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A KRONECKER-TYPE THEOREM FOR COMPLEX POLYNOMIALS IN SEVERAL VARIABLES

BY
C. J. SMYTH

ABSTRACT. We give a classification result for "extreme-monic" polynomials in several variables having measure 1. The result implies a recent several-variable generalization, by D. W. Boyd, of Kronecker's classical theorem (that all zeros of a monic integral polynomial, with non-zero constant term and measure 1, are roots of unity).

Introduction. For a monic polynomial $P(z)$ with integer coefficients and $P(0)\neq 0$, the classical Kronecker theorem [4] states that if all zeros of $P(z)$ lie in $|z|\leq 1$, they are all roots of unity.

In this paper we generalize (Theorem 1) to several variables the following result: if $P(z)\in\mathbb{C}[z]$ is monic with $|P(0)|=1$ and measure (defined below) 1, then all zeros of P lie on $|z|=1$. This result is an immediate consequence of equation (1) below. In more than one variable, however, the result is somewhat deeper, since, for instance, it enables Boyd's [1, Theorem 1] recent several-variable generalization of Kronecker's theorem to be derived from it as a corollary (Corollary 1). This theorem had strengthened an earlier result of the same type by Montgomery and Schinzel [6, Theorem 2].

The method of this paper is based on a correspondence between a polynomial $F\in\mathbb{C}[\mathbf{z}]=\mathbb{C}[z_1,\ldots,z_n]$ and a certain convex set $\mathscr{C}(F)$ in $\mathbb{R}^n$. We show that under suitable conditions the faces of $\mathscr{C}(F)$ correspond to factors of F. This fact is used as a basis for an induction argument.

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Definitions and results. For $\mathbf{z}=(z_1,\ldots,z_n)$ and $F(\mathbf{z})=\sum_{\mathbf{j}\in J}a(\mathbf{j})z_1^{j_1}\cdots z_n^{j_n}\in\mathbb{C}[\mathbf{z}]$, we define a body $\mathscr{C}(F)$ in $\mathbb{R}^n$ to be the convex hull of the $\mathbf{j}\in J$ with $a(\mathbf{j})\neq 0$ (Clarke [2] called $\mathscr{C}(F)$ the *exponent polytope* of F). For $F\in\mathbb{C}[\mathbf{z}]$, the measure $M(F)$ is

$$\exp\left[\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\log|F(e^{i\theta_1},\ldots,e^{i\theta_n})|\,d\theta_1\cdots d\theta_n\right].$$

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By Jensen's Theorem,

$$(1)\qquad M\Big(a_0 \prod_{i=1}^{m} (z-\alpha_1)\Big) = |a_0| \prod_{i=1}^{m} \max(|\alpha_i|, 1)$$

for polynomials in one variable z.

A one-variable polynomial $P(z)$ is said to be unit-monic if it is monic with $|P(0)|=1$. More generally, $F \in \mathbb{C}[\mathbf{z}]$ is said to be *extreme-monic* if $|a(\mathbf{j})|=1$ for all extreme points $\mathbf{j}$ of $\mathscr{C}(F)$. In a similar manner to Boyd [1], we define a polynomial $F \in \mathbb{C}[\mathbf{z}]$ to be *extended unit-monic* (resp. *extended cyclotomic*) if it is of the form $F(\mathbf{z}) = z_1^{b_1} \cdots z_n^{b_n} P(z_1^{v_1} \cdots z_n^{v_n})$, where P is a unit-monic (resp. cyclotomic) polynomial in one variable, the v_i are integers and the b_i are chosen minimally such that $F(z)$ is a polynomial in $z_1, \ldots, z_n$.

