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3-DIMENSIONAL AFFINE HYPERSURFACES IN $\mathbb{R}^4$ WITH PARALLEL CUBIC FORM

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§ 1. Introduction

In this paper, we study 3-dimensional locally strongly convex affine hypersurfaces in $\mathbb{R}^4$. Since the publication of Blaschke's book [B] in the early twenties, it is well-known that on a nondegenerate affine hypersurface M there exists a canonical transversal vector field called the affine normal. The second fundamental form associated to the affine normal is called the affine metric. In the special case that M is locally strongly convex, this affine metric is a Riemannian metric. Also, using the affine normal, by the Gauss formula one can introduce an affine connection on M, called the induced connection ∇. So on M, we can consider two connections, namely the induced affine connection ∇ and the Levi Civita connection $\hat{\nabla}$ of the affine metric h.

The cubic form C is defined by $C = \nabla h$. The classical Berwald theorem states that the cubic form vanishes identically if and only if M is an open part of a nondegenerate quadric. Here, we will consider the condition that the cubic form is parallel with respect to Levi Civita connection of the affine metric, i.e. $\hat{\nabla} C = 0$. For surfaces, this condition has been studied by M. Magid and K. Nomizu in [MN]. There, they proved the following theorem.

THEOREM [MN]. *Let M be a Blaschke surface in $\mathbb{R}^3$ with $\hat{\nabla} C = 0$. Then either M is an open part of a nondegenerate quadric* (*i.e.* $C = 0$) *or M is affine equivalent to an open part of one of the following surfaces*:

(i) $xyz = 1$,

(ii) $x(y^2 + z^2) = 1$,

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(iii) $z = xy + \frac{1}{3}y^3$, (*the Cayley surface*).

A generalization of this theorem to higher order derivatives of the cubic form is given in [V2]. In this paper, we will extend this theorem to 3-dimensional affine locally strongly convex hypersurfaces. The Main Theorem that we prove is the following.

MAIN THEOREM. *Let M be a 3-dimensional affine locally strongly convex hypersurface in $\mathbb{R}^4$ with $\hat{\nabla}C = 0$. Then either M is an open part of a locally strongly convex quadric (i.e. $C = 0$) or M is affine equivalent to an open part of one of the following two hypersurfaces*:

(i) $xyzw = 1$,

(ii) $(y^2 - z^2 - w^2)^3 x^2 = 1$.

The condition that C is parallel with respect to the induced affine connection ∇ is treated in [NP2], for surfaces, and in [V1] for 3-dimensional affine hypersurfaces. A partial classification of higher order parallel surfaces, i.e. surfaces which satisfy $\nabla^n C = 0$, for some integer number n, can be found in [DV].

Finally, the authors would like to thank Professor K. Nomizu, for many valuable lectures and discussions on affine differential geometry. Nomizu's lecture notes [N] are a modern approach to affine differential geometry. We mostly follow his notations. We also thank the referee for his valuable comments.

§ 2. Preliminaries

Let $f: M^3 \to \mathbb{R}^4$ be an immersion of a connected differentiable 3-dimensional manifold into the affine space $\mathbb{R}^4$ equipped with its usual flat connection D and a parallel volume element ω and let ξ be an arbitrary local transversal vector field to $f(M^3)$. For any vector fields X, Y, X_1, X_2, X_3, we write

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X, Y)\xi\,, \tag{2.1}$$

$$\theta(X_1, X_2, X_3) = \omega(f_*X_1, f_*X_2, f_*X_3, \xi)\,, \tag{2.2}$$

thus defining an affine connection ∇, a symmetric $(0, 2)$-type tensor h, called the second fundamental form, and a volume element θ. We say that f is nondegenerate if h is nondegenerate (and this condition is independent of the choice of transversal vector field ξ). In this case, it is known (see [N], [NP1]) that there is a unique choice (up to sign) of transversal vector field such that the induced connection ∇, the induced second fundamental form h and the induced volume element θ satisfy the following conditions:

$$\text{(i)} \qquad \nabla\theta = 0\,,$$

$$\text{(ii)} \qquad \theta = \omega_h\,,$$

where ω_h is the metric volume element induced by h. We call ∇ the induced affine connection, ξ the affine normal and h the affine metric. By combining (i) and (ii), we obtain the apolarity condition which states that $\nabla\omega_h = 0$. A nondegenerate immersion equipped with this special transversal vector field is called a Blaschke immersion. Throughout this paper, we will always assume that f is a Blaschke immersion. If h is positive (or negative) definite, the immersion is called locally strongly convex. Notice that if h is negative definite, we can always replace ξ by $-\xi$, thus making the new affine metric positive definite. Therefore, if we say that M is locally strongly convex, we will always assume that ξ is chosen so that h is positive definite.

