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THE UNIQUENESS OF ELEMENTARY EMBEDDINGS

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Abstract. Much of the theory of large cardinals beyond a measurable cardinal concerns the structure of elementary embeddings of the universe of sets into inner models. This paper seeks to answer the question of whether the inner model uniquely determines the elementary embedding.

§1. Introduction. Much of the theory of large cardinals beyond a measurable cardinal concerns the structure of elementary embeddings of the universe of sets into inner models. This paper seeks to answer the question of whether the inner model uniquely determines the elementary embedding.

The question cannot be answered assuming ZFC alone: in unpublished work, exposited in Section 3, Woodin observed that it is consistent that there are distinct normal ultrafilters with the same ultrapower. He proved, however, that definable embeddings of the universe into the same model must agree *on the ordinals,* and under a strong version of the HOD Conjecture, he proved the same result for arbitrary elementary embeddings. Woodin conjectured that the result can be proved in second-order set theory (NBG) with the Axiom of Choice. The first theorem of this paper confirms his conjecture:

THEOREM 3.5. *Any two elementary embeddings from the universe into the same inner model agree on the ordinals.*

In Section 4, we prove stronger uniqueness properties of elementary embeddings assuming global large cardinal axioms. To avoid repeating the same hypothesis over and over, we introduce the following terminology:

DEFINITION 1.1. If δ is an ordinal, we say *the uniqueness of elementary embeddings holds above* δ if for any inner model M, there is at most one elementary embedding from the universe into M with critical point greater than δ.

We say *the uniqueness of elementary embeddings holds* if it holds above 0. The uniqueness of elementary embeddings is formulated in the language of second-order set theory.

It turns out that the uniqueness of elementary embeddings holds above sufficiently large cardinals:

THEOREM 4.20. *The uniqueness of elementary embeddings holds above the least extendible cardinal.*

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The hypothesis of Theorem 4.20 seems to be optimal: for example, Theorem 4.21 shows that it is consistent with a proper class of supercompact cardinals that the uniqueness of elementary embeddings fails above every cardinal.

Our analysis also yields the uniqueness of elementary embeddings (above 0) from other hypotheses. The independence results of [2, 8, 12] are often taken to show that Reitz's Ground Axiom has no consequences. The following theorem indicates that this may not be completely true:

THEOREM 4.19 (Ground Axiom). *If there is a proper class of strongly compact cardinals, then the uniqueness of elementary embeddings holds.*

The conclusion of this result (that is, the outright uniqueness of elementary embeddings) cannot be proved from any large cardinal axiom by the proof of Theorem 3.1 and the Lévy–Solovay theorem [11]. It is unclear whether the strongly compact cardinals are necessary, though: it seems unlikely, but it could be that the Ground Axiom alone suffices to prove the result. With this in mind, we conclude the section by proving a similar result from a different hypothesis.

THEOREM 4.32 (Ground Axiom). *If there is a proper class of strong cardinals, then the uniqueness of ultrapower embeddings holds.*

In Section 5, we consider the situation under the Ultrapower Axiom (UA), which turns out to be quite simple:

THEOREM 5.2 (UA). *The uniqueness of elementary embeddings holds.*

Finally, we use Theorem 4.20 to analyze a principle called the Weak Ultrapower Axiom (Weak UA) under the assumption of an extendible cardinal. Weak UA states that any two ultrapowers of the universe of sets have a common internal ultrapower. Before this work, we knew of no consequences of Weak UA. Here we sketch the proofs of some results indicating that above an extendible cardinal, Weak UA is almost as powerful as UA:

THEOREM 5.13 (Weak UA). *If κ is extendible, then V is a generic extension of HOD by a forcing in V_κ.*

The reader familiar with Vopěnka's theorem will note that this is just a fancy way of saying that every set is ordinal definable from some fixed parameter $x \in V_\kappa$. We also show that UA holds in HOD for embeddings with critical point greater than or equal to κ (see Theorem 5.18 for a precise statement). We do not know how to show this is true in V! Combining this with some proofs from [6] allows us to prove the GCH above the first extendible cardinal under Weak UA.

THEOREM 5.20 (Weak UA). *If κ is extendible, then for all cardinals $\lambda \geq \kappa$, $2^\lambda = \lambda^+$.*

§2. Preliminaries.

2.1. Ultrapower embeddings and extender embeddings. If P and Q are models of ZFC, $i : P \to Q$ is an elementary embedding, $X \in P$ is a set, and $a \in i(X)$, then there is a minimum elementary substructure of Q containing $i[P] \cup \{a\}$; namely, the substructure

$$H^Q(i[P] \cup \{a\}) = \{i(f)(a) : f \in P, f : X \to P\}.$$

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The fact that $H^Q(i[P] \cup \{a\})$ is an elementary substructure of Q is a consequence of the Axiom of Choice and the Axiom of Collection, applied in P, and is really just a restatement of Łós's Theorem applied to the P-ultrafilter $U = \{A \subseteq X : A \in P, a \in i(A)\}$.

A similar argument shows that for any $X \in P$ and $B \subseteq i(X)$,

$$H^Q(i[P] \cup B) = \{i(f)(a) : a \in [B]^{<\omega},\ f \in P,\ f : [X]^{<\omega} \to P\}$$

is an elementary substructure of Q. As a consequence of this, if $i : P \to Q$ is a *cofinal* elementary embedding, in the sense that every $a \in Q$ belongs to $i(X)$ for some $X \in P$, then for any $B \subseteq Q$,

$$H^Q(i[P] \cup B) = \{i(f)(a) : a \in [B]^{<\omega},\ f \in P\}$$

is an elementary substructure of Q. (Here f ranges over all functions in P such that $a \in \operatorname{dom}(i(f))$.)

Definition 2.1. Suppose P and Q are models of set theory. An elementary embedding $i : P \to Q$ is an *ultrapower embedding* of P if there is some $X \in P$ and some $a \in i(X)$ such that $Q = H^Q(i[P] \cup \{a\})$.

An elementary embedding $i : P \to Q$ is an ultrapower embedding if and only if there is a P-ultrafilter U and an isomorphism $k : \mathrm{Ult}(P, U) \to Q$ such that $k \circ j_U = i$.

Definition 2.2. An elementary embedding $i : P \to Q$ is an *extender embedding* if there is a set A such that $Q = H^Q(i[P] \cup i(A))$.

Again, an elementary embedding is an extender embedding if and only if it is isomorphic to the ultrapower of P by a P-extender (using only functions in P).

A *generator* of an elementary embedding $i : P \to Q$ is an ordinal ξ of Q such that $\xi \notin H^Q(i[P] \cup \xi)$. By the well-ordering theorem, $Q = H^Q(i[P] \cup \mathrm{Ord}^Q)$, and if Q is wellfounded, it follows easily that $Q = H^Q(i[P] \cup G)$ where G is the class of generators of i. Thus a cofinal elementary embedding is an extender embedding if and only if its generators are bounded in Q.

Definition 2.3. Given an elementary embedding $i : P \to Q$ and a set $a \in Q$, we let $\lambda_i(a)$ denote the least P-cardinality of a set X such that $a \in i(X)$.

Note that $\lambda_i(a)$ may not be defined, either because there is no X such that $a \in i(X)$ or because there is no minimum cardinality of such a set. We will be focused solely on cofinal elementary embeddings of wellfounded models, in which case $\lambda_i(a)$ will always be defined.

By the well-ordering theorem, $\lambda_i(a)$ (if infinite) is the least ordinal λ of P such that $a \in H^Q(i[P] \cup i(\lambda))$. This immediately implies the following:

Lemma 2.4. *Suppose $i : P \to Q$ is an elementary embedding, a is a point in Q, and $\lambda = \lambda_i(a)$. If $\lambda > 1$, then $i(\lambda) \neq \sup i[\lambda]$.*

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§3. The uniqueness of embeddings on the ordinals.

3.1. Woodin's results.

THEOREM 3.1 (Woodin). *If it is consistent that there is a measurable cardinal, then it is consistent that the uniqueness of elementary embeddings fails.*

PROOF. Suppose κ is a measurable cardinal, and assume without loss of generality that $2^\kappa = \kappa^+$. Let $\langle \dot{\mathbb{P}}_{\alpha\beta} : \alpha \leq \beta < \kappa \rangle$ be the Easton support iteration where $\dot{\mathbb{P}}_{\alpha\alpha}$ is trivial unless α is an inaccessible non-Mahlo cardinal, in which case $\dot{\mathbb{P}}_{\alpha\alpha} = \text{Add}(\alpha, 1)$. Thus $\dot{\mathbb{P}}_{0\alpha}$ names a partial order in V, which we denote by $\mathbb{P}_{0\alpha}$.

Let $G \subseteq \mathbb{P}_{0\kappa}$ be a V-generic filter, let $G_{0\alpha} \subseteq \mathbb{P}_{0\alpha}$ be the restriction of G to $\mathbb{P}_{0\alpha}$, let $\mathbb{P}_{\alpha\beta} = (\mathbb{P}_{\alpha\beta})_{G_{0\alpha}}$, and let $G_{\alpha\beta} \subseteq \mathbb{P}_{\alpha\beta}$ be the $V[G_{0\alpha}]$-generic filter induced by G.

In V, let U be a normal ultrafilter on κ. We claim that in $V[G]$, there are distinct normal ultrafilters U_0 and U_1 extending U such that $\text{Ult}(V[G], U_0) = \text{Ult}(V[G], U_1)$. Let $j : V \to M$ be the ultrapower of V by U. Let

$$\langle \dot{\mathbb{P}}_{\alpha\beta} : \alpha \leq \beta < j(\kappa) \rangle = j(\langle \dot{\mathbb{P}}_{\alpha\beta} : \alpha \leq \beta < \kappa \rangle).$$

Let $\mathbb{P}_{\kappa,j(\kappa)} = (\dot{\mathbb{P}}_{\kappa,j(\kappa)})_G$. Then since $M[G]$ is closed under κ-sequences in $V[G]$, $\mathbb{P}_{\kappa,j(\kappa)}$ is $\leq\kappa$-closed. Moreover, the set of maximal antichains of $\mathbb{P}_{\kappa,j(\kappa)}$ that belong to $M[G]$ has cardinality κ^+ in $V[G]$. Therefore working in $V[G]$, one can construct an $M[G]$-generic filter $G_{\kappa,j(\kappa)} \subseteq \mathbb{P}_{\kappa,j(\kappa)}$. Note that $j[G_{0\kappa}] \subseteq G_{0\kappa} * G_{\kappa,j(\kappa)}$. Letting $H_0 = G_{0\kappa} * G_{\kappa,j(\kappa)}$, this implies that j extends to an elementary embedding $j_0 : V[G] \to M[H_0]$ such that $j_0(G) = H_0$.