Our main result is

THEOREM 1. *Let $F \in \mathbb{C}[\mathbf{z}]$. Then F is extreme-monic with $M(F)=1$ iff F is a product of $\rho z_1^{d_1} \cdots z_n^{d_n}$ and extended unit-monic polynomials. Here $d_1, \ldots, d_n$ are integers, and $|\rho|=1$.*

COROLLARY 1. (Boyd [1]). *Let $F \in \mathbb{Z}[\mathbf{z}]$. Then $M(F)=1$ iff F is a product of $\pm z_1^{d_1} \cdots z_n^{d_n}$ and extended cyclotomic polynomials.*

As a by-product of the proof of Theorem 1 we obtain

THEOREM 2. *For any k-dimensional face $\mathscr{C}'$ of $\mathscr{C}(F)$ $(0 \le k < n)$, we have $M(F) \ge M(F(\mathscr{C}'))$. Here $F(\mathscr{C}') = \sum_{\mathbf{j} \in J \cap \mathscr{C}'} a(\mathbf{j}) z_1^{j_1} \cdots z_n^{j_n}$.*

In particular $(k=0)$

COROLLARY 2. *$M(F) \ge |a(\mathbf{j})|$ for every extreme point $\mathbf{j}$ of $\mathscr{C}(F)$.*

Auxiliary results. For the proof, we need the corollary to the following

LEMMA 1. *Let $\mathscr{C}_1$, $\mathscr{C}_2$ be closed convex polyhedra in $\mathbb{R}^n$, and $\mathscr{C}_1 + \mathscr{C}_2 = \{\mathbf{j}^{(1)} + \mathbf{j}^{(2)} \mid \mathbf{j}^{(i)} \in \mathscr{C}_i (i=1,2)\}$. Then*

(i) *Every extreme point of $\mathscr{C}_1 + \mathscr{C}_2$ can be expressed as a sum $\mathbf{j}^{(1)} + \mathbf{j}^{(2)}$, $\mathbf{j}^{(i)} \in \mathscr{C}_i (i=1,2)$, in only one way. Further such $\mathbf{j}^{(i)}$ are extreme points of $\mathscr{C}_i (i=1,2)$.*

(ii) *For every extreme point $\mathbf{j}^{(1)}$ of $\mathscr{C}_1$ there is an extreme point $\mathbf{j}^{(2)}$ of $\mathscr{C}_2$ such that $\mathbf{j}^{(1)} + \mathbf{j}^{(2)}$ is an extreme point of $\mathscr{C}_1 + \mathscr{C}_2$.*

The lemma is essentially Theorem 15 of [3].

COROLLARY 3. *Let $F_0 = F_1 F_2$, where $F_0, F_1, F_2 \in \mathbb{C}[\mathbf{z}]$, and $F_i(\mathbf{z}) = \sum_{\mathbf{j} \in J_i} a_i(\mathbf{j}) z_1^{j_1} \cdots z_n^{j_n} (i=0,1,2)$. Then*

(i) $\mathscr{C}(F_1F_2) = \mathscr{C}(F_1) + \mathscr{C}(F_2)$

(ii) *if any two of the F_i are extreme monic, so is the third.*

Proof. Clearly $\mathscr{C}(F_1F_2) \subseteq \mathscr{C}(F_1) + \mathscr{C}(F_2)$. Since the $a_i(\mathbf{j})$ are non-zero for $\mathbf{j} \in J_i$, Lemma 1 (i) shows that any extreme point of $\mathscr{C}(F_1) + \mathscr{C}(F_2)$ is uniquely expressible in the form $\mathbf{j}^{(1)} + \mathbf{j}^{(2)}$, for some extreme points $\mathbf{j}^{(i)}$ of $\mathscr{C}(F_i)(i = 1, 2)$. Hence

$$a_0(\mathbf{j}^{(1)} + \mathbf{j}^{(2)}) = a_1(\mathbf{j}^{(1)})a_2(\mathbf{j}^{(2)}) \tag{2}$$

so that $a_0(\mathbf{j}^{(1)} + \mathbf{j}^{(2)}) \neq 0$, and $\mathbf{j}^{(1)} + \mathbf{j}^{(2)} \in \mathscr{C}(F_1F_2)$. Thus all extreme points of $\mathscr{C}(F_1) + \mathscr{C}(F_2)$ belong to $\mathscr{C}(F_1F_2)$, which proves (i).

From (2) we see that F_0 is extreme monic if F_1 and F_2 are. Now suppose that F_0 and F_1 are extreme monic. Then Lemma 1 (i), shows that for each extreme point $\mathbf{j}^{(1)}$ of $\mathscr{C}(F_1)$ there is an extreme point $\mathbf{j}^{(2)}$ of $\mathscr{C}(F_2)$ such that $\mathbf{j}^{(1)} + \mathbf{j}^{(2)}$ is an extreme point of $\mathscr{C}(F_0)$, so that (2) again holds. Hence $|a_2(\mathbf{j}^{(2)})| = 1$ and F_1 is also extreme-monic.