Condition (i) implies that $D_X\xi$ is tangent to $f(M^3)$ for any tangent vector X to M. Hence, we can define a (1, 1)-tensor field S, called the affine shape operator by

$$D_X\xi = -f_*(SX)\,. \tag{2.3}$$

M is called an affine sphere if $S = \lambda I$. We define the affine mean curvature H by $H = 1/n\,\mathrm{trace}(S)$. The following fundamental equations of Gauss, Codazzi and Ricci are given by

(2.4) $R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY$ (Equation of Gauss)

(2.5) $(\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z)$ (Equation of Codazzi for h)

(2.6) $(\nabla_X S)Y = (\nabla_Y S)X$ (Equation of Codazzi for S)

(2.7) $h(X, SY) = h(SX, Y)$ (Equation of Ricci).

If $\dim(M) \geq 2$ and M is an affine sphere, it follows from (2.6) that λ is constant. If $\lambda \neq 0$, we say that M is a proper affine sphere and if $\lambda = 0$, we call M an improper affine sphere. From (2.5) it follows that the cubic form $C(X, Y, Z) = (\nabla h)(X, Y, Z)$ is symmetric in X, Y, Z. The Theorem of Berwald states that C vanishes identically if and only if M is an open part of a nondegenerate quadric.

Let $\hat{\nabla}$ denote the Levi Civita connection of the affine metric h. The difference tensor K is defined by

$$K(X, Y) = \nabla_X Y - \hat{\nabla}_X Y,$$

for vector fields X and Y on M. Notice that K is symmetric in X and Y. We also write $K_X Y = K(X, Y)$. From [N], we have that

$$h(K_X Y, Z) = -\frac{1}{2} C(X, Y, Z) \tag{2.8}$$

$$\hat{R}(X, Y)Z = \frac{1}{2}(h(Y, Z)SX - h(X, Z)SY + h(SY, Z)X - h(SX, Z)Y) - [K_X, K_Y]Z \tag{2.9}$$

where $\hat{R}$ denotes the curvature tensor of $\hat{\nabla}$. Notice also that the apolarity condition together with (2.8) implies that trace $K_X = 0$ for every tangent vector X. In the special case that M is an affine sphere, i.e. $S = \lambda I$, equation (2.9) becomes

$$\hat{R}(X, Y)Z = \lambda(h(Y, Z)X - h(X, Z)Y) - [K_X, K_Y]Z. \tag{2.10}$$

Further, if M is an affine sphere, we have from [N] that

$$(\hat{\nabla}_Y K)(X, Z) = (\hat{\nabla}_X K)(Y, Z), \tag{2.11}$$

where $(\hat{\nabla}_Y K)(X, Z) = \hat{\nabla}_Y(K(X, Z)) - K(\hat{\nabla}_Y X, Z) - K(X, \hat{\nabla}_Y Z)$. Finally, we need the following results from [BNS], [Y].

THEOREM 2.1 [BNS]. *Let M be an n-dimensional Blaschke hypersurface in $\mathbb{R}^{n+1}$. If $\hat{\nabla} C = 0$, then M is an affine sphere.*

THEOREM 2.2 [Y]. *Let M^3 be a locally strongly convex affine hypersphere in $\mathbb{R}^4$ such that the affine metric h has constant sectional curvature. Then M is an open part of a quadric or M is affine equivalent to an open part of $x_1 x_2 x_3 x_4 = 1$*

A generalization of this last theorem to arbitrary dimensions is given in [VLS].

