + \lambda x)^{\beta} \mathbf{1}_{\{x \geq 0\}}$ with parameters $\alpha, \lambda \geq 0$ and $\beta > -\frac{1}{2}$, $\beta \neq 0$ as in Gander and Stephens (2007, p. 635), where they use this function g for stochastic volatility models driven by Lévy processes. Then the sample paths of S have jumps and S can exhibit short or long memory increments, depending on β.

Remark 2.1. (i) Like an FLP, the GFLP S has stationary increments and $S(0) = 0$ holds. Moreover, it inherits the symmetry from the driving Lévy process, i.e. $S(-t) \overset{\mathrm{D}}{=} -S(t)$ for $t \geq 0$. A novelty of GFLPs is that they combine processes that can have jumps without having independent increments, and without losing the symmetry or the stationary increments. Moreover, while fractional Lévy processes always exhibit long memory behavior, the class of GFLPs can model both short and long memory.

(ii) Continuity of $\int_0^t L(u) g'(t - u)\,\mathrm{d}u$ in S_1 and S_2 as proved in Lemma 2.1 also follows from the Kolmogorov–Čhentsov theorem.

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3. Stochastic integrals with respect to a GFLP

Recall that the Riemann–Liouville fractional integrals $I_{\pm}^{\alpha}$ are defined for $\alpha \in (0, 1)$ by

$$(I_{-}^{\alpha}h)(u) = \frac{1}{\Gamma(\alpha)} \int_{u}^{\infty} h(t)(t-u)^{\alpha-1}\,\mathrm{d}t, \qquad (I_{+}^{\alpha}h)(u) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{u} h(t)(u-t)^{\alpha-1}\,\mathrm{d}t$$

for functions $h\colon \mathbb{R} \to \mathbb{R}$, provided that the integrals exist for almost all $u \in \mathbb{R}$. For details, see, e.g. Samko *et al.* (1993).

As a motivation for what follows, note that for $t > 0$,

$$g(t-u) - g(-u) = \int_{\mathbb{R}} \mathbf{1}_{(0,t]}(v) g'(v-u)\,\mathrm{d}v.$$

We use now g' for an extension of the classical Riemann–Liouville kernel function and define for appropriate functions h,

$$(I_{-}^{g}h)(u) := \int_{u}^{\infty} h(v)g'(v-u)\,\mathrm{d}v = \int_{\mathbb{R}} h(v)g'(v-u)\,\mathrm{d}v. \tag{3.1}$$

In what follows we assume that S is a GFLP driven by a Lévy process L as in Definition 2.1. Starting from the fact that

$$S(t) = \int_{\mathbb{R}} (I_{-}^{g}\mathbf{1}_{(0,t]})(x)L(\mathrm{d}x), \qquad t \in \mathbb{R}, \tag{3.2}$$

we shall define a stochastic integral for a function h in a similar way as in Marquardt (2006, Section 5). Since g' is continuous on $(0, \infty)$ by Assumption 2.1, the integral (3.2) is well defined as

$$(I_{-}^{g}\mathbf{1}_{(0,t]})(x) = -\int_{t}^{0} g'(v-x)\,\mathrm{d}v = g(t-x) - g(-x).$$

For a fixed function g as above, define

$$\widetilde{\mathcal{H}} := \left\{ h\colon \mathbb{R} \to \mathbb{R}_{+} \colon \int_{\mathbb{R}} (I_{-}^{g}h)^2(u)\,\mathrm{d}u < \infty \right\},$$

where $I_{-}^{g}h$ is as in (3.1). The proof of the following result is analogous to that of Marquardt (2006, Proposition 5.1).

Proposition 3.1. *Suppose that g satisfies Assumption 2.1 and that for its derivative g',*

$$\int_{0}^{1} |g'(s)|\,\mathrm{d}s + \int_{1}^{\infty} (g'(s))^2\,\mathrm{d}s < \infty$$

also holds. If $h\colon \mathbb{R} \to \mathbb{R}_{+}$ satisfies $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $h \in \widetilde{\mathcal{H}}$.

Proof. Starting from the fact that $I_{-}^{g}h \in L^2(\mathbb{R})$ if and only if

$$\left| \int_{\mathbb{R}} \varphi(u)(I_{-}^{g}h)(u)\,\mathrm{d}u \right| \le C\|\varphi\|_{L^2(\mathbb{R})}$$

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for all $\varphi \in L^2(\mathbb{R})$ for some $C > 0$, it suffices to show that for all $\varphi \in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} \int_0^{\infty} |\varphi(u) g'(s) h(s+u)| \, \mathrm{d}s \, \mathrm{d}u \le C \|\varphi\|_{L^2(\mathbb{R})}.$$

This holds if

$$I_1 = \int_{\mathbb{R}} \int_0^{1} |\varphi(u) g'(s) h(s+u)| \, \mathrm{d}s \, \mathrm{d}u \le C \|\varphi\|_{L^2(\mathbb{R})},$$
$$I_2 = \int_{\mathbb{R}} \int_1^{\infty} |\varphi(u) g'(s) h(s+u)| \, \mathrm{d}s \, \mathrm{d}u \le C \|\varphi\|_{L^2(\mathbb{R})}.$$

Applying Fubini's theorem and the Hölder inequality, we obtain

$$I_1 = \int_0^1 |g'(s)| \int_{\mathbb{R}} |\varphi(u) h(s+u)| \, \mathrm{d}u \, \mathrm{d}s \le \|\varphi\|_{L^2(\mathbb{R})} \|h\|_{L^2(\mathbb{R})} \int_0^1 |g'(s)| \, \mathrm{d}s < \infty.$$

Furthermore, setting $t = s + u$ and using again Fubini's theorem and the Hölder inequality,

$$\begin{aligned} I_2 &= \int_{\mathbb{R}} |h(t)| \int_1^{\infty} |\varphi(t-s) g'(s)| \, \mathrm{d}s \, \mathrm{d}t \\ &\le \int_{\mathbb{R}} \|\varphi\|_{L^2(\mathbb{R})} \bigg(\int_1^{\infty} (g'(s))^2 \, \mathrm{d}s \bigg)^{1/2} |h(t)| \, \mathrm{d}t \\ &\le \|\varphi\|_{L^2(\mathbb{R})} \|h\|_{L^1(\mathbb{R})} \bigg(\int_1^{\infty} (g'(s))^2 \, \mathrm{d}s \bigg)^{1/2} \\ &< \infty. \end{aligned}$$

Based on Proposition 3.1 we can equip the space of functions $\mathcal{H}^0 := \{h \colon \mathbb{R} \to \mathbb{R}_+ \colon h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\}$ with the norm

$$\|h\|_{\mathcal{H}} := \bigg(\mathbb{E}[(L(1))^2] \int_{\mathbb{R}} (I_-^g h)^2(u) \, \mathrm{d}u \bigg)^{1/2},$$

and define the space $\mathcal{H}$ as the completion of $\mathcal{H}^0$ with respect to this norm.

We shall need an additional condition on g, given in the following assumption.

Assumption 3.1. *In addition to Assumption 2.1, assume that g is monotone on $(0, \infty)$, i.e. $g' > 0$ or $g' < 0$ on $(0, \infty)$. We call g' a* kernel function.

Assumption 3.1 implies that the sign of $g'h$ is fixed on the whole of $\mathbb{R}$ and, thus, $\|\cdot\|_{\mathcal{H}}$ defines in fact a norm. For more details on such spaces for the classical Riemann–Liouville kernel, see Pipiras and Taqqu (2000).

From the proof of Proposition 3.1, it follows immediately that for $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\|h\|_{\mathcal{H}} \le C(\|h\|_{L^1(\mathbb{R})} + \|h\|_{L^2(\mathbb{R})}).$$

Next we define a stochastic integral with integrator S, which gives the correspondence between the space $\mathcal{H}$ and that of stochastic integrals in $L^2(\Omega)$.