Notice that one obtains a second $M[G]$-generic filter $G^*_{\kappa,j(\kappa)} \subseteq \mathbb{P}_{\kappa,j(\kappa)}$ by flipping the bits of each component of $G_{\kappa,j(\kappa)}$. Let $H_1 = G_{0\kappa} * G^*_{\kappa,j(\kappa)}$. Obviously, $M[H_0] = M[H_1]$. Moreover, $j[G] \subseteq H_1$, so j extends to an elementary embedding $j_1 : V[G] \to M[H_1]$ such that $j_1(G) = H_1$. Letting U_0 and U_1 be the normal ultrafilters of $V[G]$ derived from j_0 and j_1 respectively, we have $\text{Ult}(V[G], U_0) = M[H_0] = M[H_1] = \text{Ult}(V[G], U_1)$. Since $j_0(G) \neq j_1(G)$, $U_0 \neq U_1$.

In any case, $j_0 : V[G] \to M[H_0]$ and $j_1 : V[G] \to M[H_1]$ witness the failure of the uniqueness of elementary embeddings. ⊣

THEOREM 3.2. *It is consistent that there exist distinct normal ultrafilters U_0 and U_1 with the same ultrapower M such that $j_{U_0}(U_0) = j_{U_1}(U_1)$.*

PROOF. Let U be a normal ultrafilter on a measurable cardinal κ and let $j : V \to M$ denote its ultrapower. Assume $2^\kappa = \kappa^+$. Let $\mathbb{P}$ be the Easton product $\prod_{\delta \in I} \text{Add}(\delta, 1)$ where I is the set of inaccessible non-Mahlo cardinals less than κ.

Let $\mathbb{Q} = j(\mathbb{P})$ and let $\mathbb{Q}/\mathbb{P}$ denote the product $\prod_{\delta \in j(I) \setminus \kappa} \text{Add}(\delta, 1)$ as computed in M. Thus $\mathbb{Q} \cong \mathbb{P} \times (\mathbb{Q}/\mathbb{P})$. Since $\kappa \notin j(I)$, $\mathbb{Q}/\mathbb{P}$ is $\leq\kappa$-closed in M, and hence $\mathbb{Q}/\mathbb{P}$ is $\leq\kappa$-closed and in V. Also $\mathbb{Q}/\mathbb{P}$ is $j(\kappa)$-cc in M, and so one can enumerate the maximal antichains $\langle A_\alpha : \alpha < \kappa^+ \rangle$ of $\mathbb{Q}/\mathbb{P}$ that belong to M using that $|j(\kappa)| = \kappa^+$.

Fix a well-order $\preceq$ of $\mathbb{Q}/\mathbb{P}$, and let $\langle p_\alpha : \alpha < \kappa^+ \rangle$ be a continuous descending sequence in $\mathbb{Q}/\mathbb{P}$ defined by letting $p_{\alpha+1}$ be the $\preceq$-least element of $\mathbb{Q}/\mathbb{P}$ below p_α and an element of A_α. Then $G = \{p \in \mathbb{Q}/\mathbb{P} : \exists\alpha\, p_\alpha \leq p\}$ is an M-generic filter.

We denote by σ_α the involution of $\mathbb{P}$ that flips the bits of the Cohen sets added to cardinals above α. We overload notation by denoting the involution of $\mathbb{Q}$ that flips the bits of the Cohen sets added to cardinals above α in exactly the same way.

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Now we pass to a forcing extension: let $H \subseteq \mathbb{P}$ be a V-generic filter. In $V[H]$, we extend j in two different ways. Let $j_0 : V[H] \to M[H \times G]$ be the unique extension of j such that $j_0(H) = H \times G$. Let $j_1 : V[H] \to M[H \times G]$ be the unique extension of j such that $j_1(H) = \sigma_\kappa(H \times G)$.

Let U_i be the normal ultrafilter on κ derived from j_i using κ. We claim U_0 and U_1 are as desired. Note that $j_i = j_{U_i}$ since $M \cup \{H \times G\} \subseteq H^{M[H\times G]}(j_i[V[G]] \cup \{\kappa\})$, and hence $M[H \times G] = H^{M[H\times G]}(j_i[V[G]] \cup \{\kappa\})$. Therefore $U_0 \neq U_1$, since $j_0 \neq j_1$. On the other hand, to show $j_0(U_0) = j_1(U_1)$, it suffices to show that $j_0(j_0) = j_1(j_1)$, and for this we just need that $j_0(j_0)$ and $j_1(j_1)$ agree on $H \times G$. This is a consequence of the following computation:

$$\begin{aligned}
j_1(j_1)(H \times G) &= j_1(j_1)(\sigma_\kappa(j_1(H)))\\
&= \sigma_\kappa(j_1(j_1(H)))\\
&= \sigma_\kappa(j_1(\sigma_\kappa(H \times G)))\\
&= \sigma_\kappa \circ \sigma_{j(\kappa)}(j_1(H \times G))\\
&= \sigma_\kappa \circ \sigma_{j(\kappa)}(\sigma_\kappa(H \times G) \times j_1(G))\\
&= \sigma_\kappa \circ \sigma_{j(\kappa)}(\sigma_\kappa(H \times G) \times j_0(G))\\
&= \sigma_\kappa \circ \sigma_\kappa(H \times G \times j_0(G))\\
&= H \times G \times j_0(G)\\
&= j_0(H) \times j_0(G)\\
&= j_0(H \times G)\\
&= j_0(j_0(H))\\
&= j_0(j_0)(j_0(H))\\
&= j_0(j_0)(H \times G).
\end{aligned}$$ ⊣

Given this independence result, the following theorem is quite counterintuitive:

THEOREM 3.3 (Woodin). *Assume $V = \mathrm{HOD}$. Then the uniqueness of elementary embeddings holds.*

It is worth pondering why one cannot refute this theorem by first forcing the failure of the uniqueness of elementary embeddings as in Theorem 3.1 and then forcing $V = \mathrm{HOD}$ by some highly closed coding forcing. For definable elementary embeddings, Woodin proved more:

THEOREM 3.4 (Woodin). *Suppose $j_0, j_1 : V \to M$ are definable elementary embeddings from the universe into the same inner model. Then for every ordinal α, $j_0(\alpha) = j_1(\alpha)$.*

PROOF. Fix a number n (in the metatheory), and we will prove the theorem for Σ_n-definable elementary embeddings. Towards a contradiction, let α be the least ordinal such that there exist Σ_n-definable elementary embeddings $j_0, j_1 : V \to M$ such that $j_0(\alpha) \neq j_1(\alpha)$.

Notice that α is definable in V without parameters. To see this, let $U \subseteq V \times V$ be a universal Σ_n-class. Note that α is the least ordinal such that there exist sets p_0 and p_1 such that for $n \in \{0, 1\}$, the class $j_n = \{a : (p_n, a) \in U\}$ forms an Σ_0-elementary embedding from V to an inner model $M_n = \bigcup \mathrm{ran}(j_n)$, and $M_0 = M_1$

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but $j_0(\alpha) \neq j_1(\alpha)$. (Here we use the fact that any Σ_0-elementary embedding from the universe of sets into an inner model is in fact fully elementary; it is a first-order property of p_n that j_n is Σ_0-elementary since the Σ_0-satisfaction predicate of V is definable. This is a result due to Gaifman; see [9, Proposition 5.1].)

Therefore if $k_0, k_1 : V \to N$ are elementary embeddings, then $k_0(\alpha)$ is the unique ordinal defined in N by the formula defining α in V, and similarly for $k_1(\alpha)$. Hence $k_0(\alpha) = k_1(\alpha)$, contradicting the definition of α. ⊣

3.2. Uniqueness of embeddings on the ordinals. Theorem 3.4 raises an interesting second-order question. Working in second-order set theory, suppose $j_0, j_1 : V \to M$ are elementary embeddings. Must j_0 and j_1 agree on the ordinals? Woodin conjectured that the answer is yes. Here we verify his conjecture.

THEOREM 3.5. *Any two embeddings from the universe of sets into the same inner model agree on the ordinals.*

Roughly speaking, we proceed by reducing the question to the case of definable embeddings (in fact, ultrapower embeddings).

DEFINITION 3.6. An elementary embedding $j : V \to M$ is *almost an ultrapower embedding* if for every set $B \subseteq M$, there is some $a \in M$ such that $B \subseteq H^M(j[V] \cup \{a\})$.

I am grateful to Moti Gitik and the anonymous referee for pointing out an error in the proof of the following theorem as it appeared in an early draft of this paper, which has now been corrected:

THEOREM 3.7. *Suppose $j_0, j_1 : V \to M$ are elementary embeddings. Then there exist elementary embeddings $i_0, i_1 : V \to N$ and an elementary embedding $k : N \to M$ such that i_0 and i_1 are almost ultrapower embeddings and $j_0 = k \circ i_0$ and $j_1 = k \circ i_1$.*

PROOF. Suppose $j_0, j_1 : V \to M$ are elementary embeddings. Let $X = H^M(j_0[V] \cup j_1[V])$, let N be the transitive collapse of X, and let $k : N \to M$ be the inverse of the transitive collapse map. Let $i_0, i_1 : V \to N$ be the collapses of j_0, j_1; that is $i_0 = k^{-1} \circ j_0$ and $i_1 = k^{-1} \circ j_1$.

We claim that for all sets A, there is a point $g \in i_1[V]$ such that $i_1[A] \subseteq H^N(i_0[V] \cup \{g\})$. To see this, let B be a set of cardinality $|A|$ such that $i_0 \restriction B = i_1 \restriction B$. Note that such a set exists because i_0 and i_1 have an ω-closed unbounded class of common fixed points. Let $f : B \to A$ be a surjection. Let $g = i_1(f)$. For all $a \in A$, $a = f(b)$ for some $b \in B$, and so

$$i_1(a) = i_1(f)(i_1(b)) = g(i_0(b)) \in H^N(i_0[V] \cup \{g\}).$$

Thus $i_1[A] \subseteq H^N(i_0[V] \cup \{g\})$.

We now show that i_0 is almost an ultrapower embedding. Fix $B \subseteq N$. Since $N = H^N(i_0[V] \cup i_1[V])$, there is a set A such that $B \subseteq H^N(i_0[V] \cup i_1[A])$. The previous paragraph yields $g \in i_1[V]$ such that $i_1[A] \subseteq H^N(i_0[V] \cup \{g\})$. Hence $B \subseteq H^N(i_0[V] \cup \{g\})$, as desired. ⊣

Under favorable cardinal arithmetic hypotheses, one can remove the word "almost" in the statement of the previous theorem. We will say here that the *eventual*

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singular cardinals hypothesis (eventual SCH) holds if for all sufficiently large strong limit cardinals λ of cofinality ω, $2^\lambda = \lambda^+$. (By Silver's theorem, this also implies $2^\lambda = \lambda^+$ for strong limit singular cardinals of uncountable cofinality, but we will not need this, and this form will be more convenient. Our eventual SCH is a bit weaker than the more natural version asserting that for all sufficiently large singular λ, $\lambda^{\mathrm{cf}(\lambda)} = 2^{\mathrm{cf}(\lambda)} \cdot \lambda^+$.)