§ 3. Proof of the theorem

Throughout this section, we will always assume that M is a 3-dimensional, locally strongly convex affine hypersurface in $\mathbb{R}^4$ which has parallel cubic form, i.e. which satisfies $\hat{\nabla} C = 0$. Notice that (2.8) implies that this is equivalent with $\hat{\nabla} K = 0$. From Theorem 2.1, we deduce that M is an affine sphere. First, we remark that if the cubic form C vanishes identically, then from the Berwald theorem it follows that M is an open part of a nondegenerate locally strongly convex quadric. Hence from now on, we will assume that C does not vanish identically. Since C is parallel with respect to $\hat{\nabla}$, it follows that C vanishes nowhere.

We now choose an orthonormal basis with respect to the affine metric h at the point p in the following way. Let $UM_p = \{u \in TM_p \mid h(u, u) = 1\}$. Since M is locally strongly convex, UM_p is compact. We define a function f on UM_p by

$$f(u) = h(K_u u, u),$$

for $u \in UM_p$. Notice that because of (2.8), the function f does not vanish identically. Let e_1 be an element of UM_p at which the function f attains an absolute maximum. Thus $f(e_1) > 0$. Let $v \in UM_p$ such that $\langle v, e_1 \rangle = 0$. Then, we define a real function g by $g(t) = f(\cos(t)e_1 + \sin(t)v)$. Since g attains an absolute maximum at $t = 0$, we have that $g'(0) = 0$ and $g''(0) \leq 0$. Using (2.8) these equations give

$$h(K_{e_1}e_1, v) = 0, \tag{3.1}$$

$$h(K_{e_1}e_1, e_1) - 2h(K_{e_1}v, v) \geq 0, \tag{3.2}$$

for all v satisfying $\langle v, e_1 \rangle = 0$. Hence e_1 is an eigenvector of K_{e_1}, say with eigenvalue λ_1. Then, we choose e_2, e_3 as the other eigenvectors of K_{e_1} with eigenvalues respectively λ_2 and λ_3. Using this, (2.8) and the apolarity we obtain the following formulas for the difference tensor.

$$\begin{aligned}
K_{e_1}e_1 &= \lambda_1 e_1, \\
K_{e_1}e_2 &= \lambda_2 e_2, \\
K_{e_1}e_3 &= \lambda_3 e_3, \\
K_{e_2}e_2 &= \lambda_2 e_1 + ae_2 + be_3, \\
K_{e_2}e_3 &= be_2 - ae_3, \\
K_{e_3}e_3 &= \lambda_3 e_1 - ae_2 - be_3,
\end{aligned}$$

where $a, b \in \mathbb{R}$ and, because of apolarity, $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Further, since $f(e_1) > 0$, we have $\lambda_1 > 0$ and from (3.2) it follows that $\lambda_1 \geq 2\lambda_i$, where $i = 2, 3$. Furthermore, by changing the sign of e_2 or e_3, if necessary, we may assume that $a, b \geq 0$. The next two lemmas will improve further our choice of orthonormal basis.

LEMMA 3.1. *If $\lambda_2 = \lambda_3$, then we can choose e_2 and e_3 in such a way that $b = 0$.*

Proof. If $\lambda_2 = \lambda_3$, then every $u \in UM_p$ which is orthogonal to e_1 is an eigenvector of K_{e_1} with eigenvalue $\lambda_2 = \lambda_3$. Hence, the choice of e_2 and e_3, which we made earlier was not unique. So we can still choose e_2 as *a* vector in which the function f restricted to $B = \{u \in UM_p \mid h(u, e_1) = 0\}$ attains its maximal value. Finally, we pick e_3 such that $\{e_1, e_2, e_3\}$ is an h-orthonormal basis. Since, f, restricted to B, attains a maximal value in e_2 we have $h(K_{e_2}e_2, e_3) = 0$. Hence $b = 0$. □

LEMMA 3.2. *For $i = 1, 2$, we have $\lambda_1 > 2\lambda_i$.*

Proof. Let us assume that $\lambda_1 \leq 2\lambda_2$. We will derive a contradiction. Since then $\lambda_1 = 2\lambda_2$, we have $\lambda_3 = -\frac{3}{2}\lambda_1$. Now, we put $u = (1/\sqrt{2})(-e_1 - e_3)$. Then

$$f(u) = \frac{1}{2\sqrt{2}}(-f(e_1) - 3h(K_{e_1}e_1, e_3) - 3h(K_{e_1}e_3, e_3) - f(e_3)$$
$$= \frac{1}{2\sqrt{2}}\left(-\lambda_1 + \frac{9}{2}\lambda_1 + b\right).$$

Hence we obtain that $f(u) > \lambda_1$. This contradicts the fact the function f attains an absolute maximum in e_1.