Theorem 3.1. *Let S be the GFLP as defined in Definition 2.1 and suppose that g' satisfies Assumption 3.1. Let $h \in \mathcal{H}$. Then the left-hand side integral is defined in the $L^2(\Omega)$ sense and it holds that*

$$\int_{\mathbb{R}} h(u)\,\mathrm{d}S(u) = \int_{\mathbb{R}} (I_-^g h)(u)\,\mathrm{d}L(u). \tag{3.3}$$

Moreover, the following isometry holds:

$$\left\| \int_{\mathbb{R}} h(u)\,\mathrm{d}S(u) \right\|^2_{L^2(\Omega)} = \|h\|^2_{\mathcal{H}}.$$

Proof. We assume w.l.o.g. that $\mathbb{E}[(L(1))^2] = 1$. To construct the integral $\int_{\mathbb{R}} h(t)\,\mathrm{d}S(t)$ for $h \in \mathcal{H}$ we proceed as usual. For the indicator function $\varphi(\cdot) = \mathbf{1}_{(0,t]}(\cdot)$ for $t > 0$, we calculate

$$\int_{\mathbb{R}} \varphi(u)\,\mathrm{d}S(u) = \int_{\mathbb{R}} \mathbf{1}_{(0,t]}(u)\,\mathrm{d}S(u) = S(t)$$

and for the right-hand side of (3.3), we obtain

$$\begin{aligned}\int_{\mathbb{R}} (I_-^g \varphi)(u)\,\mathrm{d}L(u) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{(0,t]}(s) g'(s-u)\,\mathrm{d}s\,\mathrm{d}L(u) \\ &= \int_{\mathbb{R}} (g(t-u) - g(-u))\,\mathrm{d}L(u) \\ &= S(t).\end{aligned}$$

Let $\varphi\colon \mathbb{R} \to \mathbb{R}_+$ be a step function, i.e. $\varphi(t) = \sum_{i=1}^{n-1} a_i \mathbf{1}_{(t_i,t_{i+1}]}(t)$, where $a_i \in \mathbb{R}_+$ for $i = 1, \ldots, n-1$ and $-\infty < t_1 < \cdots < t_n < \infty$. Note that $\varphi \in \mathcal{H}$. Define

$$\int_{\mathbb{R}} \varphi(t)\,\mathrm{d}S(t) = \sum_{i=1}^{n-1} a_i (S(t_{i+1}) - S(t_i)).$$

Then the right-hand side of (3.3) can be expressed as

$$\begin{aligned}\int_{\mathbb{R}} (I_-^g \varphi)(u)\,\mathrm{d}L(u) &= \iint \sum_{j=1}^{n-1} a_j \mathbf{1}_{(t_j,t_{j+1}]}(s) g'(s-u)\,\mathrm{d}s\,\mathrm{d}L(u) \\ &= \int \sum_{j=1}^{n-1} a_j \int_{t_j}^{t_{j+1}} g'(s-u)\,\mathrm{d}s\,\mathrm{d}L(u) \\ &= \sum_{j=1}^{n-1} a_j (S(t_{j+1}) - S(t_j)).\end{aligned}$$

Moreover, for all step functions φ, from (2.2), it follows that

$$\begin{aligned}\left\| \int_{\mathbb{R}} \varphi(u)\,\mathrm{d}S(u) \right\|^2_{L^2(\Omega)} &= \mathbb{E}\Big[\Big(\int_{\mathbb{R}} (I_-^g \varphi)(u)\,\mathrm{d}L(u)\Big)^2\Big] \\ &= \int_{\mathbb{R}} (I_-^g \varphi)^2(u)\,\mathrm{d}u \\ &= \|\varphi\|^2_{\mathcal{H}}.\end{aligned} \tag{3.4}$$

Since the nonnegative step functions are dense in $\mathcal{H}$, there exists a sequence $(\varphi_k)_{k\in\mathbb{N}}$ of such functions such that $\|\varphi_k - h\|_{\mathcal{H}} \to 0$ as $k \to \infty$. It follows from the isometry property (3.4) that the integrals converge in $L^2(\Omega)$ and the isometry property is preserved when going to the limit. Finally, (3.4) implies that the integral $\int_{\mathbb{R}} h(t)\,\mathrm{d}S(t)$ is the same for all sequences of step functions converging to h.

The second-order properties of integrals, which are driven by GFLPs can be calculated directly. It is useful to observe that $L^2(\Omega)$ and $\mathcal{H}$ are inner product spaces with the inner products given for $h_1, h_2 \in \mathcal{H}$ by

$$\left\langle \int_{\mathbb{R}} h_1(u)\,\mathrm{d}S(u), \int_{\mathbb{R}} h_2(u)\,\mathrm{d}S(u) \right\rangle_{L^2(\Omega)} = \langle h_1, h_2\rangle_{\mathcal{H}}.$$

The inner product in $L^2(\Omega)$ is the covariance, whereas an interpretation of the inner product in $\mathcal{H}$ can be found in the next proposition.

Proposition 3.2. *Let S be the GFLP as in Definition 2.1 and suppose that g' satisfies Assumption 3.1. Let $h_1, h_2 \in \mathcal{H}$. Then*

$$\operatorname{cov}\left[\int_{\mathbb{R}} h_1(v)\,\mathrm{d}S(v), \int_{\mathbb{R}} h_2(u)\,\mathrm{d}S(u)\right] = \int_{\mathbb{R}}\int_{\mathbb{R}} h_1(u)h_2(v)\Gamma(u,v)\,\mathrm{d}u\,\mathrm{d}v,$$

where

$$\Gamma(u,v) = \frac{\partial^2 \operatorname{cov}[S(u), S(v)]}{\partial u \partial v} = \mathbb{E}[(L(1))^2]\int_{\mathbb{R}} g'(u-w)g'(v-w)\,\mathrm{d}w. \tag{3.5}$$

In particular,

$$\langle h_1, h_2\rangle_{\mathcal{H}} = \mathbb{E}[(L(1))^2]\int_{\mathbb{R}}\int_{\mathbb{R}} h_1(u)h_2(v)\int_{\mathbb{R}} g'(u-w)g'(v-w)\,\mathrm{d}w\,\mathrm{d}u\,\mathrm{d}v.$$

Proof. Set w.l.o.g. $\mathbb{E}[(L(1))^2] = 1$. It suffices to prove the identities for the indicator functions $h_1 = \mathbf{1}_{(0,s]}$ and $h_2 = \mathbf{1}_{(0,t]}$ for $0 < s < t$. For $s < 0$ or $t < 0$, we use the stationarity of the increments and the symmetry of S to obtain

$$\begin{aligned}
\operatorname{var}[S(t)] &= \|S(t)\|^2_{L^2(\Omega)} \\
&= \int (g(t-u) - g(-u))^2\,\mathrm{d}u \\
&= \int_{\mathbb{R}}\left(\int_u^\infty \mathbf{1}_{(0,t]}(v)g'(v-u)\,\mathrm{d}v\right)^2\mathrm{d}u \\
&= \|\mathbf{1}_{(0,t]}\|^2_{\mathcal{H}},
\end{aligned}$$

$$\begin{aligned}
\operatorname{cov}[S(s), S(t)] &= \langle S(s), S(t)\rangle_{L^2(\Omega)} \\
&= \int_{\mathbb{R}} \{g(s-w) - g(-w)\}\{g(t-w) - g(-w)\}\,\mathrm{d}w \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{(0,s]}(u)\mathbf{1}_{(0,t]}(v)\int_{\mathbb{R}} g'(v-w)g'(u-w)\,\mathrm{d}w\,\mathrm{d}u\,\mathrm{d}v \\
&= \langle \mathbf{1}_{(0,s]}, \mathbf{1}_{(0,t]}\rangle_{\mathcal{H}},
\end{aligned}$$

where we have used Fubini's theorem for the second to last identity, which is justified by the definition of $\mathcal{H}$.

Remark 3.1. Assumption 3.1 is necessary to guarantee the isometry between the space of stochastic integrals with respect to S and the function space of integrands $\mathcal{H}$, which depend on g. Note, however, that the stochastic integral with integrator S can be defined on the larger space $\widetilde{\mathcal{H}}$.

Next we define the OU process driven by a GFLP.

Definition 3.1. Let S be the GFLP as in Definition 2.1 and suppose that g' satisfies Assumption 3.1. Let $\lambda, \gamma > 0$.

(i) For an initial finite random variable $V(0)$, we define an *OU process driven by a GFLP* as

$$V(t) := \mathrm{e}^{-\lambda t}\biggl(V(0) + \gamma \int_0^t \mathrm{e}^{\lambda u}\,\mathrm{d}S(u)\biggr), \qquad t \in \mathbb{R}. \tag{3.6}$$

(ii) If the initial random variable is given by $V(0) = \gamma \int_{-\infty}^0 \mathrm{e}^{\lambda u}\,\mathrm{d}S(u)$ a.s., the OU process driven by a GFLP is stationary and we denote its stationary version by

$$\overline{V}(t) = \gamma \int_{-\infty}^t \mathrm{e}^{-\lambda(t-u)}\,\mathrm{d}S(u), \qquad t \in \mathbb{R}. \tag{3.7}$$

(iii) Recall that when S is replaced by the FBM B^H for $H \in (\frac{1}{2}, 1)$ in (3.6) and (3.7), we obtain the *fractional (Gaussian) Ornstein–Uhlenbeck* (FOU) process (cf. Lemma 2.1 of Cheridito *et al.* (2003) or Pipiras and Taqqu (2000)). We denote the stationary FOU process by $\overline{Y} = \{\overline{Y}(t)\}_{t\in\mathbb{R}}$. It will appear as a limit process in Section 4.