LEMMA 3.8 (Eventual SCH). *Any elementary embedding from the universe into an inner model closed under ω-sequences is an extender embedding.*

PROOF. Suppose not, and fix an elementary embedding $j : V \to M$ such that M is closed under ω-sequences but j is not an extender embedding. Then the class $\{\lambda_j(a) : a \in M\}$ is unbounded (see Definition 2.3). Otherwise, let γ be its supremum. Then

$$M = H^M(j[V] \cup j(\gamma))$$

contrary to the fact that j is not an extender embedding.

Let η be such that for all strong limit cardinals $\lambda > \eta$ of countable cofinality, $2^\lambda = \lambda^+$. By recursion, construct a sequence of points $a_n \in M$ such that $\lambda_j(a_0) > \eta$, $\lambda_j(a_{n+1}) > j(2^{\lambda_j(a_n)})$. Let $\lambda = \sup_{n<\omega} \lambda_j(a_n)$.

Note that λ is a strong limit cardinal of countable cofinality, so $2^\lambda = \lambda^+$. Moreover, $j[\lambda] \subseteq \lambda$, and so since $j(\omega) = \omega$, $j(\lambda) = \lambda$. In particular, j is continuous at λ. Also $j(\lambda^+) = (j(\lambda)^+)^M \leq \lambda^+$. In particular, j is continuous at λ^+.

Let $a = \langle a_n : n < \omega\rangle$. Then $a \in M$. In fact, since $a \subseteq H^M(j[V] \cup j(\lambda))$, $a \in H^M(j[V] \cup j({}^\omega\lambda))$. So $\lambda_j(a) \leq \lambda^\omega = \lambda^+$. On the other hand since $\langle \lambda_j(a_n) : n < \omega\rangle$ is cofinal in λ, $\lambda_j(a) \geq \lambda$. Thus $\lambda_j(a)$ is either λ or λ^+. Since j is continuous at λ and λ^+, this contradicts Lemma 2.4. ⊣

COROLLARY 3.9 (Eventual SCH). *If $j : V \to M$ is almost an ultrapower embedding, then j is an ultrapower embedding.*

PROOF. It is easy to see that M is closed under ω-sequences, and therefore Lemma 3.8 implies that j is an extender embedding. Essentially by definition, an extender embedding that is almost an ultrapower embedding is indeed an ultrapower embedding. ⊣

COROLLARY 3.10 (Eventual SCH). *If $j_0, j_1 : V \to M$ are elementary embeddings, then $j_0 \restriction \mathrm{Ord} = j_1 \restriction \mathrm{Ord}$.*

PROOF. Apply Theorem 3.7 to reduce to the case that j_0 and j_1 are almost ultrapowers. Then apply Corollary 3.9 to conclude that they are in fact ultrapower embeddings. Finally, since ultrapower embeddings are definable, apply Woodin's theorem (Theorem 3.4) to conclude the corollary. ⊣

We now turn to the proof of the uniqueness of elementary embeddings on the ordinals without SCH, for which it is convenient to introduce the notion of the *tightness function* of an elementary embedding.

DEFINITION 3.11. Suppose $j : V \to M$ is an elementary embedding and X is a set. Then $t_j(X)$ denotes the minimum M-cardinality of a set $A \in M$ such that $j[X] \subseteq A$.

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The tightness function turns out to depend only on the cardinality of its argument:

LEMMA 3.12. *Suppose $j : V \to M$ is an elementary embedding. If $|X| \leq |Y|$, then $t_j(X) \leq t_j(Y)$.*

PROOF. Let $f : Y \to X$ be a surjection. For any $A \in M$, if $j[Y] \subseteq A$, then $j[X] \subseteq j(f)[A]$. As a consequence $t_j(X) \leq t_j(Y)$. ⊣

We therefore will focus on $t_j(\lambda)$ where λ is a cardinal.

We want to get into the situation where we can apply Corollary 3.9, and for this we need Solovay's argument proving SCH above a strongly compact cardinal.

LEMMA 3.13 (Solovay). *Suppose λ is a singular strong limit cardinal of countable cofinality and there is an elementary embedding $j : V \to M$ such that j is discontinuous at λ^+. Then $2^\lambda = \lambda^+$.*

SKETCH. We may assume that j is the ultrapower of the universe by an ultrafilter on λ^+. Note that $t_j(\lambda^+) = \text{cf}^M(\sup j[\lambda^+]) < j(\lambda)$; this follows from an argument due to Ketonen [10], though this more specific case is given in the author's thesis [4, Theorem 7.2.12]. Also $t_j(\lambda^\omega) = (t_j(\lambda))^\omega \leq (t_j(\lambda^+))^\omega < j(\lambda)$. Assume towards a contradiction that $\lambda^\omega > \lambda^+$. Then $t_j(\lambda^{++}) \leq t_j(\lambda^\omega) < j(\lambda)$. But this implies j is discontinuous at λ^{++}, which contradicts that j is the ultrapower of the universe by an ultrafilter on λ^+. ⊣

A more complete proof appears in [4, Lemma 7.2.18].

PROOF OF THEOREM 3.5. We start with a simple observation. Suppose δ is a regular cardinal such that $j_0(\delta) < j_1(\delta)$. Then j_1 is discontinuous at δ. To see this, let X be a set of common fixed points of j_0 and j_1 such that $|X| = \delta$. Then $j_1[X] = j_0[X]$ is covered by $j_0(X)$, which has size $j_0(\delta)$ in M. By Lemma 3.12, $j_1[\delta]$ is covered by a set $B \in M$ such that $|B|^M = j_0(\delta)$. It follows that $\sup j_1[\delta] \neq j_1(\delta)$: otherwise $\text{cf}^M(j_1(\delta)) \leq j_0(\delta)$, contradicting that $j_1(\delta)$ is regular in M.

Assume towards a contradiction that there is an ordinal α such that $j_0(\alpha) \neq j_1(\alpha)$. Without loss of generality, assume $j_0(\alpha) < j_1(\alpha)$.

Assume towards a contradiction that for cofinally many strong limit cardinals λ of countable cofinality, $2^\lambda > \lambda^+$. Let λ be the αth strong limit cardinal of countable cofinality for which $2^\lambda > \lambda^+$. Then $j_0(\lambda)$ is the $j_0(\alpha)$th such cardinal in M, and $j_1(\lambda)$ is the $j_1(\alpha)$th. Hence $j_0(\lambda) < j_1(\lambda)$. As a consequence, $j_0(\lambda^+) < j_1(\lambda^+)$. So j_1 is discontinuous at λ^+ by the claim. It therefore follows by Lemma 3.13 that $2^\lambda = \lambda^+$, which is a contradiction.

Applying Corollary 3.10, $j_0 \restriction \text{Ord} = j_1 \restriction \text{Ord}$, contrary to assumption. ⊣

§4. The uniqueness of embeddings above large cardinals. Intuitively, an ultrafilter U on a set X is a "generalized element" of X. In this section, we study the generalization of ordinal definability that arises from this intuition: namely, definability from ultrafilters on ordinals. Since it turns out that *every* set is definable from an ultrafilter on an ordinal (Proposition 4.1), it is natural in the context of large cardinals to study the sets definable from increasingly complete such ultrafilters. After all, the ordinal definable sets are precisely the sets definable from *principal* ultrafilters on ordinals, or in other words, from ultrafilters that are κ-complete for all cardinals κ. The analysis of this concept leads to a proof of the uniqueness of elementary embeddings above an extendible cardinal.

4.1. Completely definable sets. Suppose κ is an infinite cardinal. A set is κ-*completely definable* if it is definable in the structure $(V, \in)$ from a κ-complete ultrafilter on an ordinal. A set is *completely definable* if it is δ-completely definable for all infinite cardinals δ. The class of κ-completely definable sets is denoted by $\mathrm{CD}(\kappa)$ and the class of completely definable sets by CD.

PROPOSITION 4.1. *Every set is ω-completely definable.*

PROOF. We will prove the stronger statement that every subset of an ordinal λ belongs to $L[U]$ for some ultrafilter U on λ. Since $P_\omega(\lambda) \subseteq L$ and $L \vDash |P_\omega(\lambda)| = |\lambda|$ whenever λ is infinite, it suffices to show that every subset of $S \subseteq \lambda$ belongs to $L[U]$ for some ultrafilter U on $P_\omega(\lambda)$.

The key is that there is a *constructible* independent family $\langle A_\alpha \rangle_{\alpha<\lambda}$ of subsets of $P_\omega(\lambda)$; namely, let $A_\alpha = \{\sigma \in P_\omega(\lambda) : \alpha \in \sigma\}$.

Now let F be the filter on $P_\omega(\lambda)$ generated by $\{A_\alpha\}_{\alpha\in S} \cup \{\lambda \setminus A_\alpha\}_{\alpha\notin S}$, and let U be any ultrafilter on $P_\omega(\lambda)$ extending F. Then $S \in L[U]$ since

$$\alpha \in S \iff A_\alpha \in U$$

and the sequence $\langle A_\alpha \rangle_{\alpha<\lambda}$ belongs to $L[U]$, being constructible. ⊣

Since the proof of the previous proposition turns on the strong compactness of ω, one might expect that under large cardinal axioms, for example if $\kappa > \omega$ is strongly compact, every set is definable from a κ-complete ultrafilter on an ordinal. But in fact, no matter what large cardinal axioms one assumes, it is consistent that there is a set that is *not* ω_1-completely definable. This is because if g is Cohen generic over V, then g is not ω_1-completely definable in $V[g]$. Yet all known large cardinal axioms are upwards absolute from V to $V[g]$.

For any set X, let $\mathrm{UF}_\kappa(X)$ be the set of κ-complete ultrafilters on X. Let $\mathrm{UF}_\kappa(\mathrm{Ord}) = \bigcup_{\delta\in\mathrm{Ord}} \mathrm{UF}_\kappa(\delta)$. Note that any ordinal can be coded by a principal ultrafilter on an ordinal and any finite sequence of ultrafilters on ordinals can be coded by a single ultrafilter on an ordinal; namely, the Fubini product of the ultrafilters, which, using an (ordinal definable) pairing function, can be viewed as an ultrafilter on an ordinal. As an immediate consequence, we obtain a more familiar characterization of $\mathrm{CD}(\kappa)$:

PROPOSITION 4.2. *For any cardinal κ, $\mathrm{CD}(\kappa) = \mathrm{OD}_{\mathrm{UF}_\kappa(\mathrm{Ord})}$.*

In a somewhat artificial sense, complete definability is just a quantifier-flip away from ordinal definability: x is ordinal definable if x is definable from an ultrafilter on an ordinal that is κ-complete for all cardinals κ; x is completely definable if for all cardinal κ, x is definable from a κ-complete ultrafilter on an ordinal.

A κ-completely definable set x is *hereditarily κ-completely definable* (resp. *hereditarily completely definable*) if every element of its transitive closure is also κ-completely definable (resp. completely definable). Thus the class $\mathrm{HCD}(\kappa)$ of all hereditarily κ-completely definable sets is the largest transitive subclass of $\mathrm{CD}(\kappa)$, and the class HCD of all hereditarily completely definable sets is the largest transitive subclass of CD.