LEMMA 3.3. *Let M^3 be a locally strongly convex affine hypersurface in $\mathbb{R}^4$ for which $\hat{\nabla}C = 0$ but $C \neq 0$. Then M is a hyperbolic affine sphere, i.e. $S = \lambda I$ with $\lambda < 0$. Furthermore, let $\{e_1, e_2, e_3\}$ be an orthonormal basis as defined above. Then either one of the following holds:*

(i) $K(e_1, e_2) = \lambda_1 e_1$ $\qquad K(e_1, e_2) = -\frac{1}{2}\lambda_1 e_2$

$K(e_2, e_2) = -\frac{1}{2}\lambda_1(e_1 - \sqrt{2}\, e_2)$ $\qquad K(e_1, e_3) = -\frac{1}{2}\lambda_1 e_3$

$K(e_3, e_3) = -\frac{1}{2}\lambda_1(e_1 + \sqrt{2}\, e_2)$ $\qquad K(e_2, e_3) = -\frac{1}{\sqrt{2}}\lambda_1 e_3$

(ii) $K(e_1, e_1) = \lambda_1 e_1$ $\qquad K(e_1, e_2) = -\frac{1}{2}\lambda_1 e_2$

$K(e_2, e_2) = -\frac{1}{2}\lambda_1 e_1$ $\qquad K(e_1, e_3) = -\frac{1}{2}\lambda_1 e_3$

$K(e_3, e_3) = -\frac{1}{2}\lambda_1 e_1$ $\qquad K(e_2, e_3) = 0$,

where $\lambda_1 = 2\sqrt{-\lambda/3}$.

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Proof. Since $\hat{\nabla} K = 0$, we get $\hat{R} \cdot K = 0$; we obtain for vector fields X, Y, Z, W that

$$0 = \hat{R}(X, Y)K(Z, W) - K(\hat{R}(X, Y)Z, W) - K(Z, \hat{R}(X, Y)W). \tag{3.3}$$

Applying this formula for $X = Z = W = e_1$, $Y = e_i$, $i = 2, 3$, then gives

$$0 = \hat{R}(e_1, e_i)\lambda_1 e_1 - 2K(\hat{R}(e_1, e_i)e_1, e_1). \tag{3.4}$$

By using (2.10), we see that

$$\begin{aligned} \hat{R}(e_1, e_i)e_1 &= -\lambda e_i - [K_{e_1}, K_{e_i}]e_1 \\ &= -\lambda e_i - \lambda_i^2 e_i + \lambda_1 \lambda_i e_i \\ &= (-\lambda - \lambda_i^2 + \lambda_1 \lambda_i)e_i. \end{aligned}$$

By substituting this into (3.4) we see that

$$(\lambda_1 - 2\lambda_i)(-\lambda - \lambda_i^2 + \lambda_1 \lambda_i) = 0.$$

By applying Lemma 3.2 this gives

$$-\lambda - \lambda_i^2 + \lambda_1 \lambda_i = 0. \tag{3.5}$$

By subtracting the equations obtained for $i = 2, 3$, we see that

$$(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_2 - \lambda_3) = 0.$$

Since it follows from Lemma 3.2 that $\lambda_1 - \lambda_2 - \lambda_3 \neq 0$, we obtain that $\lambda_2 = \lambda_3$. Hence by Lemma 3.1, we may assume that $b = 0$. Since by apolarity also $\lambda_1 = -\lambda_2 - \lambda_3$, (3.5) becomes

$$-\lambda - \frac{3}{4}\lambda_1^2 = 0. \tag{3.6}$$

Since $\lambda_1 \neq 0$, we deduce that $\lambda < 0$. Hence M is a hyperbolic affine hypersphere. Moreover it then follows from (3.6) that $\lambda_1 = 2\sqrt{-\lambda/3}$.