We show the existence of $\overline{V}$ and formulate some properties.

Proposition 3.3. *Let S be the GFLP as in Definition 2.1 and suppose that g' satisfies Assumption 3.1. Let $\lambda > 0$ and set w.l.o.g. $\gamma = 1$. For all $t \in \mathbb{R}$ the stochastic integral*

$$\overline{V}(t) := \int_{-\infty}^t \mathrm{e}^{-\lambda(t-u)}\,\mathrm{d}S(u) = \int_{-\infty}^t (I_-^g \mathrm{e}^{-\lambda(t-\cdot)})(u)\,\mathrm{d}L(u)$$

exists in the $L^2(\Omega)$ sense. Furthermore, for all $s, t \in \mathbb{R}$, we have $\mathbb{E}[\overline{V}(t)] = 0$ and

$$\operatorname{cov}[\overline{V}(s), \overline{V}(t)] = \int_{-\infty}^t \int_{-\infty}^s \mathrm{e}^{-\lambda(t-u)}\mathrm{e}^{-\lambda(s-v)}\Gamma(u,v)\,\mathrm{d}u\,\mathrm{d}v,$$

where Γ is given in (3.5). Moreover, the ch.f. of $\overline{V}(t_1), \dots, \overline{V}(t_m)$ for $t_1 < \cdots < t_m$ is given by

$$\mathbb{E}\biggl[\exp\biggl\{\sum_{j=1}^m \mathrm{i}\theta_j \overline{V}(t_j)\biggr\}\biggr] = \exp\biggl\{\int_{\mathbb{R}} \psi\biggl(\sum_{j=1}^m \theta_j \int_{-\infty}^{t_j} \mathrm{e}^{-\lambda(t_j-v)} g'(v-s)\,\mathrm{d}v\biggr)\,\mathrm{d}s\biggr\},$$

where $\theta_j \in \mathbb{R}$, $j = 1, \dots, m$, and ψ is given in (2.1).

Proof. By Theorem 3.1 and Proposition 3.2 the existence of the integral and the autocovariance function is a consequence of the fact that $\mathrm{e}^{-\lambda(t-\cdot)}\mathbf{1}_{\{t\geq\cdot\}} \in \mathcal{H}$. The ch.f. follows from Proposition 2.1 by observing that the $f_t(s)$ there is replaced by

$$h_t(s) = \int_{-\infty}^t \mathrm{e}^{-\lambda(t-v)} g'(v-s)\,\mathrm{d}v, \qquad s \in \mathbb{R}.$$

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4. Limit theory for OU processes driven by time-scaled GFLPs

Throughout this section we work with a GFLP S as in Definition 2.1 and assume additionally that $\mathbb{E}[(L(1))^2] = 1$. Moreover, we suppose that S has the kernel function g' satisfying Assumption 3.1 so that Theorem 3.1 and Proposition 3.2 apply. For $x > 0$ we denote $\sigma^2(x) := \operatorname{var}[S(x)]$ and define the *time-scaled GFLP* $S_x = \{S_x(t)\}_{t\in\mathbb{R}}$ by

$$S_x(t) := \frac{S(xt)}{\sigma(x)}, \qquad t \in \mathbb{R}. \tag{4.1}$$

Recall the definition of Γ from (3.5) and of $\widetilde{\Gamma}$ from (2.3). Note that the equality in (3.2) carries over to the time-scaled GFLP as follows. For $x > 0$, we have

$$S(xt) = \int \mathbf{1}_{(0,tx]}(v)\,\mathrm{d}S(v) = \int_{\mathbb{R}} (I_-^g \mathbf{1}_{(0,tx]})(u)\,\mathrm{d}L(u), \qquad t \geq 0.$$

Consequently, we can formulate the following Lemma.

Lemma 4.1. *For $x > 0$ let S_x be the time-scaled GFLP as defined in (4.1) and assume that g' satisfies Assumption 3.1. Then for $s, t \in \mathbb{R}$, we have*

$$\widetilde{\Gamma}_x(s,t) := \operatorname{cov}[S_x(s), S_x(t)] = \frac{\operatorname{cov}[S(xs), S(xt)]}{\operatorname{var}[S(x)]} = \frac{\widetilde{\Gamma}(xs, xt)}{\sigma^2(x)} = \frac{\langle \mathbf{1}_{(0,xs]}, \mathbf{1}_{(0,xt]}\rangle_{\mathcal{H}}}{\|\mathbf{1}_{(0,x]}\|^2_{\mathcal{H}}},$$

$$\widetilde{\Gamma}_x(t,t) := \operatorname{var}[S_x(t)] = \frac{\|\mathbf{1}_{(0,xt]}\|^2_{\mathcal{H}}}{\|\mathbf{1}_{(0,x]}\|^2_{\mathcal{H}}},$$

$$\begin{aligned}\Gamma_x(s,t) &= \frac{\partial^2}{\partial s \partial t}\operatorname{cov}[S_x(s), S_x(t)] \\ &= \frac{1}{\sigma^2(x)} \frac{\partial^2}{\partial s \partial t} \operatorname{cov}[S(xs), S(xt)] \\ &= \frac{x^2 \Gamma(xs, xt)}{\sigma^2(x)}.\end{aligned} \tag{4.2}$$

Proof. We prove (4.2) for $t > 0$, the other equations are proved analogously. For $s, t > 0$, we have (for $s < 0$ or $t < 0$ we use the symmetry of S_x)

$$\left\| \int_{\mathbb{R}} \mathbf{1}_{(0,t]}(u)\,\mathrm{d}S_x(u) \right\|^2_{L^2(\Omega)} = \|S_x(t)\|^2_{L^2(\Omega)} = \frac{\|S(xt)\|^2_{L^2(\Omega)}}{\|S(x)\|^2_{L^2(\Omega)}} = \frac{\|\mathbf{1}_{(0,tx]}\|^2_{\mathcal{H}}}{\|\mathbf{1}_{(0,x]}\|^2_{\mathcal{H}}}.$$

Lemma 4.1 provides a general principle by using the same construction of the integral as in Theorem 3.1.

Theorem 4.1. *For $x > 0$ let S_x be the time-scaled GFLP as defined in (4.1) and suppose that g' satisfies Assumption 3.1.*

(i) *Then for $h \in \mathcal{H}$,*

$$\int_{\mathbb{R}} h(u)\,\mathrm{d}S_x(u) = \int_{\mathbb{R}} h^x(u)\,\mathrm{d}L(u), \tag{4.3}$$

in the $L^2(\Omega)$ sense, where

$$h^x(u) = \frac{x}{\sigma(x)} \int_{\mathbb{R}} h(v) g'((xv - u)_+)\,\mathrm{d}v. \tag{4.4}$$

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(ii) *Assume that* $h_s, h_t \in \mathcal{H}$ *for* $s, t \in \mathbb{R}$. *Then*

$$\operatorname{cov}\Big(\int h_s(u)\,\mathrm{d}S_x(u), \int h_t(u)\,\mathrm{d}S_x(u)\Big) = \iint h_t(u)h_s(v)\Gamma_x(u,v)\,\mathrm{d}u\,\mathrm{d}v,$$

where

$$\Gamma_x(u,v) = \frac{x^2\Gamma(xs,xt)}{\sigma^2(x)} = \frac{x^2}{\sigma^2(x)}\int g'((ux-w)_+)g'((vx-w)_+)\,\mathrm{d}w.$$

(iii) *Defining* h_t^x *as in (4.4) with* h *replaced by* h_t, *the ch.f. of*

$$\int h_{t_1}(u)\,\mathrm{d}S_x(u), \dots, \int h_{t_m}(u)\,\mathrm{d}S_x(u) \quad \text{for } t_1 < \cdots < t_m$$

is given by

$$\mathbb{E}\Big[\exp\Big\{\mathrm{i}\sum_{j=1}^{m}\theta_j \int h_{t_j}(u)\,\mathrm{d}S_x(u)\Big\}\Big] = \exp\Big\{\int \psi\Big(\sum_{j=1}^{m}\theta_j h_{t_j}^x(u)\Big)\,\mathrm{d}u\Big\},$$

where $\theta_j \in \mathbb{R}$, $j = 1, \dots, m$, *and* ψ *is as in (2.1).*

Proof. To prove (4.3) it suffices to take an (interval)-indicator function as in the proof of Theorem 3.1, which we omit here. Theorem 4.1(ii) follows from Proposition 2.1. Finally, Theorem 4.1(iii) follows from the fact that

$$\Big(\int h_{t_1}(u)\,\mathrm{d}S_x(u), \dots, \int h_{t_m}(u)\,\mathrm{d}S_x(u)\Big) \stackrel{\mathrm{D}}{=} \Big(\int h_{t_1}^x(u)\,\mathrm{d}L(u), \dots, \int h_{t_m}^x(u)\,\mathrm{d}L(u)\Big),$$

where $h_t^x(u) = (x/\sigma(x))\int h_t(v)g'(xv-u)\,\mathrm{d}v$.