PROPOSITION 4.3. *For any cardinal* κ, $\mathrm{HCD}(\kappa)$ *is an inner model of* ZF. *In fact*, $\mathrm{HCD}(\kappa) = \mathrm{HOD}_{\mathrm{UF}_\kappa(\mathrm{Ord})}$.

PROOF. That $\mathrm{HCD}(\kappa) = \mathrm{HOD}_{\mathrm{UF}_\kappa(\mathrm{Ord})}$ is immediate by Proposition 4.2. The structure $\mathrm{HOD}_{\mathrm{UF}_\kappa(\mathrm{Ord})}$ is a model of ZF since $\mathrm{UF}_\kappa(\mathrm{Ord})$ is itself definable from an ordinal. ⊣

Let κ_α denote the supremum of the first α measurable cardinals. We have a decreasing sequence of inner models:

$$V = \mathrm{HCD}(\omega) \supseteq \mathrm{HCD}(\omega_1) = \mathrm{HCD}(\kappa_1) \supseteq \mathrm{HCD}(\kappa_2) \supseteq \cdots \supseteq \mathrm{HCD}(\kappa_\alpha) \supseteq \cdots \supseteq \mathrm{HCD} \supseteq \mathrm{HOD}.$$

One reason the κ-completely definable sets are interesting is that for certain large cardinals κ, $\mathrm{HCD}(\kappa)$ is a model of ZFC.

THEOREM 4.4. *If* κ *is a strongly compact cardinal, then* $\mathrm{HCD}(\kappa)$ *is a model of* ZFC.

For this we will use the following facts.

LEMMA 4.5. *A set* x *is* κ*-completely definable if and only if there is an ultrapower embedding* $j : V \to M$ *with* $crit(j) \geq \kappa$ *such that* x *is definable in the structure* $(V, \in, j)$ *from ordinal parameters.*

PROOF. For the forwards direction, note that any κ-complete ultrafilter W on an ordinal is definable in the structure $(V, \in, j_W)$ from the ordinal $[\mathrm{id}]_W$; hence any set definable in V from W is definable from ordinal parameters in the structure $(V, \in, j_W)$.

For the converse, note that if $j : V \to M$ is an ultrapower embedding with $\mathrm{crit}(j) \geq \kappa$, then (by the well-ordering theorem) there is a κ-complete ultrafilter W on an ordinal such that $j = j_W$. ⊣

THEOREM 4.6 (Kunen). *Suppose* $\mathcal{U}$ *is a fine ultrafilter on* $P_\kappa(P(\delta))$ *and* W *is a* κ*-complete ultrafilter on* δ. *Then there is some* $\alpha < j_\mathcal{U}(\delta)$ *such that* $W = \{A \subseteq \delta : M_\mathcal{U} \vDash \alpha \in j_\mathcal{U}(A)\}$.

PROOF. Let $\sigma = [\mathrm{id}]_\mathcal{U}$. Since $\mathcal{U}$ is a fine ultrafilter on $P_\kappa(P(\delta))$, $j_\mathcal{U}[P(\delta)] \subseteq \sigma \subseteq j_\mathcal{U}(P(\delta))$ and $|\sigma|^{M_\mathcal{U}} < j_\mathcal{U}(\kappa)$. Let $B = j_\mathcal{U}(W) \cap \sigma$, so that $B \in M_\mathcal{U}$, $j_\mathcal{U}[W] \subseteq B \subseteq j_\mathcal{U}(W)$, and $|B|^{M_\mathcal{U}} < j_\mathcal{U}(\kappa)$. Since W is κ-complete, $j_\mathcal{U}(W)$ is $j_\mathcal{U}(\kappa)$-complete, and hence $\bigcap B \in j_\mathcal{U}(W)$, and in particular, there is some $\alpha \in \bigcap B$. Using that $j_\mathcal{U}[W] \subseteq B$, it is easy to see that $W \subseteq \{A \subseteq \delta : M_\mathcal{U} \vDash \alpha \in j_\mathcal{U}(A)\}$, and so by the maximality of W, equality holds. ⊣

COROLLARY 4.7. *Suppose* $\kappa \leq \delta$ *are cardinals and* κ *is* 2^δ*-strongly compact. Then there is a* κ*-completely definable well-order of* $\mathrm{UF}_\kappa(\delta)$.

PROOF. Since κ is 2^δ-strongly compact, there is a κ-complete fine ultrafilter $\mathcal{U}$ on $P_\kappa(P(\delta))$. Theorem 4.6 permits us to define a function $g : \mathrm{UF}_\kappa(\delta) \to j_\mathcal{U}(\delta)$ by setting $g(W)$ equal to the least $\alpha < j_\mathcal{U}(\delta)$ such that $W = \{A \subseteq \delta : \alpha \in j_\mathcal{U}(A)\}$. Then g is an injection and g is κ-completely definable by Lemma 4.5. Set $W_0 \preceq W_1$ if $g(W_0) \leq g(W_1)$. Then $\preceq$ is a well-order of $\mathrm{UF}_\kappa(\delta)$ since it order-embeds into the well-order $j_\mathcal{U}(\delta)$, and $\preceq$ is κ-completely definable since $\preceq$ is definable from g. ⊣

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Proof of Theorem 4.4. By Corollary 4.7, for any ordinal δ, $\text{UF}_\kappa(\delta)$ admits a κ-completely definable well-order. As a consequence, the class $\text{OD}_{\text{UF}_\kappa(\delta)}$ admits a κ-completely definable well-order. By Proposition 4.3, $\text{CD}(\kappa) = \text{OD}_{\text{UF}_\kappa(\text{Ord})} = \bigcup_{\delta\in\text{Ord}} \text{OD}_{\text{UF}_\kappa(\delta)}$.

Now fix an ordinal α, and we will show that there is a well-order of $\text{HCD}(\kappa) \cap V_\alpha$ in $\text{HCD}(\kappa)$. Since $\text{CD}(\kappa)$ is the increasing union of the classes $\text{OD}_{\text{UF}_\kappa(\delta)}$, the pigeonhole principle implies that for any ordinal α, $\text{CD}(\kappa) \cap V_\alpha = \text{OD}_{\text{UF}_\kappa(\delta)} \cap V_\alpha$ for some cardinal $\delta \geq \kappa$. The restriction of any κ-completely definable well-order of $\text{OD}_{\text{UF}_\kappa(\delta)}$ to $\text{CD}(\kappa) \cap V_\alpha$ yields a κ-completely definable well-order of the latter set. Restricting further, $\text{HCD}(\kappa) \cap V_\alpha \subseteq \text{CD}(\kappa) \cap V_\alpha$ admits a κ-completely definable well-order. This well-order is trivially hereditarily κ-completely definable (its transitive closure is equal to $\text{HCD}(\kappa) \cap V_\alpha$, at least if α is a limit ordinal), and there is a well-order of $\text{HCD}(\kappa) \cap V_\alpha$ in $\text{HCD}(\kappa)$, as claimed.

Since $\text{HCD}(\kappa)$ satisfies that for any α, $\text{HCD}(\kappa) \cap V_\alpha$ is well-orderable, $\text{HCD}(\kappa)$ satisfies the Axiom of Choice. ⊣

Note that while $\text{HCD}(\kappa)$ is an inner model of ZFC whenever κ is strongly compact, it is not provable in ZFC that the entire class $\text{HCD}(\kappa)$ can be definably well-ordered from any parameter whatsoever. (Indeed, by Theorem 4.10, this holds if and only if V itself can be definably well-ordered from a parameter.)

We now show that when κ is strongly compact, $\text{HCD}(\kappa)$ is a very large model. In fact, $\text{HCD}(\kappa)$ is a ground of the universe, in the sense of set theoretic geology. Recall that if $N \subseteq M$ are models of set theory, N is said to be a *ground of M* if there is a partial order $\mathbb{P} \in N$ and an N-generic filter $G \subseteq \mathbb{P}$ in M such that $M = N[G]$.

For any set x, let $\text{CD}(\kappa)_x$ denote the class of sets that are κ-completely definable from x, and let $\text{HCD}(\kappa)_x$ denote the class of all sets hereditarily κ-completely definable from x.

Proposition 4.8. *If κ is strongly compact and x is a set such that $V_\kappa \subseteq \text{CD}(\kappa)_x$, then $V = \text{HCD}(\kappa)_x$.*

Proof. We first claim that for every strong limit cardinal $\lambda > \kappa$ of cofinality at least κ, there is a κ-independent family of λ-many subsets of λ that belongs to $\text{HCD}(\kappa)_x$. To see this, let $j : V \to M$ be an ultrapower embedding with $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. Then M is closed under κ-sequences. We claim that $V_{j(\kappa)} \cap M \subseteq \text{HCD}(\kappa)_x$. By elementarity,

$$V_{j(\kappa)} \cap M \subseteq j(V_\kappa) \subseteq j(\text{CD}(\kappa)_x) = \text{CD}^M(j(\kappa))_{j(x)} \subseteq \text{CD}^M(\kappa)_{j(x)}.$$

So it suffices to show that $\text{CD}^M(\kappa)_{j(x)} \subseteq \text{CD}(\kappa)_x$.

Note that M and $j(x)$ are definable over the structure $(V, \in, j)$ from x. Also every ultrapower embedding $i : M \to N$ with $\text{crit}(i) \geq \kappa$ is definable over $(V, \in, j, k)$ from ordinal parameters for some ultrapower embedding $k : V \to N$ with $\text{crit}(k) \geq \kappa$. For this, take $k = i \circ j$, let $\alpha \in \text{Ord}$ be a seed of j (so $M = H^M(j[V] \cup \{\alpha\})$), and let $\beta = i(\alpha)$. Then given $a \in M$, $i(a)$ can be computed by choosing any $f \in V$ such that $a = j(f)(\alpha)$ and noting that

$$i(a) = i(j(f)(\alpha)) = k(f)(\beta).$$

This defines i in the structure $(V, \in, j, k)$ from the ordinals α and β.

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Since M, $j(x)$, and every ultrapower embedding of M are each ordinal definable from x over a structure of the form $(V, \in, j, k)$ where j and k are ultrapower embeddings with critical point at least κ, a slight generalization of Lemma 4.5 yields $\mathrm{CD}^M(\kappa)_{j(x)} \subseteq \mathrm{CD}(\kappa)_x$.

Since λ is a strong limit cardinal of cofinality at least κ, the same is true in M by the downwards absoluteness of Π_1 formulas. In particular, $\lambda^{<\kappa} = \lambda$ in M. Since M is a model of ZFC, M satisfies that there is a κ-independent family of λ-many subsets of λ. Since M is closed under $<\kappa$-sequences, this family of sets really is κ-independent. Since $V_{j(\kappa)} \cap M \subseteq \mathrm{HCD}(\kappa)_x$, and $\lambda < j(\kappa)$, this κ-independent family belongs to $\mathrm{HCD}(\kappa)_x$.