Using the previous results, we find that

$$\begin{aligned} \hat{R}(e_2, e_3)e_1 &= -[K_{e_2}, K_{e_3}]e_1 \\ &= -\lambda_3 K(e_2, e_3) + \lambda_2 K(e_3, e_2) = 0 \\ \hat{R}(e_2, e_3)e_2 &= -\lambda e_3 - K_{e_2}K_{e_3}e_2 + K_{e_3}K_{e_2}e_2 \\ &= (-\lambda - 2a^2 + \lambda_2 \lambda_3)e_3. \end{aligned}$$

So if we then substitute $X = Z = W = e_2$ and $Y = e_3$ in (3.3), we get

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$$\begin{aligned} 0 &= \hat{R}(e_2, e_3)(\lambda_2 e_1 + a e_2) - 2(-\lambda - 2a^2 + \lambda_2\lambda_3)K(e_2, e_3) \\ &= 3a(-\lambda - 2a^2 + \lambda_2\lambda_3)e_3 \\ &= 3a\Big(-\lambda - 2a^2 + \frac{1}{4}\lambda_1^2\Big)e_3 \\ &= 3a\Big(-2a^2 - \frac{4}{3}\lambda\Big)e_3 . \end{aligned}$$

Hence $a = 0$ or $a = \sqrt{-2\lambda/3}$. □

LEMMA 3.4. *If Lemma* 3.3 (i) *holds at a point p then all sectional curvatures* (*w.r.t.* $\hat{R}$ *and* h) *are zero. Moreover* $h(K, K) = 6\lambda^2$. *If Lemma* 3.3 (ii) *holds at a point p then* $h(K, K) = (10/3)\lambda^2$.

Proof. From (2.10) and Lemma 3.1, we obtain that

$$\begin{aligned} \hat{R}(e_1, e_2)e_2 &= \hat{R}(e_1, e_3)e_3 = \hat{R}(e_2, e_3)e_3 = 0 , \\ \hat{R}(e_1, e_2)e_3 &= \hat{R}(e_2, e_3)e_1 = \hat{R}(e_3, e_1)e_2 = 0 . \end{aligned}$$

Linearization then implies that $\hat{R} = 0$. The remaining claim follows straightforwardly from Lemma 3.3. □

Since $h(K, K)$ is different for the cases (i) and (ii), it follows that Lemma 3.3 (i) holds at every point p of M or Lemma 3.3 (ii) holds at every point p of M. Notice that if Lemma 3.3 (i) holds at every point p of M, then from Lemma 3.4 it follows that M has constant zero sectional curvature. Applying Theorem 2.2 then shows that M is affine equivalent to an open part of $xyzw = 1$. So from now on, we will assume that Lemma 3.3 (ii) holds at every point p of M. The following lemma then shows that we can extend the basis we found differentiably to a neighbourhood.

LEMMA 3.5. *Let M be an affine* 3-*dimensional locally strongly convex affine hypersurface in* $\mathbb{R}^4$ *with* $\hat{\nabla}C = 0$. *Assume that Lemma* 3.3 (ii) *holds at every point of M. Then around any point, there exists a local basis* $\{E_1, E_2, E_3\}$, *orthonormal with respect to h, such that*

$$\begin{aligned} K(E_1, E_1) &= \lambda_1 E_1 , & K(E_1, E_2) &= -\frac{1}{2}\lambda_1 E_2 , \\ K(E_2, E_2) &= -\frac{1}{2}\lambda_1 E_1 , & K(E_1, E_3) &= -\frac{1}{2}\lambda_1 E_3 , \\ K(E_3, E_3) &= -\frac{1}{2}\lambda_1 E_1 , & K(E_2, E_3) &= 0 , \end{aligned}$$

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where $\lambda_1 = 2\sqrt{-\lambda/3}$.