Proof of the Theorems. Take $F(\mathbf{z}) = \sum_{\mathbf{j} \in J} a(\mathbf{j}) z_1^{j_1} \cdots z_n^{j_n} \in \mathbb{C}[\mathbf{z}]$, consider a k-dimensional face $\mathscr{C}'$ of $\mathscr{C}(F)$, for some $k: 0 \leq k < n$, and choose a hyperplane $\mathscr{H}$ containing $\mathscr{C}'$. Since $J \subseteq \mathbb{Z}^n \subset \mathbb{R}^n$, $\mathscr{H}$ can be chosen so that it has a normal vector $\mathbf{v}_1 = (\mathrm{v}_{11}, \mathrm{v}_{21}, \ldots, \mathrm{v}_{n1})$, where the $v_{i1}(i = 1, \ldots, n)$ are coprime integers. We can then find, by a classical result of Hermite, a square matrix $V = (v_{il})$ with integral entries, determinant 1 and first column $\mathbf{v}_1^{\mathbf{T}}$. Hence we can change variables by defining new variables $w_l(l = 1, \ldots, n)$ by $z_i = \prod_{l=1}^n w_l^{v_{il}}(i = 1, \ldots, n)$, and then putting $G(\mathbf{w}) = F(\mathbf{z})$, where $\mathbf{w} = (w_1, \ldots, w_n)$. Then since $\prod_{i=1}^n z_i^{j_i} = \prod_{l=1}^n w_l(\sum_{i=1}^n j_i v_{il})$, $G(\mathbf{w}) = \sum_{\mathbf{k} \in K} a(\mathbf{k}V^{-1}) w_1^{k_1} \cdots w_n^{k_n}$, where $K = \{\mathbf{j}V \mid \mathbf{j} \in J\}$.

With these new variables $\mathbf{w}$, we define $\mathscr{C}(G)$ to be the convex hull of the $\mathbf{k} \in K$ with $a(\mathbf{k}V^{-1}) \neq 0$. Now, for some integer m, $\mathscr{H} = \{\mathbf{j} \mid \sum_{i=1}^n v_{i1} j_i = m\}$. So the face $\mathscr{C}'(G) = \{\mathbf{j}V \mid \mathbf{j} \in \mathscr{C}'\}$ of $\mathscr{C}(G)$ is in the hyperplane $\mathscr{H}V = \{\mathbf{k} = jV \mid \sum v_{i1} j_i = m\} = \{\mathbf{k} \mid k_1 = m\}$.

We now write $G(\mathbf{w})$ as a sum of terms $G_l(w_2, \ldots, w_n)w_1^l$, where the $G_l(w_2, \ldots, w_n)$ are polynomials in $w_2^{\pm 1}, \ldots, w_n^{\pm 1}$, and l runs over a finite set of integers either (i) all $\leq m$, or (ii) all $\geq m$. By replacing w_1 by w_1^{-1}, if necessary, we may assume that (i) occurs, with L the least value of l. Then

$$\begin{aligned} G(\mathbf{w}) &= w_1^L\{G_m w_1^{m-L} + G_{m-1} w_1^{m-L-1} + \cdots + G_L\}. \\ &= w_1^L G_m \{w_1^{m-L} + (G_{m-1}/G_m) w_1^{m-L-1} + \cdots + (G_L/G_m)\} \\ &\qquad \text{for } \quad G_m \neq 0 = w_1^L G_m H \text{ say,} \end{aligned}$$

where H is a rational function of $w_1, \ldots, w_n$.