Finally, fix a set of ordinals A. We will prove that $A \in \mathrm{HCD}(\kappa)_x$. Fix $\lambda \geq \sup A$ such that $\lambda^{<\kappa} = \lambda$, and let $\langle S_\alpha : \alpha < \lambda\rangle \in \mathrm{HCD}(\kappa)_x$ be a κ-independent family of subsets of λ. Let F be the κ-complete filter on λ generated by

$$B = \{S_\alpha : \alpha \in A\} \cup \{\lambda \setminus S_\alpha : \alpha \notin A\}.$$

Since κ is strongly compact, there is a κ-complete ultrafilter U extending F. Let $W = U \cap \mathrm{HCD}(\kappa)_x$. Since U is a κ-complete ultrafilter on an ordinal, $W \in \mathrm{HCD}(\kappa)_x$. Now $A = \{\alpha : S_\alpha \in W\}$, and so $A \in \mathrm{HCD}(\kappa)_x$.

Since we have shown that every set of ordinals belongs to $\mathrm{HCD}(\kappa)_x$, the Axiom of Choice yields that $V = \mathrm{HCD}(\kappa)_x$. ⊣

We thank the referee for pointing out an error in the original proof of the above theorem.

Theorem 4.9 (Vopenka). *Suppose C is a class such that for all ordinals α, there is a well-order of $C \cap V_\alpha$ in OD_C. Then for any set of ordinals x, HOD_C is a ground of $\mathrm{HOD}_{C\cup\{x\}}$.*

Sketch. Suppose $x \subseteq \beta$. By our assumption, there is some ordinal γ such that there is an OD_C bijection $f : \gamma \to P^{\mathrm{OD}_C}(P(\beta))$. Let $\mathbb{B}$ be the Boolean algebra on γ induced by the Boolean algebra structure on $P^{\mathrm{OD}_C}(P(\beta))$. Then $\mathbb{B}$ is a complete Boolean algebra in HOD_C. Let $U \subseteq P^{\mathrm{OD}_C}(P(\beta))$ be the principal ultrafilter on $P(\beta)$ concentrated at x; that is,

$$U = \{X \in P^{\mathrm{OD}_C}(P(\beta)) : x \in X\}.$$

Let $G = f^{-1}[U]$. Then $G \subseteq \mathbb{B}$ is a HOD_C-generic ultrafilter. Moreover, one can check that $\mathrm{HOD}_{C\cup\{x\}} = \mathrm{HOD}_C[G]$. ⊣

Theorem 4.10. *If κ is strongly compact, then $\mathrm{HCD}(\kappa)$ is a ground of V.*

Proof. The hypotheses of Theorem 4.9 hold for $C = \mathrm{UF}_\kappa(\mathrm{Ord})$ by Corollary 4.7. Fix $x \subseteq \kappa$ such that $V_\kappa \subseteq L[x]$. Then $\mathrm{HCD}(\kappa)$ is a ground of $\mathrm{HCD}(\kappa)_x$ by Theorem 4.9 and $V = \mathrm{HCD}(\kappa)_x$ by Proposition 4.8, so $\mathrm{HCD}(\kappa)$ is a ground of V. ⊣

The next proposition follows from the proof of Theorem 4.9.

Proposition 4.11. *Suppose κ is strongly compact, and let $\delta = (2^{2^\kappa})^+$. Then $\mathrm{HCD}(\kappa)$ is a ground of V for a forcing in $\mathrm{HCD}(\kappa)$ of cardinality less than δ.*

This yields a new consequence of the Ground Axiom:

THEOREM 4.12 (Ground Axiom). *Assume there is a proper class of strongly compact cardinals. Then* $V = \text{HCD}$.

One can also use Theorem 4.10 to prove that large cardinals are downwards absolute to $\text{HCD}(\kappa)$. We will use [4].

THEOREM 4.13. *Suppose* κ *is strongly compact and* M *is an inner model with the* κ*-cover property. Then* M *has the* κ*-approximation property if and only if every* κ*-complete ultrafilter on an ordinal is amenable to* M.

An inner model M is a *weak extender model for the supercompactness of* κ if for all $\lambda \geq \kappa$, there is a normal fine ultrafilter $\mathcal{U}$ on $P_\kappa(\lambda)$ such that $P_\kappa(\lambda) \cap M \in \mathcal{U}$ and $\mathcal{U} \cap M \in M$.

THEOREM 4.14. *If* κ *is strongly compact, then* $\text{HCD}(\kappa)$ *has the* κ*-approximation and cover properties. Moreover, if* κ *is supercompact, then* $\text{HCD}(\kappa)$ *is a weak extender model for the supercompactness of* κ.

PROOF. It suffices to show the κ-cover property by Theorem 4.13. We proceed by showing that for all cardinals $\lambda \geq \kappa$, there is a κ-complete fine ultrafilter on $P_\kappa(\lambda)$ concentrating on $\text{HCD}(\kappa)$. It follows that $\text{HCD}(\kappa)$ has the cover property for subsets of λ: if $\sigma \in P_\kappa(\lambda)$, then since $\mathcal{U}$ is fine, $\{\tau \in P_\kappa(\lambda) : \sigma \subseteq \tau\} \in \mathcal{U}$ and hence intersects the $\mathcal{U}$-large set $P_\kappa(\lambda) \cap \text{HCD}(\kappa)$; in other words, for some $\tau \in \text{HCD}(\kappa)$, $\sigma \subseteq \tau$.

It suffices to find such an ultrafilter for all regular λ large enough that $\text{HCD}(\kappa)$ is stationary correct at λ. For such a λ, there is a stationary partition $\langle S_\alpha : \alpha < \lambda \rangle$ of S^λ_ω that belongs to $\text{HCD}(\kappa)$. Let $j : V \to M$ be an elementary embedding such that $\text{crit}(j) = \kappa$ and $\text{cf}^M(\sup j[\lambda]) < j(\kappa)$. Let $\langle T_\alpha : \alpha < j(\lambda)\rangle = j(\langle S_\alpha : \alpha < \lambda\rangle)$, and in M, let σ be the set of $\alpha < \lambda$ such that T_α reflects to $\sup j[\lambda]$.

Then $j[\lambda] \subseteq \sigma$ since $j[S_\alpha] \subseteq j(S_\alpha)$ and $j[S_\alpha]$ is truly stationary in $\sup j[\lambda]$. Moreover, $|\sigma|^M < j(\kappa)$. To see this, fix a closed cofinal set $C \subseteq \sup j[\lambda]$ in M of ordertype less than $j(\kappa)$. Define $f : \sigma \to C$ by $f(\alpha) = \min C \cap T_\alpha$. Then f is injective since $\langle T_\alpha : \alpha < j(\lambda)\rangle$ is a partition. Hence $|\sigma|^M \leq |C|^M < j(\kappa)$.

Let $\mathcal{U}$ be the ultrafilter on $P_\kappa(\lambda)$ derived from j using σ. Then since $j[\lambda] \subseteq \sigma$, $\mathcal{U}$ is a κ-complete fine ultrafilter. Since $\sigma \in j(\text{HCD}(\kappa))$, $P_\kappa(\lambda) \cap \text{HCD}(\kappa) \in \mathcal{U}$.

If κ is supercompact, we could have assumed $j[\lambda] \in M$, in which case, one can prove $\sigma = j[\lambda]$. Then $\mathcal{U}$ is a normal fine ultrafilter on $P_\kappa(\lambda)$. Note that $\mathcal{U}$ is definable from j, so $\mathcal{U} \in \text{CD}(\kappa)$ and hence $\mathcal{U} \cap \text{HCD}(\kappa)$ belongs to $\text{HCD}(\kappa)$. This suffices to conclude that $\text{HCD}(\kappa)$ is a weak extender model for the supercompactness of κ. ⊣

This has a number of surprising corollaries. For example, if E is an $\text{HCD}(\kappa)$-extender with $\text{crit}(j_E) \geq \kappa$ and $j_E(A) \cap [\text{length}(E)]^{<\omega} \in \text{HCD}(\kappa)$ for all $A \subseteq [\text{length}(E)]^{<\omega}$, then by a theorem of Woodin, E actually belongs to $\text{HCD}(\kappa)$, despite the fact that $\text{HCD}(\kappa)$ is defined in terms of ultrafilters and not extenders. Is there a direct proof of this fact?

We turn now to the structure of HCD itself under large cardinal assumptions. The proof is based on the proof of Usuba's theorem [13], although the result does not literally follow from his theorem.

THEOREM 4.15. *Suppose κ is an extendible cardinal. Then* $\mathrm{HCD}(\kappa) = \mathrm{HCD}$.

PROOF. Suppose A is a κ-completely definable set of ordinals. We will show that for any cardinal $\gamma \geq \kappa$, A is γ-completely definable. For this, fix a regular cardinal $\delta \geq \max\{\gamma, \sup(A)\}$ and a cardinal $\lambda > \delta$ such that $A \in (\mathrm{HCD}(\kappa))^{V_\lambda}$. Let $j : V_{\lambda+1} \to V_{j(\lambda)+1}$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. Note that $(\mathrm{HCD}(j(\kappa)))^{V_{j(\lambda)}} \subseteq \mathrm{HCD}(\delta)$. But $j(A)$ and $j \restriction \delta$ belong to $(\mathrm{HCD}(j(\kappa)))^{V_{j(\lambda)}}$, the latter by the stationary splitting argument from Theorem 4.14. It follows that $A = j^{-1}[j(A)] \in (\mathrm{HCD}(j(\kappa)))^{V_{j(\lambda)}} \subseteq \mathrm{HCD}(\delta) \subseteq \mathrm{HCD}(\gamma)$. ⊣

4.2. Embeddings above an extendible cardinal. We will need the following consequence of Kunen's commuting ultrapowers lemma:

THEOREM 4.16 (Kunen). *Suppose $j : V \to M$ is the ultrapower embedding associated with an extender in V_δ and $i : V \to N$ is an ultrapower embedding with* $\mathrm{crit}(i) \geq \delta$. *Then $j(i) = i \restriction M$.*

For a proof, see Lemma 4.34.