An important step in the proof of convergence of an OU process driven by a time-scaled GFLP is the convergence of the covariance function and its second derivative. This requires that g' is *regularly varying*, i.e. for all $u > 0$,

$$\lim_{x\to\infty}\frac{g'(xu)}{g'(x)} = u^{\rho-1} \tag{4.5}$$

holds for $\rho \in (0, \frac{1}{2})$, and we write $g' \in \mathrm{RV}_{\rho-1}$, where RV denotes regular variation. Such properties have also been used in Klüppelberg and Mikosch (1995) and Klüppelberg and Kühn (2004) to prove convergence of scaled shot-noise processes to self-similar Gaussian processes, in particular, to an FBM. Condition (4.5) on g implies that, in particular, $\operatorname{cov}[S(s), S(t)]$ is bivariate regularly varying with index $1 + 2\rho$ and, hence, that $\sigma^2 \in \mathrm{RV}_{1+2\rho}$. For more details on RV; see Bingham *et al.* (1987). The following result exploits these properties.

Theorem 4.2. *For* $x > 0$ *let* S_x *be the time-scaled GFLP as defined in (4.1) and suppose that* g' *satisfies Assumption 3.1 and that* $g' \in \mathrm{RV}_{\rho-1}$ *for* $\rho \in (0, \frac{1}{2})$. *Define* $H := \rho + \frac{1}{2}$. *Then for*

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every $s, t \in \mathbb{R}$,

$$\lim_{x\to\infty} \Gamma_x(s,t) = \lim_{x\to\infty} \frac{x^2 \int_{-\infty}^{s\wedge t} g'(x(s-w)_+) g'(x(t-w)_+)\,\mathrm{d}w}{\sigma^2(x)}$$
$$= \frac{\partial^2}{\partial t \partial s} \operatorname{cov}(B^{\rho+1/2}(s), B^{\rho+1/2}(t))$$
$$= H(2H-1)|t-s|^{2H-2}, \tag{4.6}$$
$$\lim_{x\to\infty} \widetilde{\Gamma}_x(s,t) = \operatorname{cov}(B^{\rho+1/2}(s), B^{\rho+1/2}(t)) = \tfrac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}). \tag{4.7}$$

Proof. We use the second moment expressions from Theorem 3.2. To prove (4.6) write

$$\Gamma_x(s,t) = \left(\frac{xg'(x)}{g(x)}\right)^2 \frac{\int_{-\infty}^{s\wedge t} g'(x(s-w)_+) g'(x(t-w)_+)/(g'(x))^2\,\mathrm{d}w}{\int_{-\infty}^{1} \{g(x(1-v)_+) - g(x(-v)_+)\}^2/g^2(x)\,\mathrm{d}v}. \tag{4.8}$$

Then by Karamata's theorem (cf. Bingham *et al.* (1987, Theorem 1.5.11)),

$$\lim_{x\to\infty} \frac{xg'(x)}{g(x)} = \rho$$

and $g \in \mathrm{RV}_\rho$. We first show convergence of the numerator in (4.8) by deriving bounds in the spirit of Potter (cf. Bingham *et al.* (1987, Theorem 1.5.6)). For $0 < \varepsilon < (\frac{1}{2} - \rho) \wedge \rho$, we have $x^{1-\varepsilon} g'(x) \in \mathrm{RV}_{\rho-\varepsilon}$ and $\rho - \varepsilon \in (0, \frac{1}{2})$. Hence, for every $\delta > 0$ there exists some x_0 such that for all $x \geq x_0$ and $|s-w| \leq M$ for some $M > 0$,

$$\left|\frac{g'(x(s-w))}{g'(x)}\right| = \frac{(x(s-w)_+)^{1-\varepsilon} g'(x(s-w)_+)}{(s-w)_+^{1-\varepsilon} x^{1-\varepsilon} g'(x)} \leq \frac{\delta + (s-w)_+^{\rho-\varepsilon}}{(s-w)_+^{1-\varepsilon}} \leq c_M (s-w)_+^{\varepsilon-1},$$

where $c_M > 0$ is some constant depending on M. On the other hand, for $|s-w| > M$,

$$\left|\frac{g'(x(s-w)_+)}{g'(x)}\right| \leq (1+\varepsilon)(s-w)_+^{\rho-1+\varepsilon}$$

for sufficiently large x (cf. Resnick (1987, Propositions 0.5 and 0.8)).

If we choose M appropriately, it follows that

$$\int_{-\infty}^{s\wedge t} \frac{g'(x(s-w)_+) g'(x(t-w)_+)}{(g'(x))^2}\,\mathrm{d}w \leq (1+\varepsilon)^2 \int_{-\infty}^{s-M} (s-w)_+^{\rho-1+\varepsilon} (t-w)_+^{\rho-1+\varepsilon}\,\mathrm{d}w$$
$$+ c_M^2 \int_{s-M}^{s\wedge t} (s-w)_+^{-1+\varepsilon} (t-w)_+^{-1+\varepsilon}\,\mathrm{d}w.$$

Now we apply Lebesgue's dominated convergence theorem to the numerator of (4.8) and obtain convergence of this numerator to that of (4.6). As for the denominator of (4.6), its convergence follows (as also the convergence of $\widetilde{\Gamma}_x$ in (4.7)) by a dominated convergence argument as in the proof of Klüppelberg and Kühn (2004, Theorem 3.2).

Since S_x is a time-changed version of S, $\mathbb{E}[S_x(t)] = 0$ and $\operatorname{var}[S_x(t)] = \sigma^2(xt)/\sigma^2(x)$ hold for all $t \in \mathbb{R}$. Hence, we can define the following time-scaled version of V.

Definition 4.1. For $x > 0$ let S_x be a time-scaled GFLP as defined in (4.1) and suppose that g' satisfies Assumption 3.1.

(i) For $\lambda, \gamma > 0$ we define the OU process $V_x = \{V_x(t)\}_{t \in \mathbb{R}}$ driven by the time-scaled GFLP S_x by

$$V_x(t) := \mathrm{e}^{-\lambda t}\left(V_x(0) + \gamma \int_0^t \mathrm{e}^{\lambda u}\, \mathrm{d}S_x(u)\right), \qquad t \geq 0.$$

(ii) If the initial random variable is given by $V_x(0) = \gamma \int_{-\infty}^0 \mathrm{e}^{\lambda u}\, \mathrm{d}S_x(u)$ a.s. then V_x is stationary and we denote the stationary process by

$$\overline{V}_x(t) := \gamma \int_{-\infty}^t \mathrm{e}^{-\lambda(t-u)}\, \mathrm{d}S_x(u), \quad t \in \mathbb{R}. \tag{4.9}$$

The following is a consequence of Theorem 4.1 and Proposition 2.1. We have set again $\gamma = 1$ for simplicity.