Now $\log M(F) = 1/(2\pi)n \int_0^{2\pi} \cdots \int_0^{2\pi} \log |F(e^{i\theta_1}, \ldots, e^{i\theta_n})| \, d\theta_1 \cdots d\theta_n$. On changing variables with the transformation $(\theta_1, \ldots, \theta_n) = (\phi_1, \ldots, \phi_n)V$, with Jacobian $\det V = 1$, we have

$$\log M(F) = \log M(G) = \log M(G_m) + \log M(H). \tag{3}$$

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Now $\log M(H) = 1/(2\pi)n \int_0^{2\pi} \cdots \int_0^{2\pi} d\theta_2 \cdots d\theta_n \int_0^{2\pi} \log |H| \, d\theta_1 \geq 0$, by Jensen's Theorem. So $\log M(F) \geq \log M(G_m)$. On applying the above transformation to $F(\mathscr{C}')$ (as defined in the statement of Theorem 2), we see that $M(F(\mathscr{C}')) = M(G_m)$, and $\log M(F) \geq \log M(F(\mathscr{C}'))$. This proves Theorem 2.

To complete the proof of Theorem 1, first note that it is trivial in one direction—i.e. if f is a product of $\rho z_1^{d_1} \cdots z_n^{d_n}$ and extended unit-monic polynomials, then $M(F) = 1$, by Jensen's Theorem, and F is extreme-monic, by Corollary 3. We therefore assume that F is extreme-monic with $M(F) = 1$, and have to prove that F is a product of $\rho z_1^{d_1} \cdots z_n^{d_n}$ and extended unit-monic polynomials.

We use double induction on the number of variables n of F and the number r of irreducible factors of F in $\mathbb{C}[\mathbf{z}]$ (excluding trivial factors $\rho z_1^{d_1} \cdots z_n^{d_n}$). The result is clearly true if $n = 1$, or if $r = 0$, in which case $F = a(\mathbf{j}) z_1^{j_1} \cdots z_n^{j_n}$ for some single point $\mathbf{j}$. We now assume the truth of the result for all $n' < n$ and $r' < r$, where $n \geq 2$, $r \geq 1$. Let F be a polynomial in n variables with r irreducible factors. The main step in the proof is to show that for the polynomial G, as defined earlier, the rational function H is in fact a polynomial in $w_1, w_2^{\pm 1}, \ldots, w_n^{\pm 1}$. To show this, first note that as F is extreme monic, G is extreme monic, and hence G_m is extreme-monic, as $\mathscr{C}(G_m)$ is a face of $\mathscr{C}(G)$. Now $1 = M(G) \geq M(G_m) \geq 1$, so that $M(G_m) = 1$. As G_m is a function of $w_2, \ldots, w_n$, the induction hypothesis therefore shows that G_m is a product of $\rho w_2^{d_2} \cdots w_n^{d_n}$ and extended unit-monic polynomials. However, we can also ensure that G_m is not just of the form $\rho w_2^{d_2} \cdots w_n^{d_n}$, but does in fact contain extended unit-monic factors. To do this it is simply necessary to choose the face $\mathscr{C}'$ of $\mathscr{C}(F)$ so that it contains at least two points of J, but is not the whole of $\mathscr{C}(F)$. This is always possible if the points of $\mathscr{C}'$ do not lie on a single line. However, we can assume this, for if the points of $\mathscr{C}'$ were collinear, we could, by a change of variables, express F as a product of a monomial and a polynomial in one variable. This would mean that we could take $n = 1$, while we are assuming $n \geq 2$.

We have from (3) that

$$0 = \log M(G_m) = \log M(H). \tag{4}$$

We can now show that

Lemma 2. *Under our previous assumptions, H is a polynomial.*

Proof. Assume H is not a polynomial. Then G_m does not divide some G_k. Since G_m is extended unit monic we can choose a factor of G_m of the form $w_2^{a_2} \cdots w_n^{a_n} - \alpha$, with $|\alpha| = 1$, which also does not divide G_k. Further, since $w_2^{a_2} \cdots w_n^{a_n} - \alpha = \prod_{j=1}^{h} (w_2^{a_2/h} \cdots w_n^{a_n/h} - e^{2\pi i j/h} \alpha^{1/h})$ where $h = (a_2, \ldots, a_n)$, we can assume that $h = 1$. We then change variables, keeping w_1 fixed, so that $w_2^{a_2} \cdots w_n^{a_n}$ becomes a new variable. Assuming that this has already been done, we are now able to assume that G_m has a factor $w_2 - \alpha$ not dividing G_k.