LEMMA 4.17. *If $j_0, j_1 : V \to M$ are elementary embeddings associated with extenders in V_δ, then $j_0 \restriction \mathrm{CD}(\delta) = j_1 \restriction \mathrm{CD}(\delta)$.*

PROOF. Suppose $x \in \mathrm{CD}(\delta)$. By Lemma 4.5, x is definable from finitely many ordinal parameters $\vec{\alpha}$ in the structure $(V, \in, i)$ for some ultrapower embedding $i : V \to N$ with $\mathrm{crit}(i) \geq \delta$. By Theorem 3.4, $j_0(\vec{\alpha}) = j_1(\vec{\alpha})$, and by Theorem 4.16, $j_0(i) = i \restriction M = j_1(i)$. Hence $j_0(x) = j_1(x)$. ⊣

THEOREM 4.18. *Suppose there is a proper class of strongly compact cardinals. Then any two elementary embeddings from the universe into the same inner model agree on a ground.*

PROOF. Fix elementary embeddings $j_0, j_1 : V \to M$. By Theorem 3.7 and Corollary 3.9, there are ultrapower embeddings $i_0, i_1 : V \to N$ and an elementary embedding $k : N \to M$ such that $j_0 = k \circ i_0$ and $j_1 = k \circ i_1$. Let δ be a strongly compact cardinal such that i_0 and i_1 are the embeddings associated with ultrafilters in V_δ. By Lemma 4.17, i_0 and i_1 agree on $\mathrm{CD}(\delta)$, and hence j_0 and j_1 agree on $\mathrm{CD}(\delta)$. Hence j_0 and j_1 agree on $\mathrm{HCD}(\delta)$, which is a ground by Theorem 4.10. ⊣

THEOREM 4.19. *Assume the Ground Axiom and a proper class of strongly compact cardinals. Then the uniqueness of elementary embeddings holds.*

THEOREM 4.20. *The uniqueness of elementary embeddings holds above the least extendible cardinal.*

PROOF. By Theorem 3.7, Corollary 3.9, and the fact that the eventual singular cardinals hypothesis holds (Lemma 3.13), it suffices to prove the uniqueness of elementary embeddings for ultrapower embeddings $j_0, j_1 : V \to M$ whose critical points lie above the least extendible cardinal κ. By Lemma 4.17, $j_0 \restriction \mathrm{HCD} = j_1 \restriction \mathrm{HCD}$. By Theorem 4.15, $\mathrm{HCD}(\kappa) = \mathrm{HCD}$. Hence j_0 and j_1 agree on $\mathrm{HCD}(\kappa)$. Let i be their common restriction to $\mathrm{HCD}(\kappa)$, and note that i is definable over $\mathrm{HCD}(\kappa)$ by the downwards Lévy–Solovay theorem, using that by Proposition 4.11, V is a generic extension of $\mathrm{HCD}(\kappa)$ by a forcing of size less than the critical point of i.

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Hence i extends uniquely to any forcing extension of $\mathrm{HCD}(\kappa)$ by a forcing of size less than $\mathrm{crit}(i)$. But by Proposition 4.11, V is such a forcing extension. Thus $j_0 = j_1$, since each extends i. ⊣

It would be interesting to know whether there is a combinatorial proof of this theorem avoiding the use forcing and ordinal definability.

We say the uniqueness of elementary embeddings *fails cofinally* if it fails above every cardinal.

THEOREM 4.21. *It is (relatively) consistent with a proper class of supercompact cardinals that the uniqueness of elementary embeddings fails cofinally.*

SKETCH. First class force to make every supercompact cardinal κ indestructible by κ-directed closed forcing. Let $\mathbb{P}$ the (class) Easton product of the forcings adding a Cohen subset of every inaccessible non-Mahlo cardinal κ. This preserves the supercompacts by standard arguments. Moreover, for each κ of Mitchell order 1, one can factor $\mathbb{P}$ as $\mathbb{P}_{\kappa,\infty} \times \mathbb{P}_{0,\kappa}$ and run a essentially the same argument as Theorem 3.1 in $V^{\mathbb{P}_{\kappa,\infty}}$ to prove the uniqueness of elementary embeddings fails at κ in $V^{\mathbb{P}}$. ⊣

Note that a model with a proper class of strongly compact cardinals in which the uniqueness of elementary embeddings fails cofinally must have a proper class of grounds by Theorem 4.18.

4.3. Application: the Kunen inconsistency for ultrapowers. This section concerns the following open question.

QUESTION 4.22. *Suppose $j : V \to M$ is an elementary embedding such that M is closed under ω-sequences. Can there be a nontrivial elementary embedding $k : M \to M$?*

Note that the requirement that M be closed under ω-sequences is necessary since given any elementary embedding j, one can construct an iterated ultrapower $j_{0\omega} : V \to M_\omega$ such that j restricts to an elementary embedding from M_ω to itself.

DEFINITION 4.23. The Rudin–Keisler order is defined on extenders E and F by setting $E <_{\mathrm{RK}} F$ if there is a nontrivial elementary embedding $k : M_E \to M_F$ such that $k \circ j_E = j_F$.

Combinatorially, $E <_{\mathrm{RK}} F$ if there is a nonidentity function $g : \mathrm{length}(E) \to \mathrm{length}(F)$ such that $E_a = F_{g[a]}$.

The Rudin–Keisler order is a preorder, and it is well-known that if E is the extender of an ultrafilter then $E \not<_{\mathrm{RK}} E$. In other words, if $j : V \to M$ is an ultrapower embedding, then there is no nontrivial elementary embedding $k : M \to M$ such that $k \circ j = j$. In fact, a theorem of Solovay states that the Rudin–Keisler order is *wellfounded* on countably complete ultrafilters. We will generalize this to extenders. The issue, however, is that the Rudin–Keisler order is *not* wellfounded, or even irreflexive, on arbitrary (wellfounded) extenders (see the remarks following Question 4.22).

DEFINITION 4.24. An extender E is said to be *countably closed* if its associated ultrapower M_E is closed under ω-sequences.

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THEOREM 4.25. *The Rudin–Keisler order is wellfounded on countably closed extenders.*

Actually, in the spirit of this paper, we prove a slightly stronger second-order theorem (Theorem 4.28), although Lemma 3.8 suggests that this extra strength is an illusion.

DEFINITION 4.26. Suppose P and Q are models of ZF and $j : P \to Q$ is a cofinal elementary embedding. For $a, b \in Q$, set $a \leq_j b$ if there is a structure $\mathcal{M} \in P$ such that $b \in j(\mathcal{M})$ and a is definable in $j(\mathcal{M})$ using b as a parameter. For any $a \in Q$, let $v_j(a)$ denote the least ordinal v such that $a \leq_j v$.

By reflection, one can prove the schema: if a is definable in Q from b and parameters in $j[P]$, then $a \leq_j b$. If P and Q are models of ZFC, one can prove that $v_j(a)$ is defined for all $a \in Q$ using the Well-ordering Theorem. Using Los's Theorem, one can show that for any $b \in Q$, the set $X_b = \{a \in Q : a \leq_j b\}$ is an elementary substructure of Q.

The following remarkable fact about elementary embeddings of transitive models of ZFC may be due to Solovay. In any case, it is closely related to his proof of the wellfoundedness of the Rudin–Keisler order on countably complete ultrafilters.

LEMMA 4.27 (Folklore). *Suppose P and Q are wellfounded models of* ZFC *and $j : P \to Q$ is an elementary embedding. Then $\leq_j$ is wellfounded and has rank bounded by* $\text{Ord} \cap Q$.

PROOF. Let $v = v_j(a)$. We first show that $v \leq_j a$. Let D be the P-ultrafilter derived from j using a, let $k : M_D^P \to Q$ be the canonical factor embedding, and let $\bar{a} = k^{-1}(a)$. Let $\bar{v} = v_{j_D}(\bar{a})$. Thus $\bar{a} \leq_{j_D} \bar{v}$ and moreover $\bar{v} \leq_{j_D} \bar{a}$ since this is true of every $x \in M_D^P$. It follows that $k(\bar{v}) \leq_j a$ and $a \leq_j k(\bar{v})$. The latter fact implies $k(\bar{v}) \geq v$. Assume towards a contradiction that $k(\bar{v}) > v$. Fix a structure $\mathcal{M} \in P$ such that $v \in j(\mathcal{M})$ and a is definable from v in $j(\mathcal{M})$. Then Q satisfies that there is an ordinal $\xi < k(\bar{v})$ such that a is definable from ξ in $j(\mathcal{M})$. By elementarity, M_D^P satisfies that there is an ordinal $\xi < \bar{v}$ such that $\bar{a}$ is definable from ξ in $j_D(\mathcal{M})$. This contradicts that $\bar{v} = v_{j_D}(\bar{a})$.

It follows that the function $v_j : Q \to \text{Ord}^Q$ ranks the preorder $\leq_j$. Indeed, suppose $a <_j b$ (in the sense that $a \leq_j b$ but $b \not\leq_j a$). Obviously $v_j(a) \leq v_j(b)$ (since $\leq_j$ is transitive). But $v_j(a) \neq v_j(b)$ or else $b \leq_j v_j(b) = v_j(a) \leq_j a$. Thus $v_j(a) < v_j(b)$. ⊣

THEOREM 4.28. *Suppose $\langle j_n : n < \omega \rangle$ is a sequence of elementary embeddings $j_n : V \to M_n$ where M_n is closed under ω-sequences. Suppose $\langle i_{n,m} : m \leq n < \omega \rangle$ is a sequence of elementary embeddings such that for all $\ell \leq m \leq n < \omega$,*

$$i_{m,\ell} \circ i_{n,m} = i_{n,\ell}$$
$$i_{n,m} \circ j_n = j_m.$$

Then for all sufficiently large $m \leq n < \omega$, $M_m = M_n$ and $i_{n,m}$ is the identity.

PROOF. For each $n < \omega$, let $\kappa_n = \text{crit}(i_{n+1,n})$. Consider the sequence $s = \langle i_{n,0}(\kappa_n) : n < \omega \rangle$. For each $n < \omega$, let $s_n = s \restriction [n, \omega)$. Since M_0 is closed under ω-sequences, $s_n \in M_0$ for all $n < \omega$.

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Clearly $s_{n+1} \leq_{j_0} s_n$. We claim that $s_n \not\leq_{j_0} s_{n+1}$. Note that $s_{n+1} \in i_{n+1,0}[M_{n+1}]$. Since $j_0[V] \subseteq i_{n+1,0}[M_{n+1}]$ and $i_{n+1,0}[M_{n+1}]$ is definably closed, $i_{n+1,0}[M_{n+1}]$ is downwards closed under $\leq_{j_0}$. Now $i_{n,0}(\kappa_n) \notin i_{n+1,0}[M_{n+1}]$ since

$$i_{n,0}^{-1}(i_{n,0}(\kappa_n)) = \kappa_n \notin i_{n+1,n}[M_{n+1}] = i_{n,0}^{-1}[i_{n+1,0}[M_{n+1}]].$$

But $i_{n,0}(\kappa_n) = s_n(n) \leq_{j_0} s_n$. Hence $s_n \notin i_{n+1,0}[M_{n+1}]$. In particular, $s_n \not\leq_{j_0} s_{n+1}$.