Proof. Let $p \in M$. We take the orthonormal basis $\{e_1, e_2, e_3\}$ given by Lemma 3.3 (ii). We extend this basis, by parallel translation along geodesics (with respect to $\hat{\nabla}$) through p to a normal neighbourhood around p. By the properties of parallel translation this gives an h-orthonormal basis defined on a neighbourhood of p. Since $\hat{\nabla} K = 0$, it also follows that K has the desired form at every point of a normal neighbourhood. □

LEMMA 3.6. *Let M be as in Lemma 3.5, let $p \in M$ and let $\{E_1, E_2, E_3\}$ be the local orthonormal basis given by Lemma 3.5. Then for any vector field X on M we have that*

$$\hat{\nabla}_X E_1 = 0 .$$

Moreover (M, h), considered as a Riemannian manifold, is locally isometric to $\mathbb{R} \times H$, where H is the hyperbolic plane of constant negative curvature $\frac{4}{3}\lambda$. Also, after identification, the local vector field E_1 is tangent to $\mathbb{R}$.

Proof. Let $p \in M$. We take the h-orthonormal basis given by Lemma 3.5. Since $\hat{\nabla} K = 0$, we have that

$$\begin{aligned} 0 &= (\hat{\nabla}_{E_i} K)(E_1, E_1) \\ &= \lambda_1 \hat{\nabla}_{E_i} E_1 - 2K(\hat{\nabla}_{E_i} E_1, E_1) , \end{aligned}$$

for $i = 1, 2, 3$. Since $\hat{\nabla}_{E_i} E_1$ is h-orthogonal to E_1, this last equation implies that

$$0 = 2\lambda_1 \hat{\nabla}_{E_i} E_1 .$$

In order to show that M is locally isometric to $\mathbb{R} \times H$, we define two local distributions T_0 and T_1 by

$$\begin{aligned} T_0 &: q \longmapsto T_0|_q = \operatorname{span}\{E_1(q)\} , \\ T_1 &: q \longmapsto T_1|_q = \{v \in TM_q \mid h(v, E_1(q)) = 0\} . \end{aligned}$$

Since $\hat{\nabla}_X E_1 = 0$, we have $\hat{\nabla}_{T_0} T_0 \subset T_0$ and $\hat{\nabla}_{T_1} T_0 \subset T_0$. Since T_0 and T_1 are h-orthogonal this then implies that also $\hat{\nabla}_X T_1 \subset T_1$ for any vector field X. Therefore, it follows from the de Rham decomposition theorem ([KN]) that (M, h) is locally isometric to $\mathbb{R} \times H$, where H is a surface. Moreover since $E_1 \in T_0$, after identification E_1 is tangent to the $\mathbb{R}$-component.

Finally, we notice from (2.10) and Lemma 3.5 that

$$\hat{R}(E_2, E_3)E_3 = \frac{4}{3}\lambda E_2 .$$

Hence H has constant negative curvature $\frac{4}{3}\lambda$ and therefore, H is locally isometric to the hyperbolic plane. □

Finally, we have the following lemma.

LEMMA 3.7. *Let M be as in Lemma 3.5. Then, M is affine equivalent to an open part of the affine hypersurface described by*

$$(y^2 - z^2 - w^2)^3 x^2 = 1 .$$

Proof. By Lemma 3.3, we know that $\lambda < 0$. Hence, by applying a suitable homothetic transformation, we may assume that $\lambda = -1$. Let $p \in M$ and let $\{E_1, E_2, E_3\}$ be the basis given by Lemma 3.5. First, we notice that if we put $U_2 = \cos\theta E_2 + \sin\theta E_3$ and $U_3 = -\sin\theta E_2 + \cos\theta E_3$, then the new h-orthonormal basis $\{E_1, U_2, U_3\}$ also satisfies Lemma 3.5.

Further, we will denote the immersion of M into $\mathbb{R}^4$ by x. Then, after applying a translation, we may assume that $\xi = x$. Next, by Lemma 3.6, we know that M is h-isometric to $\mathbb{R} \times H$, where H is the hyperbolic plane with constant negative curvature $-\frac{4}{3}$, and E_1 is tangent to the $\mathbb{R}$-component. So, using the standard parametrization of the hypersphere model of H, we see that there exist local coordinates $\{u, v, w\}$ on M, such that $E_1 = x_w$, and such that x_u and $(1/\sinh(2/\sqrt{3}u))x_v$, together with x_w form an h-orthonormal basis. So by the remark made in the beginning of the proof, we may assume that $E_2 = x_u$ and $\sinh(2/\sqrt{3}u)E_3 = x_v$. A straightforward computation then also shows that

$$\hat{\nabla}_{x_u} x_u = 0 ,$$
$$\hat{\nabla}_{x_u} x_v = \hat{\nabla}_{x_v} x_u = \frac{2}{\sqrt{3}} \coth\Big(\frac{2}{\sqrt{3}} u\Big) x_v$$
$$\hat{\nabla}_{x_v} x_v = -\frac{2}{\sqrt{3}} \sinh\Big(\frac{2}{\sqrt{3}} u\Big)\cosh\Big(\frac{2}{\sqrt{3}} u\Big) x_u .$$