Proposition 4.1. *For $x > 0$ let S_x be the time-scaled GFLP as defined in (4.1) and suppose that g' satisfies Assumption 3.1.*

(i) *For $t \in \mathbb{R}$, we have*

$$\overline{V}_x(t) = \int_{-\infty}^t \mathrm{e}^{-\lambda(t-u)}\, \mathrm{d}S_x(u) = \frac{x}{\sigma(x)} \int_{\mathbb{R}} \int_{-\infty}^t \mathrm{e}^{-\lambda(t-v)} g'(xv-u)\, \mathrm{d}v\, \mathrm{d}L(u).$$

(ii) *For $s, t \in \mathbb{R}$, we have $\mathbb{E}[\overline{V}_x(t)] = 0$ and*

$$\operatorname{cov}[\overline{V}_x(s), \overline{V}_x(t)] = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{-\lambda(t-u)} \mathbf{1}_{(-\infty,t]}(u) \mathrm{e}^{-\lambda(s-v)} \mathbf{1}_{(-\infty,s]}(v) \Gamma_x(u,v)\, \mathrm{d}u\, \mathrm{d}v, \tag{4.10}$$

where

$$\Gamma_x(u,v) = \frac{x^2}{\sigma^2(x)} \int_{\mathbb{R}} g'(xu-w) g'(xv-w)\, \mathrm{d}w.$$

(iii) *The ch.f. of $\overline{V}_x(t_1), \overline{V}_x(t_2), \dots, \overline{V}_x(t_m)$ for $t_1 < t_2 < \cdots < t_m$ is given by*

$$\mathbb{E}\left[\exp\left\{\mathrm{i} \sum_{j=1}^m \theta_j \overline{V}_x(t_j)\right\}\right] = \exp\left\{\int_{\mathbb{R}} \psi\left(\sum_{j=1}^m \theta_j \frac{x}{\sigma(x)} \int_{-\infty}^{t_j} \mathrm{e}^{-\lambda(t_j - v)} g'(xv-u)\, \mathrm{d}v\right) \mathrm{d}u\right\},$$

where $\theta_j \in \mathbb{R}$ for $j = 1, \dots, m$ and ψ is given in (2.1).

By extending the earlier work of Lane (1984), who proved a central limit theorem for the Poisson shot-noise process, it was shown in Klüppelberg and Kühn (2004, Theorem 3.2) that, if the driving Lévy process is compound Poisson, then the GFLP S_x converges weakly to B^H in the Skorokhod space $D(\mathbb{R}_+)$ equipped with the metric of uniform convergence on compacts. Since the limit process has continuous sample paths, by Billingsley (1999, Theorem 6.6) it is equivalent to prove weak convergence with respect to the Skorohod d_∞^0-metric on $D(\mathbb{R}_+)$, which induces the J_1 topology. For a definition of d_∞^0, see, e.g. Billingsley (1999, Equation (16.4)).

We extend this result two-fold. First, we generalize the driving compound Poisson process to a Lévy process and, secondly, we consider the convergence of stochastic volatility models driven by a GFLP in Section 5.

Theorem 4.3. *For $x > 0$ let $\overline{V}_x$ be the stationary OU process (4.9) driven by a time-scaled GFLP S_x as in (4.1) and suppose that g' satisfies Assumption 3.1 and $g' \in \mathrm{RV}_{\rho-1}$ for $\rho \in (0, \frac{1}{2})$. Define $H := \rho + \frac{1}{2}$. Let $\overline{Y}$ be the stationary FOU process from Definition 3.1(iii) with $H \in (\frac{1}{2}, 1)$. Then*

$$\overline{V}_x \stackrel{\mathrm{D}}{\to} \overline{Y} \quad \text{as } x \to \infty,$$

where convergence holds in the Skorokhod space $D(\mathbb{R}_+)$ equipped with the metric which induces the Skorokhod J_1 topology.

Proof. According to Billingsley (1999, Theorems 16.7 and 13.1) we have to show weak convergence of the finite-dimensional distributions and tightness of $(\overline{V}_(t)|_{[0,T]})_{t\in\mathbb{R}}$ for every $T > 0$.

We start by proving convergence of the finite-dimensional distributions. Let $0 = t_1 < t_2 < \cdots < t_m < T$ and $\theta_j \in \mathbb{R}$ for $j = 1, \ldots, m$. Recall that, from Proposition 4.1(iii) the ch.f. of $\overline{V}_x(t)$,

$$\begin{aligned}\mathbb{E}\bigg[\exp\bigg\{\mathrm{i}\sum_{j=1}^{m}\theta_j\overline{V}_x(t_j)\bigg\}\bigg] &= \exp\bigg\{\int_{\mathbb{R}}\int_{\mathbb{R}}\phi(y\sum_{j=1}^{m}\theta_j h_{t_j}^x(u))\nu(\mathrm{d}y)\,\mathrm{d}u\bigg\} \\ &= \exp\bigg\{\int_{\mathbb{R}}\int_{\mathbb{R}}x\phi\bigg(y\sum_{j=1}^{m}\theta_j h_{t_j}^x(xu)\bigg)\nu(\mathrm{d}y)\,\mathrm{d}u\bigg\},\end{aligned} \tag{4.11}$$

where $\phi(x) = \mathrm{e}^{\mathrm{i}x} - 1 - \mathrm{i}x$, and we set

$$h_t^x(s) := \frac{x}{\sigma(x)}\int_{-\infty}^{t}\mathrm{e}^{-\lambda(t-v)}g'(xv - s)\,\mathrm{d}v.$$

For the proof we use a Taylor expansion (Lemma 3.2 of Petrov (1995)) to $x\phi(\cdot)$ in (4.11) and we shall show that

$$x\phi\bigg(y\sum_{j=1}^{m}\theta_j h_{t_j}^x(xw)\bigg) \sim -\frac{y^2}{2}x\bigg(\sum_{j=1}^{m}\theta_j h_{t_j}^x(xw)\bigg)^2 \quad \text{as } x \to \infty. \tag{4.12}$$

Then since $\int_{\mathbb{R}} y^2\nu(\mathrm{d}y) = \mathbb{E}[(L(1))^2] = 1$, we shall show that

$$\int_{\mathbb{R}} x\bigg(\sum_{j=1}^{m}\theta_j h_{t_j}^x(xw)\bigg)^2\mathrm{d}w \to \sum_{j,k}^{m}\theta_j\theta_k\,\mathrm{cov}(\overline{Y}(t_j)\overline{Y}(t_k)) \quad \text{as } x \to \infty, \tag{4.13}$$

which implies that the finite-dimensional distributions converge to the corresponding Gaussian distributions.

In order to prove tightness, first, in view of (4.10) and Theorem 4.2 we prove that for $s, t \geq 0$,

$$\lim_{x\to\infty}\int xh_s^x(xw)h_t^x(xw)\,\mathrm{d}w = \lim_{x\to\infty}\mathrm{cov}(\overline{V}_x(s), \overline{V}_x(t)) = \mathrm{cov}(\overline{Y}(s), \overline{Y}(t)). \tag{4.14}$$

In view of (4.10), since $\Gamma_x(u, v)$ should converge to the unbounded function $|u - v|^{2H-2}$ for $H = \rho + \frac{1}{2}$, there is some difficulty in applying the dominated convergence theorem directly, i.e. to find a dominant function. Alternatively, we work with the following representation, which is obtained from integration by parts:

$$\begin{aligned} h_t^x(xw) &= \frac{g(x(t-w))}{\sigma(x)} - \lambda \int_{-\infty}^t \mathrm{e}^{-\lambda(t-v)} \frac{g(x(v-w))}{\sigma(x)}\,\mathrm{d}v \\ &= \frac{f_{tx}(xw)}{\sigma(x)} - \lambda \int_{-\infty}^t \mathrm{e}^{-\lambda(t-v)} \frac{f_{xv}(xw)}{\sigma(x)}\,\mathrm{d}v, \end{aligned}$$

where we have set $f_t(w) = g(t-w) - g(-w)$. Now we apply dominated convergence to each term of the following representation:

$$\begin{aligned} &\int x h_s^x(xw) h_t^x(xw)\,\mathrm{d}w \\ &\quad = \widetilde{\Gamma}_x(s,t) - \lambda \int_{-\infty}^s \mathrm{e}^{-\lambda(s-u)} \int_{\mathbb{R}} \frac{x f_{xu}(xw) f_{xt}(xw)}{\sigma^2(x)}\,\mathrm{d}w\,\mathrm{d}u \\ &\quad\quad - \lambda \int_{-\infty}^t \mathrm{e}^{-\lambda(t-u)} \int_{\mathbb{R}} \frac{x f_{xu}(xw) f_{xs}(xw)}{\sigma^2(x)}\,\mathrm{d}w\,\mathrm{d}u \\ &\quad\quad + \lambda^2 \int_{-\infty}^s \mathrm{e}^{-\lambda(s-u)} \int_{-\infty}^t \mathrm{e}^{-\lambda(t-v)} \int_{\mathbb{R}} \frac{x f_{xu}(xw) f_{xv}(xw)}{\sigma^2(x)}\,\mathrm{d}w\,\mathrm{d}u\,\mathrm{d}v. \end{aligned} \tag{4.15}$$

From Theorem 4.2, we obtain

$$\lim_{x\to\infty} \widetilde{\Gamma}_x(s,t) = \operatorname{cov}(B^{\rho+1/2}(s), B^{\rho+1/2}(t)).$$