Thus for all $n < \omega$, $s_{n+1} <_{j_0} s_n$, and this contradicts Lemma 4.27. ⊣

This yields Theorem 4.25:

Proof of Theorem 4.25. Take ultrapowers and apply Theorem 4.28. ⊣

Corollary 4.29. *Suppose M is an inner model closed under ω-sequences and $j : V \to M$ and $k : M \to M$ are elementary embeddings such that $k \circ j = j$. Then k is the identity.*

Proof. Apply Theorem 4.28. ⊣

Theorem 4.30. *Suppose δ is extendible, M is an inner model closed under ω-sequences, and $j : V \to M$ is an elementary embedding with critical point above δ. Then there is no nontrivial elementary embedding from M to M.*

Proof. Suppose $k : M \to M$ is an elementary embedding. Note that $k \circ j$ and j agree on the ordinals by Theorem 3.5, and therefore $\text{crit}(k \circ j) > \delta$. The theorem now follows from Theorem 4.20. ⊣

4.4. Weaker hypotheses. The models $\text{HCD}(\kappa)$ of the previous section are particularly interesting, being models of the Axiom of Choice, but in fact certain applications of these models can be carried out under hypotheses of lower consistency strength than a strongly compact cardinal. Here we will show the following:

Theorem 4.31. *Assume there is a proper class of strong cardinals. Then any pair of ultrapower embeddings from the universe into an inner model agree on a ground of V.*

We immediately obtain a proof of the uniqueness of elementary embeddings from the Ground Axiom under (consistency-wise) weaker hypotheses:

Corollary 4.32. *Assume the Ground Axiom and a proper class of strong cardinals. Then the uniqueness of ultrapower embeddings holds.*

By Theorem 3.7, the uniqueness of ultrapower embeddings implies the uniqueness of extender embeddings. In the context of the eventual SCH, one can improve this to arbitrary elementary embeddings using Corollary 3.9.

We begin by defining a ZF-ground of V on which the embeddings agree.

Definition 4.33. Suppose κ is a cardinal. A set is *κ-extender definable* if it is ordinal definable over $(V, j_0, \dots, j_{n-1})$ for finitely many extender embeddings $j_i : V \to M_i$ such that $\text{crit}(j_i) \geq \kappa$ and $M_i^{<\kappa} \subseteq M_i$. We denote the class of κ-extender definable sets by $\text{ED}(\kappa)$. The class of *hereditarily κ-extender definable sets*, denoted by $\text{HED}(\kappa)$, is the largest transitive subclass of $\text{ED}(\kappa)$.

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Everything we prove about $\mathrm{ED}(\kappa)$ can also be proven about the conceivably smaller class of sets ordinal definable from short strong extenders, or in other words from elementary embeddings $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$, $M^{<\kappa} \subseteq M$, and $M = H^M(j[V] \cup V_\lambda)$. The relationship between the two notions is unclear.

The proof uses a generalization of Kunen's commuting ultrapowers lemma:

LEMMA 4.34. *Suppose $i : V \to M$ and $j : V \to N$ are elementary embeddings such that $i(j) = j \restriction M$ and $i(v) = j(i)(v)$ for all generators v of j. Then $j(i) = i \restriction N$.*

PROOF. In general given elementary embeddings $i : V \to M$ and $j : V \to N$, one has $i \circ j = i(j) \circ i$ because $i(j(x)) = i(j)(i(x))$. But note that in our case, $i \circ j = j(i) \circ j$ since $i \circ j = i(j) \circ i = j \circ i = j(i) \circ j$. In particular, $i(N) = j(i)(N)$ since $N = j(V)$. Therefore $i, j(i) : N \to i(N)$ are elementary embeddings with the same target model.

Note that $i \restriction j[V] = j(i) \restriction j[V]$, since this is just another way of saying that $i \circ j = j(i) \circ j$. Our hypothesis states that $i \restriction G = j(i) \restriction G$ where G is the class of generators of j. Now $N = H^N(j[V] \cup G)$, and i and $j(i)$ coincide on $j[V] \cup G$. Hence $i \restriction N = j(i)$. ⊣

COROLLARY 4.35. *Suppose λ is a cardinal and $j_0, j_1 : V \to M$ are extender embeddings such that $M = H^M(j_n[V] \cup V_\alpha)$, for some $\alpha < \lambda$. Then j_0 and j_1 agree on* $\mathrm{ED}(\lambda)$.

The following theorem, generalizing Theorem 4.10, is the key:

THEOREM 4.36. *Suppose κ is strong. Then* $\mathrm{HED}(\kappa)$ *is a* ZF-*ground.*

The theorem cannot be proved in exactly the same way as Theorem 4.10 since Vopěnka's theorem does not seem to go through. But one can instead use Bukovsky's Theorem. Suppose θ is a cardinal. An inner model M is said to have the θ-*uniform cover property* if for all $X \in M$ and $f : X \to M$, there is a function $F : X \to M$ in M such that $f(x) \in F(x)$ for all $x \in X$ and $F(x)$ does not surject onto θ.

The following proposition is a version of Bukovsky's theorem which follows the proof from [1] in order to deal with ZF-grounds. Our situation is nominally different, since our definition of the θ-uniform cover property is somewhat weaker than the one employed there.

PROPOSITION 4.37. *Suppose M is an inner model of* ZF *with the θ-uniform cover property for some cardinal θ. Then every set of ordinals is generic over M.*

PROOF. Let γ be an ordinal and A a subset of γ. We will show A is generic over M.

Let $\mathcal{L}$ denote the class of infinitary propositional formulae with γ indeterminates $\langle x_\alpha : \alpha < \gamma \rangle$. Let $\mathcal{L}_M = \mathcal{L} \cap M$. Let $\lambda > \gamma$ be a Beth fixed point of cofinality at least θ. Let $\mathcal{L}_\lambda = \mathcal{L}_M \cap V_\lambda$. Let $f : P^M(\mathcal{L}_\lambda) \to \mathcal{L}_\lambda$ assign to each $\Gamma \in P^M(\mathcal{L}_\lambda)$ such that $A \vDash \bigvee \Gamma$ some $\varphi \in \Gamma$ such that $A \vDash \varphi$. Let $F : P^M(\mathcal{L}_\lambda) \to P^M(\mathcal{L}_\lambda)$ be a function in M witnessing the θ-uniform cover property for f. Note that our assumption that $\mathrm{cf}(\lambda) \geq \theta$ yields that $\bigvee F(\Gamma) \in \mathcal{L}_\lambda$ for all $\Gamma \subseteq \mathcal{L}_\lambda$.

Let T be the theory consisting of formulae of the form $\bigvee \Gamma \to \bigvee F(\Gamma)$ for nonempty $\Gamma \subseteq \mathcal{L}_\lambda$. Note that $A \vDash T$. Let $\mathbb{P}$ be the set of $\varphi \in \mathcal{L}_\lambda$ such that T does not prove $\neg\varphi$ (using any valid proof system for infinitary logic that is definable in M

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and suffices for the argument in the final paragraph of the proof of this proposition). Partially order $\mathbb{P}$ by setting $\varphi \leq \psi$ if $T \vdash \varphi \to \psi$.

Let $G \subseteq \mathbb{P}$ be the set of $\varphi \in \mathbb{P}$ such that $A \vDash \varphi$. We claim G is an M-generic filter. We leave the verification that G is a filter to the reader (see [1]).

Suppose $D \subseteq \mathbb{P}$ is a dense set that lies in M. We claim $A \vDash \bigvee D$. Otherwise T does not prove $\bigvee D$. Since $F(D) \subseteq D$, it follows that T does not prove $\bigvee F(D)$. Let $\varphi = \bigvee F(D)$. Then $\neg\varphi \in \mathbb{P}$. By density, fix $\psi \in D$ such that $\psi \leq \neg\varphi$. By contraposition, T proves $\neg\varphi$ implies $\neg(\bigvee D)$. Therefore since $T \vdash \psi \to \neg\varphi$, $T \vdash \psi \to \neg(\bigvee D)$. Since $\psi \in D$, $T \vdash \neg(\bigvee D) \to \neg\psi$, and therefore $T \vdash \psi \to \neg\psi$. As a consequence, $T \vdash \neg\psi$, contradicting that $\psi \in \mathbb{P}$. ⊣

Relativizing extender definability gives rise to the classes ED_A and HED_A for every set parameter A.

Proposition 4.38. *Suppose κ is strong and $A \subseteq \kappa$ is such that $V_\kappa \subseteq \text{HOD}_A$. Then $V = \text{HED}(\kappa)_A$.*

Proof. Fix $\lambda \geq \kappa$, and we will show $V_\lambda \subseteq \text{HED}(\kappa)_A$. For this, let $j : V \to M$ be an extender embedding such that $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$, and $M^{<\kappa} \subseteq M$. Then $j(A) \in \text{HED}(\kappa)_A$ and $V_\lambda \subseteq \text{HOD}^M_{j(A)} \subseteq \text{HED}(\kappa)_A$. ⊣

Proof of Theorem 4.36. Fix a set $A \subseteq \kappa$ such that $V_\kappa \subseteq \text{HOD}_A$. By Proposition 4.37 and Proposition 4.38, it suffices to prove that $\text{HED}(\kappa)$ has the θ-uniform cover property inside $\text{HED}(\kappa)_A$ for some θ. Let $\theta = (2^\kappa)^+$.

Suppose $f : X \to \text{HED}(\kappa)$ is $\text{ED}(\kappa)_A$, and we will find $F : X \to \text{HED}(\kappa)$ in $\text{HED}(\kappa)$ witnessing the θ-uniform cover property. Fix an extender E and a formula φ such that $f(x) = y$ if and only if $\varphi(x, y, A, E)$ holds. Define $F : X \to \text{HED}(\kappa)$ by setting

$$F(x) = \{y : \exists B \subseteq \kappa\, \varphi(x, y, B, E)\}.$$

Clearly F is $\text{ED}(\kappa)$, and so $F \in \text{HED}(\kappa)$. In V, $F(x)$ is the surjective image of $P(\kappa)$, and so in $\text{HED}(\kappa)$, $F(x)$ does not surject onto $(2^\kappa)^+$. Since $f(x) \in F(x)$ for all $x \in X$, F is as desired. ⊣

Corollary 4.39. *If κ is a strong cardinal, then* $\text{HED}(\kappa)$ *contains a ground.*

Proof. A theorem of Usuba [14] shows (in ZFC) that every ZF-ground contains a ground. ⊣

Proof of Theorem 4.31. The theorem is now immediate from Corollaries 4.35 and 4.39. ⊣

§5. Ultrapower axioms. The Ultrapower Axiom is a combinatorial principle that clarifies the theory of countably complete ultrafilters. Here we will show it implies the uniqueness of elementary embeddings. We will also consider a slight weakening of the Ultrapower Axiom called the Weak Ultrapower Axiom, which until this work was not known to have any consequences at all.

An elementary embedding $i : P \to Q$ is *close* if for all $A \in Q$, $i^{-1}[A] \in P$. If i is an ultrapower embedding, we say in this case that i is *internal*, since for ultrapower embeddings, closeness is equivalent to the existence of an ultrafilter $U \in P$ and an isomorphism $k : \text{Ult}(P, U) \to Q$ such that $k \circ j_U = i$.

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DEFINITION 5.1. The *Ultrapower Axiom* (UA) states that for any inner models P_0 and P_1 admitting internal ultrapower embeddings $j_0 : V \to P_0$ and $j_1 : V \to P_1$, there exists an inner model N admitting internal ultrapower embeddings $k_0 : P_0 \to N$ and $k_1 : P_1 \to N$ such that $k_0 \circ j_0 = k_1 \circ j_1$.