So, using the definition of K, we get the following system of differential equations, where in order to simplify the equations, we have put $c = \sqrt{3}$.

$$x_{ww} = \frac{2}{c} x_w + x , \tag{3.7}$$

$$x_{uw} = -\frac{1}{c} x_u , \tag{3.8}$$

$$x_{vw} = -\frac{1}{c} x_v \,, \tag{3.9}$$

$$x_{uu} = -\frac{1}{c} x_w + x \,, \tag{3.10}$$

$$x_{uv} = \frac{2}{c} \coth\left(\frac{2}{c} u\right) x_v \,, \tag{3.11}$$

$$\begin{aligned} x_{vv} = & -\frac{1}{c}\left(\sinh\left(\frac{2}{c} u\right)\right)^2 x_w - \frac{2}{c} \sinh\left(\frac{2}{c} u\right) \cosh\left(\frac{2}{c} u\right) x_u \\ & + \left(\sinh\left(\frac{2}{c} u\right)\right)^2 x \,. \end{aligned} \tag{3.12}$$

First, we see from (3.7) that there exist vector valued functions $P_1(u, v)$ and $P_2(u, v)$ such that

$$x = P_1(u, v) \exp(cw) + P_2(u, v) \exp\left(-\frac{1}{c} w\right).$$

From (3.8) and (3.9) it then follows that the vector valued function P_1 is independent of u and v. Hence there exists a constant vector A_1 such that $P_1(u, v) = A_1$. Next it follows from (3.10) that P_2 satisfies the following differential equation:

$$(P_2)_{uu} = \frac{4}{3} P_2 \,.$$

Hence we can write

$$P_2(u, v) = Q_1(v) \cosh\left(\frac{2}{c} u\right) + Q_2(v) \sinh\left(\frac{2}{c} u\right).$$

From (3.11), we then deduce that there exists a constant vector A_2 such that $Q_1(v) = A_2$. Finally, from (3.12), we get the following differential equation for Q_2:

$$(Q_2)_{vv} = -\frac{4}{3} Q_2 \,.$$

This last formula implies that there exist constant vectors A_3 and A_4 such that

$$Q_2(v) = A_3 \cos\left(\frac{2}{c} v\right) + A_4 \sin\left(\frac{2}{c} v\right).$$

Since M is nondegenerate, M lies linearly full in $\mathbb{R}^4$. Hence A_1, A_2, A_3, A_4

are linearly independent vectors. Thus there exist an affine transformation such that

$$x = \left(\exp(cw),\ \cosh\left(\frac{2}{c}u\right)\exp\left(-\frac{1}{c}w\right),\right.$$
$$\left.\cos\left(\frac{2}{c}v\right)\sinh\left(\frac{2}{c}u\right)\exp\left(-\frac{1}{c}w\right),\ \sin\left(\frac{2}{c}v\right)\sinh\left(\frac{2}{c}u\right)\exp\left(-\frac{1}{c}w\right)\right).$$

So clearly the image of M lies, upto an affine transformation, locally on $(y^2 - z^2 - w^2)^3x^2 = 1$. The analyticity of this last hypersurface then completes the proof. □

So, by combining this lemma with the previous results we see that a 3-dimensional locally strongly convex hypersurface M in $\mathbb{R}^4$ with $\hat{\nabla}C = 0$ is either a quadric or else satisfies Lemma 3.3 (i) at every point p or satisfies Lemma 3.3 (ii) at every point p. In the second case, we see from Lemma 3.4 that M has constant sectional curvature. So by applying Theorem 2.2, we see that M is affine equivalent to the affine hypersurface given by $xyzw = 1$. Finally, in the last case, Lemma 3.7 completes the proof.

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