For the remaining terms, we consider only the third integral, since convergence of other integrals can be proved similarly. By the Cauchy–Schwarz inequality the integrand in the third integral is dominated by

$$\mathrm{e}^{-\lambda(s-u)-\lambda(t-v)} \sqrt{\frac{\sigma^2(xu)}{\sigma^2(x)} \frac{\sigma^2(xv)}{\sigma^2(x)}}.$$

We provide a uniform upper bound for $\sigma^2(ux)/\sigma^2(x)$. Since $\sigma^2 \in \mathrm{RV}_{1+2\rho}$, for sufficiently small $\delta > 0$, the function $\gamma(x) := \sigma^2(x)|x|^{-1-2\rho-\delta}$ for $x > 0$ is regularly varying with index $-\delta$ and $\gamma(ux)/\gamma(x)$ converges to $|u|^{-\delta}$ uniformly in $|u| \in [1,\infty)$ as $x \to \infty$ (cf. Bingham *et al.* (1987, Theorem 1.5.2)). Hence, we have

$$\frac{\sigma^2(ux)}{\sigma^2(x)} = |u|^{1+2\rho+\delta} \frac{\gamma(ux)}{\gamma(x)} \le |u|^{1+2\rho+\delta}(1+|u|^{-\delta}) \le 2|u|^{1+2\rho+\delta}, \qquad |u| \in [1,\infty) \tag{4.16}$$

for sufficiently large x. Furthermore, by Karamata's theorem, $\sigma^2(ux)/\sigma^2(x) \to |u|^{1+2\rho}$ as $x \to \infty$ uniformly in $|u| \in (0,1]$, and this, together with (4.16), implies that

$$\frac{\sigma^2(ux)}{\sigma^2(x)} \le (c + c'|u|^{1+2\rho}) \mathbf{1}_{\{|u|\le 1\}} + 2|u|^{1+2\rho+\delta} \mathbf{1}_{\{|u|>1\}}$$

for constants $c, c' > 0$. Thus, the dominating function is uniformly integrable and the integral converges, i.e.

$$\lim_{x\to\infty} \int_{-\infty}^s \int_{-\infty}^t \lambda^2 \mathrm{e}^{-\lambda(s-u)-\lambda(t-v)} \sqrt{\frac{\sigma^2(xu)}{\sigma^2(x)} \frac{\sigma^2(xv)}{\sigma^2(x)}}\,\mathrm{d}u\,\mathrm{d}v < \infty.$$

Now we apply a generalized dominated convergence theorem (e.g. Kallenberg (1997, Theorem 1.21)) to (4.15) to obtain, for the third integral of (4.15) in the limit,

$$\lambda^2 \int_{-\infty}^{s} \int_{-\infty}^{t} \mathrm{e}^{-\lambda(s-u)-\lambda(t-v)} \frac{1}{2}(|u|^{2\rho+1} + |v|^{2\rho+1} - |u-v|^{2\rho+1})\,\mathrm{d}u\,\mathrm{d}v.$$

Hence, with (4.15), we conclude that

$$\begin{aligned}
&\lim_{x\to\infty} \operatorname{cov}(\overline{V}_x(s), \overline{V}_x(t)) \\
&\quad = \frac{1}{2}(|t|^{2\rho+1} + |s|^{2\rho+1} - |t-s|^{2\rho+1}) \\
&\qquad - \lambda \int_{-\infty}^{s} \mathrm{e}^{-\lambda(s-u)} \frac{1}{2}(|t|^{2\rho+1} + |u|^{2\rho+1} - |t-u|^{2\rho+1})\,\mathrm{d}u \\
&\qquad - \lambda \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-u)} \frac{1}{2}(|s|^{2\rho+1} + |u|^{2\rho+1} - |s-u|^{2\rho+1})\,\mathrm{d}u \\
&\qquad + \lambda^2 \int_{-\infty}^{s} \int_{-\infty}^{t} \mathrm{e}^{-\lambda(s-u)-\lambda(t-v)} \frac{1}{2}(|u|^{2\rho+1} + |v|^{2\rho+1} - |u-v|^{2\rho+1})\,\mathrm{d}u\,\mathrm{d}v \\
&\quad = \rho(2\rho+1) \int_{-\infty}^{s} \mathrm{e}^{-\lambda(s-u)} \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-v)} |u-v|^{2\rho-1}\,\mathrm{d}u\,\mathrm{d}v \\
&\quad = \operatorname{cov}(\overline{Y}(s), \overline{Y}(t)),
\end{aligned}$$

which proves (4.14).

We turn to the proofs of (4.12) and (4.13). From the representation

$$\frac{f_{xt}(xw)}{\sigma(x)} = \frac{1}{\sqrt{x}} \frac{\{g(x(t-w)) - g(x(-u))\}/g(x)}{\sqrt{(\int_{\mathbb{R}}(f_x(xu))^2\,\mathrm{d}u/g^2(x))}}$$

we observe that $f_{xt}(xw)/\sigma(x) = O(x^{-1/2})$ and, hence, $h_t^x(xw) = O(x^{-1/2})$. Then (4.12) follows from a Taylor expansion. For sufficiently large x, we have

$$\begin{aligned}
x\left|\phi\left(y\sum_{j=1}^{m}\theta_j h_{t_j}^x(xw)\right) + \frac{y^2}{2}\left(\sum_{j=1}^{m}\theta_j (h_{t_j}^x(xw))\right)^2\right| &= \frac{y^3}{6}\left(x^{1/3}\sum_{j=1}^{m}\theta_j h_{t_j}^x(xw)\right)^3 \\
&= O\left(\frac{1}{\sqrt{x}}\right),
\end{aligned}$$

and the right-hand side tends to 0 as $x \to \infty$. In the light of (4.14) and the same generalized dominated convergence theorem as above (e.g. Kallenberg (1997, Theorem 1.21)), it suffices for the proof of (4.13) to show that the integral of a dominating function for $x\phi(y\sum_{j=1}^{m}\theta_j h_{t_j}^x(xw))$ converges as $x \to \infty$. We choose the dominating function as

$$x\left|\phi\left(y\sum_{j=1}^{m}\theta_j h_{t_j}^x(xu)\right)\right| \le \frac{y^2}{2} x \left|\sum_{j=1}^{m}\theta_j h_{t_j}^x(xu)\right|^2 = a_x(u,y), \qquad x \in \mathbb{R}.$$

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Then the integral can be estimated as

$$\int_{\mathbb{R}}\int_{\mathbb{R}} a_x(u, y)\nu(\mathrm{d}y)\,\mathrm{d}u \le \frac{x}{2}\int_{\mathbb{R}} y^2\nu(\mathrm{d}y)\bigg\|\sum_{j=1}^m \theta_j h_{t_j}^x(x\cdot)\bigg\|^2_{L^2(\mathbb{R})}$$
$$\le \frac{x}{2}2^{m-1}\sum_{j=1}^m \|\theta_j h_{t_j}^x(x\cdot)\|^2_{L^2(\mathbb{R})}$$
$$= 2^{m-2}\sum_{j=1}^m \theta_j^2 \int_{\mathbb{R}} x(h_t^x(xu))^2\,\mathrm{d}u,$$

where we use Minkowski's inequality and the fact that $\int y^2\nu(\mathrm{d}y) = \mathbb{E}[(L(1))^2] = 1$. Since the right-hand side converges as $x \to \infty$ by (4.14), we apply the generalized dominated convergence theorem to (4.11) to obtain

$$\lim_{x\to\infty} \mathbb{E}\bigg[\exp\bigg\{\mathrm{i}\sum_{j=1}^m \theta_j \overline{V}_x(t_j)\bigg\}\bigg] = \mathbb{E}\bigg[\exp\bigg\{\mathrm{i}\sum_{j=1}^m \theta_j \overline{Y}(t_j)\bigg\}\bigg],$$

which implies convergence of the finite-dimensional distributions.