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Now, writing G_k in the form $(w_2-\alpha)A+B$, where $B\neq 0$ is a polynomial in $w_3^{\pm 1},\ldots,w_n^{\pm 1}$, it is clear that we can choose an $(n-2)$-dimensional point $(w_3^*,\ldots,w_n^*)$ with $|w_i^*|=1$ $(i=3,\ldots,n)$ and $B(w_3^*,\ldots,w_n^*)\neq 0$. Then there will be a neighbourhood $\mathcal{N}$ of $(\alpha, w_3^*,\ldots,w_n^*)$ on the $(n-1)$-dimensional unit torus such that

$$|G_k/G_m| > \binom{m-2}{k-2}+1 \tag{5}$$

on $\mathcal{N}$. Now (5) is impossible if all zeros of H, as a polynomial in w_1, lie in $|w_1|\le 1$. Hence there is an $\varepsilon>0$ such that $\int_0^{2\pi}\log|H|\,d\theta_1>\varepsilon$ for any $(w_2,\ldots,w_n)$ fixed in $\mathcal{N}$. Since $\int_0^{2\pi}\log|H|\,d\theta_1\ge 0$ for any fixed $(w_2,\ldots,w_n)$ not necessarily in $\mathcal{N}$, this implies that $\log M(H)>\varepsilon/(2\pi)n\times((n-1)$-dimensional measure of $\mathcal{N})>0$. This contradicts (4), so proves the lemma.

We have thus achieved a polynomial factorization $G=G_mH$ of G, where G_m is a function of $w_2,\ldots,w_n$, with at least one extended unit-monic factor. Hence H has fewer than r irreducible factors, which by the induction hypothesis implies that H is a product of $\rho w_1^{d_1}\cdots w_n^{d_n}$ and extended unit-monic polynomials. Thus the same is true for G, and hence, on changing variables, F is a product of $\rho z_1^{d_1}\cdots z_n^{d_n}$ and extended unit-monic polynomials.

Proof of Corollary 1. Let $F\in\mathbb{Z}[\mathbf{z}]$ and $M(F)=1$. Then for any extreme point $\mathbf{j}$ of $\mathscr{C}(F)$, $|a(\mathbf{j})|\le 1$ by Corollary 2. Hence $a(\mathbf{j})=\pm 1$, so that F is extreme-monic. Thus from Theorem 1, F can be written in the form $F(\mathbf{z})=\pm z_1^{d_1}\cdots z_n^{d_n}\prod_{s=1}^{S}(z_1^{\lambda_{s1}}z_2^{\lambda_{s2}}\cdots z_n^{\lambda_{sn}}-\theta_s)$, where $|\theta_s|=1$ $(s=1,\ldots,S)$. To show that in fact its roots are roots of unity, we proceed as follows, making use of polynomials of the type $F(z^{r_1},\ldots,z^{r_n})$, used in [6]. We take a supporting hyperplane $\sum_{i=1}^{n} r_ij_i=m>0$, with the r_i integers, meeting $\mathscr{C}(F)$ in precisely one point, an extreme point. We can also assume that none of the vectors $(\lambda_{s1},\ldots,\lambda_{sn})$ are parallel to the hyperplane, so that $\sum_{i=1}^{n}\lambda_{si}r_i\neq 0$ $(s=1,\ldots,S)$. Then either $F(z^{r_1},\ldots,z^{r_n})$ or $F(z^{-r_1},\ldots,z^{-r_n})$ is of the form $\pm z^kP(z)$ for some k and some monic polynomial $P\in\mathbb{Z}[z]$, where $P(z)$ is of the form $\prod_{s=1}^{S}(z^{k_s}-\theta_s^{\varepsilon_s})$, with all $k_s>0$, and $\varepsilon_s=\pm 1$. Hence the θ_s are all roots of unity, by Kronecker's classical Theorem.

We see that in the above proof of Corollary 1, the fact that F has integer coefficients is used in two places: (i) to show that $a(\mathbf{j})\neq 0$ and $|a(\mathbf{j})\le 1$ implies $a(\mathbf{j})=\pm 1$ for extreme points $\mathbf{j}$, and (ii) so that Kronecker's original one-variable result can be applied.

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DEPARTMENT OF MATHEMATICS
JAMES COOK UNIVERSITY OF NORTH QUEENSLAND
TOWNSVILLE, QLD AUSTRALIA