Next we prove tightness. For $0 \le s < t < \infty$ choose $T > 0$ such that $s, t \in [0, T]$. By Billingsley (1999, Equation (13.14)) it suffices to show that $\mathbb{E}[(\overline{V}_x(t)-\overline{V}_x(s))^2] \le c_T(t-s)^{1+\rho}$ for some constant $c_T > 0$. By stationarity of the increments of $\overline{V}_x$, we have

$$\overline{V}_x(t) - \overline{V}_x(s) \overset{\mathrm{D}}{=} \overline{V}_x(t-s) - \overline{V}_x(0) = (\mathrm{e}^{-\lambda(t-s)} - 1)\overline{V}_x(0) + \int_0^{t-s} \mathrm{e}^{-\lambda(t-s-u)}\,\mathrm{d}S_x(u).$$

Applying Young's inequality, we obtain

$$\mathbb{E}[(\overline{V}_x(t) - \overline{V}_x(s))^2]$$
$$\le 2(\mathrm{e}^{-\lambda(t-s)} - 1)^2\mathbb{E}[(\overline{V}_x(0))^2] + 2\mathbb{E}\bigg[\bigg(\int_0^{t-s} \mathrm{e}^{-\lambda(t-s-u)}\,\mathrm{d}S_x(u)\bigg)^2\bigg].$$

Since $|\mathrm{e}^{-\lambda(t-s)} - 1| \le c'_T(t-s)^{(1+\rho)/2}$ for $t > s$ and some constant $c'_T > 0$, it suffices to show that

$$\mathbb{E}\bigg[\bigg(\int_0^{t-s} \mathrm{e}^{-\lambda(t-s-u)}\,\mathrm{d}S_x(u)\bigg)^2\bigg] \le c''_T(t-s)^{1+\rho}$$

for some constant $c''_T > 0$. Observe that by integration by parts as in (4.15),

$$\mathbb{E}\bigg[\bigg(\int_0^{t-s} \mathrm{e}^{-\lambda(t-s-u)}\,\mathrm{d}S_x(u)\bigg)^2\bigg]$$
$$= \int_{\mathbb{R}}\bigg(\frac{x}{\sigma(x)}\int_0^{t-s} \mathrm{e}^{-\lambda(t-s-w)}g'((xu-w)_+)\,\mathrm{d}u\bigg)^2\,\mathrm{d}w$$
$$\le 2\frac{\sigma^2((t-s)x)}{\sigma^2(x)} + 2\lambda^2\mathrm{e}^{-2\lambda(t-s)}\int_0^{t-s}\int_0^{t-s} \mathrm{e}^{\lambda(u+v)}\sqrt{\frac{\sigma^2(xu)}{\sigma^2(x)}\frac{\sigma^2(xv)}{\sigma^2(x)}}\,\mathrm{d}u\,\mathrm{d}v$$
$$\le 2\frac{\sigma^2((t-s)x)}{\sigma^2(x)} + 2\lambda^2 c'''_T\,\mathrm{e}^{-2\lambda(t-s)}\bigg(\int_0^{t-s} \mathrm{e}^{\lambda u}\,\mathrm{d}u\bigg)^2$$
$$= 2\frac{\sigma^2((t-s)x)}{\sigma^2(x)} + 2c'''_T(\mathrm{e}^{-\lambda(t-s)} - 1)^2$$

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for some constant $c_T''' > 0$. Since $\sigma^2 \in \mathrm{RV}_{1+2\rho}$, similarly as in the proof of Theorem 3.2 of Klüppelberg and Kühn (2004, p. 349), the bounded function $\eta(x) := \sigma^2(x)/x^{1+\rho}$ is regularly varying with index ρ, i.e. $\eta(x(t-s))/\eta(x)$ converges to $(t-s)^\rho$ as $x \to \infty$ uniformly in $t > s$ on compact subsets of $\mathbb{R}_+$. This implies that for each $M > 0$ and $x \geq x_M$ for some sufficiently large x_M,

$$\frac{\sigma^2(x(t-s))}{\sigma^2(x)} \leq (T^\rho + 1)(t-s)^{1+\rho}.$$

This (together with the Cauchy–Schwarz inequality) implies the tightness condition of Billingsley (1999, Equation (13.14)), which completes the proof.

Remark 4.1. Convergence of the finite-dimensional distributions has also been investigated in Pipiras and Taqqu (2008, Theorem 1), who considered Poisson random measures instead of the driving Lévy process in our model. The two approaches are equivalent by viewing the general integrals in Pipiras and Taqqu (2008) as double integrals over a product space.

5. Limits of stochastic volatility models

We propose a flexible class of (bivariate) stochastic volatility (SV) models. The data are driven by BM or FBM, and the volatility process is an OU process driven by a time-scaled GFLP. This allows for different distributions by varying the driving Lévy process. It provides flexible dependence structures ranging from exponential short memory to polynomial, including long memory, and it also allows for jumps in the volatility by the behavior of g in 0. We would also like to emphasize that although the notion *volatility* is mostly used in finance, our models apply also to physical phenomena like turbulent intermittency or telecommunication measurements.

Moreover, we allow for time-scaled versions of the SV model, which gives, when we apply Theorem 4.3, in the limit a function of a FOU process with $H \in (\frac{1}{2}, 1)$. Consequently, we can adjust the model for the roughness of its sample paths, from those with jumps to continuous ones.

For $\overline{H} \in [\frac{1}{2}, 1)$ let $W^{\overline{H}}$ be FBM (BM corresponding to $\overline{H} = \frac{1}{2}$). For $x > 0$ let $\overline{V}_x$ and $\overline{Y}$ be as in Theorem 4.3. Let $f : \mathbb{R} \to \mathbb{R}_+$ be a continuous function. Then we define the SV model

$$z_x(t) := \mu t + \beta \int_0^t v_x(s)\,\mathrm{d}s + \int_0^t \sqrt{v_x(s-)}\,\mathrm{d}W^{\overline{H}}(s), \qquad v_x(t) := f(\overline{V}_x(t)). \tag{5.1}$$

The integral in the $z_x(t)$ with respect to $W^{\overline{H}}$ is for $\overline{H} > \frac{1}{2}$ a path integral as defined in Young (1936) and Mikosch and Norvaiša (2000), and requires p-variation of the sample path v_x for appropriate p. For $\overline{H} = \frac{1}{2}$ we take the usual Itô integral.

We shall show that for $x \to \infty$ the bivariate process $\{(z_x(t), v_x(t))\}_{t\geq 0}$ converges in the Skorokhod space $D(\mathbb{R}_+^2)$ to

$$z(t) := \mu t + \beta \int_0^t v(s)\,\mathrm{d}s + \int_0^t \sqrt{v(s-)}\,\mathrm{d}W^{\overline{H}}(s), \qquad v(t) := f(\overline{Y}(t)).$$

First recall that $v_x = f(\overline{V}_x)$, so that, by the continuous mapping theorem, weak convergence of v_x follows from that of $\overline{V}_x$.

Theorem 5.1. *For $x > 0$ let (z_x, v_x) be as in (5.1). Assume that $z_x = \{z_x(t)\}_{t\geq 0}$ is driven by an FBM (or BM) with $\overline{H} \in [\frac{1}{2}, 1)$. Furthermore, assume that $v_x = \{v_x(t)\}_{t\geq 0}$ is positive, has* *a.s. càdlàg sample paths, and that it is independent of* $W^{\overline{H}}$. *Suppose that for every* $T > 0$ *and* $t \in [0, T]$ *for all sufficiently large* x,

$$\mathbb{E}[(v_x(t))^2] \le M, \qquad t \in [0, T] \tag{5.2}$$

for some constant $M > 0$, *which may depend on* T. *For* $H > \frac{1}{2}$ *we additionally assume that* $\sqrt{v_x}$ *is of finite* p-*variation for* $p < 1/(1 - \overline{H})$. *If*

$$v_x \xrightarrow{\mathrm{D}} v \quad \text{as } x \to \infty$$

in the Skorkohod space $D(\mathbb{R}_+)$ *with the metric which induces the Skorokhod* J_1 *topology and if* $\sqrt{v}$ *is again of finite* p-*variation with* $p < 1/(1 - \overline{H})$, *then also*

$$(z_x, v_x) \xrightarrow{\mathrm{D}} (z, v) \quad \text{as } x \to \infty$$

in the Skorokhod space $D(\mathbb{R}_+^2)$ *with the metric which induces the Skorokhod* J_1 *topology.*

Proof. In order to prove weak convergence we show convergence of the finite-dimensional distributions and tightness. We shall often condition z_x on the σ-field

$$\mathcal{G} := \sigma\{v_x(s),\ s \in [0, T], 0 < x < \infty\}$$

so that, given $\mathcal{G}$, the process z_x is a Gaussian process. Now we take $0 = t_1 < t_2 < \cdots < t_m \le T$ and $0 = t_1' < t_2' < \cdots < t_n' \le T$ for $m, n \in \mathbb{N}$ and prove that

$$\begin{aligned}&(z_x(t_1), z_x(t_2), \dots, z_x(t_m), v_x(t_1'), v_x(t_2'), \dots, v_x(t_n'))\\ &\qquad \xrightarrow{\mathrm{D}} (z(t_1), z(t_2), \dots, z(t_m), v(t_1'), v(t_2'), \dots, v(t_n'))\end{aligned}$$

by the Cramér–Wold device. For $(\gamma_{11}, \gamma_{12}, \dots, \gamma_{1m}, \gamma_{21}, \dots, \gamma_{2n}) \in \mathbb{R}^{m+n}$ we shall show that

$$\sum_{j=1}^m \gamma_{1j} z_x(t_j) + \sum_{k=1}^n \gamma_{2k} v_x(t_k') \xrightarrow{\mathrm{D}} \sum_{j=1}^m \gamma_{1j} z(t_j) + \sum_{k=1}^n \gamma_{2k} v(t_k').$$

First observe that

$$\sum_{j=1}^m \gamma_{1j} z_x(t_j) + \sum_{k=1}^n \gamma_{2k} v_x(t_k') = \sum_{j=2}^m \biggl(\sum_{h=j}^m \gamma_{1j}\biggr)(z_x(t_j) - z_x(t_{j-1})) + \sum_{k=1}^n \gamma_{2k} v_x(t_k')$$

with $z_x(t_1) = 0$ and, hence, it suffices to show that

$$\sum_{j=2}^m \gamma_{1j}(z_x(t_j) - z_x(t_{j-1})) + \sum_{k=1}^n \gamma_{2k} v_x(t_k') \xrightarrow{\mathrm{D}} \sum_{j=2}^m \gamma_{1j}(z(t_j) - z(t_{j-1})) + \sum_{k=1}^n \gamma_{2k} v(t_k').$$

We use the independence of v_x and $W^{\overline{H}}$ and the conditional Gaussianity of both z_x and z, given the σ-field $\mathcal{G}$, to obtain the ch.f.

$$\begin{aligned}&\mathbb{E}\biggl[\exp\biggl(\mathrm{i}\lambda\biggl\{\sum_{j=2}^m \gamma_{1j}(z_x(t_j) - z_x(t_{j-1})) + \sum_{k=1}^n \gamma_{2k} v_x(t_k')\biggr\}\biggr)\biggr]\\ &\quad = \mathbb{E}\biggl[\mathbb{E}\biggl[\exp\biggl(\mathrm{i}\lambda\biggl\{\sum_{j=2}^m \gamma_{1j}(z_x(t_j) - z_x(t_{j-1})) + \sum_{k=1}^n \gamma_{2k} v_x(t_k')\biggr\}\biggr) \biggm| \mathcal{G}\biggr]\biggr]\end{aligned}$$

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$$
\begin{aligned}
&= \mathbb{E}\bigg[\exp\bigg(\mathrm{i}\lambda\sum_{k=1}^{n}\gamma_{2k}v_x(t_k')\bigg)\mathbb{E}\bigg[\exp\bigg(\mathrm{i}\lambda\sum_{j=2}^{m}\gamma_{1j}(z_x(t_j)-z_x(t_{j-1}))\bigg)\;\bigg|\;\mathcal{G}\bigg]\bigg]\\
&= \mathbb{E}\bigg[\exp\bigg(\mathrm{i}\lambda\sum_{k=1}^{n}\gamma_{2k}v_x(t_k') + \mathrm{i}\lambda\sum_{j=2}^{m}\gamma_{1j}\bigg(\mu(t_j-t_{j-1})+\beta\int_{t_{j-1}}^{t_j}v_x(u)\,\mathrm{d}u\bigg)\\
&\qquad\times\exp\bigg(-\frac{\lambda^2}{2}H(2H-1)\\
&\qquad\qquad\times\sum_{j,k}^{m}\gamma_{1j}\gamma_{1k}\int_{t_{j-1}}^{t_j}\int_{t_{k-1}}^{t_k}\sqrt{v_x(u)}\sqrt{v_x(w)}|u-w|^{2H-2}\,\mathrm{d}u\,\mathrm{d}w\bigg)\bigg)\bigg]\\
&=: \mathbb{E}[h(v_x)].
\end{aligned}
$$

Since $h(\cdot)$ is continuous, the continuous mapping theorem yields that $h(v_x) \xrightarrow{\mathrm{D}} h(v)$. Furthermore, the fact that $|h| \leq 1$ together with Kallenberg (1997, Lemma 3.11) implies that $\mathbb{E}[h(v_x)] \to \mathbb{E}[h(v)]$ as $x \to \infty$ (see also Jacod and Shiryaev (2003, Equation (3.8), Chapter VI)). Again by conditional independence, reversing the argument which led to $\mathbb{E}[h(v_x)]$ yields

$$
\mathbb{E}[h(v)] = \mathbb{E}\bigg[\exp\bigg(\mathrm{i}\lambda\bigg\{\sum_{j=2}^{m}\gamma_{1j}(z(t_j)-z(t_{j-1})) + \sum_{k=1}^{n}\gamma_{2k}v(t_k')\bigg\}\bigg)\bigg].
$$

This concludes the first part of the proof.

Secondly, we prove tightness. For the process z_x we apply the tightness condition of Billingsley (1999, Equation (13.14)). Since $W^{\overline{H}}$ has zero mean, it suffices to prove tightness of

$$
I_x^{(1)}(t) := \int_0^t v_x(s)\,\mathrm{d}s, \qquad I_x^{(2)}(t) := \int_0^t \sqrt{v_x(s)}\,\mathrm{d}W^{\overline{H}}(s), \quad t \geq 0.
$$

For $0 \leq s < t$, we have

$$
\begin{aligned}
\mathbb{E}[(I_x^{(1)}(t) - I_x^{(1)}(s))^2] &= \mathbb{E}\bigg[\int_s^t\int_s^t v_x(u)v_x(w)\,\mathrm{d}u\,\mathrm{d}w\bigg]\\
&= \int_s^t\int_s^t \mathbb{E}[v_x(u)v_x(w)]\,\mathrm{d}u\,\mathrm{d}w\\
&\leq \int_s^t\int_s^t \sqrt{\mathbb{E}[(v_x(u))^2]\mathbb{E}[(v_x(w))^2]}\,\mathrm{d}u\,\mathrm{d}w\\
&\leq M(t-s)^2,
\end{aligned}
$$

where we have used the Cauchy–Schwarz inequality and (5.2). This ensures the tightness condition for the Lebesgue integral $I_x^{(1)}$. As for tightness of the (fractional) Brownian integral $I_x^{(2)}$, recall that, given the σ-field $\mathcal{G}$, $I_x^{(2)}$ is Gaussian. We distinguish two cases.

Case 1. For $\overline{H} > \frac{1}{2}$, we calculate

$$
\begin{aligned}
\mathbb{E}[(I_x^{(2)}(t) - I_x^{(2)}(s))^2] &\leq \mathbb{E}[\mathbb{E}[(I_x^{(2)}(t) - I_x^{(2)}(s))^2 \mid \mathcal{G}]]\\
&\leq c\mathbb{E}\bigg[\int_s^t\int_s^t \sqrt{v_x(u)}\sqrt{v_x(w)}|u-w|^{2\overline{H}-2}\,\mathrm{d}u\,\mathrm{d}w\bigg]\\
&\leq cM(t-s)^{2\overline{H}},
\end{aligned}
$$

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where c is a finite positive constant, and apply Billingsley (1999, Equation (13.14)) with $\beta = \frac{1}{2}$.

Case 2. For $\overline{H} = \frac{1}{2}$, we apply the same condition with $\beta = 1$ to obtain

$$\begin{aligned}\mathbb{E}[(I_x^{(2)}(t) - I_x^{(2)}(s))^4] &= \mathbb{E}[\mathbb{E}[(I_x^{(2)}(t) - I_x^{(2)}(s))^4 \mid \mathcal{G}]] \\ &\leq c\mathbb{E}[(I_x^{(1)}(t) - I_x^{(1)}(s))^2],\end{aligned}$$

where we have used properties of the quadratic variation of the BM, and c is again a finite positive constant. Now since the limit process z is continuous, the bivariate tightness of $\{(z_x(t), v_x(t))\}_{t\geq 0}$ follows from Jacod and Shiryaev (2003, Corollary 3.33, Chapter VI).

Remark 5.1. (i) The same remark as made before Theorem 4.3 holds for the bivariate model. Since the bivariate limit process has continuous sample paths, weak convergence also holds in the Skorokhod space $D(\mathbb{R}_+^2)$ equipped with the metric of uniform convergence on compacts; see Jacod and Shiryaev (2003, Equation (1.17)(b), Chapter VI).

(ii) Assume that $v_x = \overline{V}_x$ is the stationary OU process driven by a time-scaled GFLP as defined in (4.3). If $\overline{V}_x$ satisfies (5.2) and its sample paths satisfy a p-variation condition, then Theorem 5.1 applies.

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