Although it is not immediate from our formulation here, UA is a first-order statement. In fact, it is equivalent to a Π_2-sentence.

5.1. The Ultrapower Axiom. In this subsection, we show that UA implies the uniqueness of elementary embeddings.

THEOREM 5.2. *UA implies the uniqueness of elementary embeddings.*

Typically, UA is only really useful for analyzing ultrapower embeddings. The generality of this theorem may therefore seem surprising, though not perhaps to the dutiful reader of this paper. What makes Theorem 5.2 possible are Theorem 3.7 and Lemma 3.8, which show that under cardinal arithmetic assumptions, the uniqueness of elementary embeddings reduces to the uniqueness of ultrapower embeddings.

The uniqueness of ultrapower embeddings is one of the oldest structural consequences of UA, proved during the author's dissertation work:

LEMMA 5.3 (UA). *Suppose $j_0, j_1 : V \to M$ are ultrapower embeddings. Then $j_0 = j_1$.*

PROOF. Let $(k_0, k_1) : M \to N$ be an internal ultrapower comparison of (j_0, j_1). Note that $k_0 \circ j_0 = k_1 \circ j_1$ by the definition of a comparison and k_0 and k_1 agree on the ordinals by Theorem 3.4. Since every set is constructible from a set of ordinals, it suffices to show that for all sets of ordinals A, $j_0(A) = j_1(A)$. But

$$j_0(A) = k_0^{-1}[k_0(j_0(A))] = k_0^{-1}[k_1(j_1(A))] = k_1^{-1}[k_1(j_1(A))] = j_1(A). \qquad \dashv$$

Using Lemma 5.3 and Theorem 3.7, one can now show that UA plus SCH implies the uniqueness of elementary embeddings. Once again, we will eliminate the SCH hypothesis by proving that the conclusion of Lemma 3.8 follows from UA without appealing to SCH.

LEMMA 5.4 (UA). *Suppose M is a countably closed inner model and $j : V \to M$ is an elementary embedding. Then j is an extender embedding.*

PROOF. Suppose not. Then there is a strong limit cardinal λ of cofinality ω that is closed under j and a limit of generators of j. Since j is continuous at ordinals of cofinality ω, $j(\lambda) = \lambda$. Let $\langle v_n : n < \omega\rangle$ be an increasing sequence of generators of j whose limit is λ.

Let U be the ultrafilter on λ^ω derived from j using $\langle v_n : n < \omega\rangle$. Let

$$\lambda_U = \min\{|A| : A \in U\} = \lambda_{j_U}([\mathrm{id}]_U).$$

Then $\lambda < \lambda_U \leq \lambda^\omega$. Let γ be the least cardinal greater than λ that carries a countably complete uniform ultrafilter. Clearly $\gamma \leq \lambda_U \leq \lambda^\omega$.

We claim $\gamma = \lambda^+$. The proof requires some ideas from the theory of UA. A cardinal is *Fréchet* if it carries a countably complete uniform ultrafilter. A cardinal is *isolated* if it is a Fréchet limit cardinal that is not a limit of Fréchet cardinals. (It is an open question whether it is provable from UA that all isolated cardinals are measurable.)

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Assuming towards a contradiction that $\gamma \neq \lambda^+$, then γ is an isolated cardinal by [4, Corollary 7.4.6]. Letting δ_γ be the strict supremum of the Fréchet cardinals below γ, we have that $\delta_\gamma = \lambda$ since λ is closed under j and a limit of generators of j. Since $\gamma \leq \lambda^\omega$, γ is not measurable, and so [4, Proposition 7.5.22] implies that δ_γ is regular, which is a contradiction.

Since λ^+ carries a countably complete uniform ultrafilter, Lemma 3.13 implies $2^\lambda = \lambda^+$. It follows that $\lambda_U = \lambda^+$, but $j_U(\lambda^+) \leq j(\lambda^+) = \lambda^+$, which is a contradiction. ⊣

Proof of Theorem 5.2. By Theorem 3.7, one can reduce to proving the uniqueness of embeddings that are almost ultrapowers, but by Lemma 5.4, if an embedding is almost an ultrapower, it actually is an ultrapower. By Lemma 5.3, the uniqueness of ultrapower embeddings is a consequence of UA. ⊣

5.2. The Weak Ultrapower Axiom. A model Q is an *internal ultrapower* of a model P if there is an internal ultrapower embedding from P to Q. In slogan form, the Ultrapower Axiom states: *any two ultrapowers of the universe have a common internal ultrapower.* Like so many slogans, this is not completely accurate, since the Ultrapower Axiom contains an additional requirement amounting to the commutativity of a certain diagram of ultrapowers. This discrepancy raises a number of questions.

Definition 5.5. The *Weak Ultrapower Axiom* (Weak UA) states that any two ultrapowers of the universe have a common internal ultrapower.

By ultrapower, we here mean *wellfounded* ultrapower. In 2018, Hugh Woodin raised the question: can any of the consequences of UA be proved from the Weak UA? Or does the commutativity requirement in the Ultrapower Axiom somehow contain some trace of the assumption that $V = \mathrm{HOD}$ that cannot be recovered from Weak UA?

Does the Weak Ultrapower Axiom imply the Ultrapower Axiom? Assuming the uniqueness of elementary embeddings, the answer is obviously yes.

Proposition 5.6. *UA is equivalent to the conjunction of Weak UA and the uniqueness of ultrapower embeddings.*

Using the results of this paper, one can prove some of the consequences of UA assuming just Weak UA by increasing the large cardinal hypotheses. We only sketch the proofs.

Definition 5.7. If M_0 and M_1 are inner models and α_0 and α_1 are ordinals, we write $(M_0, \alpha_0) \sim (M_1, \alpha_1)$ if there exist elementary embeddings $k_0 : M_0 \to N$ and $k_1 : M_1 \to N$ to a common inner model N such that $k_0(\alpha_0) = k_1(\alpha_1)$.

It is unclear whether this relation is first-order definable, but this will not be an issue.

Theorem 5.8. *Suppose κ is an extendible cardinal and U_0 and U_1 are κ^+-complete ultrafilters on ordinals such that $(M_{U_0}, [id]_{U_0}) \sim (M_{U_1}, [id]_{U_1})$. Then $U_0 = U_1$.*

Proof. For $n = 0, 1$, let $j_n : V \to M_n$ be the ultrapower embedding associated with U_n and let $\alpha_n = [\mathrm{id}]_{U_n}$. Let N be a model of set theory admitting elementary

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embeddings $k_0 : M_0 \to N$ and $k_1 : M_1 \to N$ such that $k_0(\alpha_0) = k_1(\alpha_1)$. Note that $k_0 \circ j_0$ and $k_1 \circ j_1$ agree on HCD by Lemma 4.17.[1] As a consequence, $U_0 \cap \mathrm{HCD} = U_1 \cap \mathrm{HCD}$:

$$\begin{aligned} A \in U_0 &\iff \alpha_0 \in j_0(A) \\ &\iff k_0(\alpha_0) \in k_0(j_0(A)) \\ &\iff k_1(\alpha_1) \in k_1(j_1(A)) \\ &\iff A \in U_1. \end{aligned}$$

Let $W = U_0 \cap \mathrm{HCD}$. Since W is κ^+-complete and $\mathrm{HCD} = \mathrm{HCD}(\kappa)$, $W \in \mathrm{HCD}$. Since V is a generic extension of HCD for a forcing of size less than the completeness of W (Proposition 4.11), the upwards Lévy–Solovay theorem [11] implies that U_0 is the filter generated by W. Similarly, U_1 is the filter generated by W, so $U_0 = U_1$. ⊣

Theorem 5.8 enables us to define a well-order of the κ^+-complete ultrafilters.

Definition 5.9. If M_0 and M_1 are inner models and α_0 and α_1 are ordinals, then

$$(M_0, \alpha_0) <_{\Bbbk} (M_1, \alpha_1)$$

if there is an inner model N admitting an elementary embedding $k_0 : M_0 \to N$ and an internal ultrapower embedding $k_1 : M_1 \to N$ with $k_0(\alpha_0) < k_1(\alpha_1)$. The *weak Ketonen order* is defined on countably complete ultrafilters U_0 and U_1 by setting $U_0 <^*_{\Bbbk} U_1$ if $(M_{U_0}, [\mathrm{id}]_{U_0}) <_{\Bbbk} (M_{U_1}, [\mathrm{id}]_{U_1})$.

Since we did not require that k_0 is an ultrapower embedding, it is unclear whether the weak Ketonen order is first-order definable, but under the large cardinal hypotheses we are assuming (or simply the eventual SCH), Corollary 3.9 implies that k_0 must be an ultrapower embedding. Actually, under Weak UA with no cardinal arithmetic hypothesis, one can show that the weak Ketonen order is always witnessed by a pair of internal ultrapower embeddings. In either context, it follows from [5, Theorem 3.5.8] that the weak Ketonen order is wellfounded. By Theorem 5.8, this yields the following:

Corollary 5.10 (Weak UA.) *If κ is an extendible cardinal, the class of κ^+-complete ultrafilters on ordinals is well-ordered by the weak Ketonen order.*

Corollary 5.11 (Weak UA). *If κ is an extendible cardinal, every κ^+-complete ultrafilter is ordinal definable.*

Theorem 5.12 (Weak UA). *If there is an extendible cardinal, then V is a generic extension of* HOD.

Proof. Since there is an extendible cardinal, V is a generic extension of HCD by Theorems 4.10 and 4.15. By Theorem 5.11, $\mathrm{HCD} = \mathrm{HOD}$. ⊣

We now bound the size of the forcing taking HOD to V. Somewhat surprisingly, one can show that it is *strictly smaller* than the least extendible.

[1] Here we use Theorem 3.7 to derive almost ultrapower embeddings $i_0, i_1 : V \to Q$ from $k_0 \circ j_0$ and $k_1 \circ j_1$ and an elementary $\ell : Q \to N$ such that $k_0 \circ j_0 = \ell \circ i_0$ and $k_1 \circ j_1 = \ell \circ i_1$. Then we argue that i_0 and i_1 are ultrapower embeddings by Corollary 3.9. Now we are in a position to apply Lemma 4.17 to i_0 and i_1, which implies that i_0 and i_1 agree on HCD, and hence so do $k_0 \circ j_0$ and $k_1 \circ j_1$.

THEOREM 5.13 (Weak UA). *Suppose there is an extendible cardinal. Then V is a generic extension of* HOD *by a forcing in V_δ where δ is the least Σ_3-reflecting cardinal.*

Note that if $(M_{U_0}, [\mathrm{id}]_{U_0}) \sim (M_{U_1}, [\mathrm{id}]_{U_1})$, then $U_0 \cap \mathrm{HOD} = U_1 \cap \mathrm{HOD}$. Combining this with the fact that the weak Ketonen order is a prewellorder whose induced equivalence relation extends $\sim$, one obtains the following:

LEMMA 5.14 (Weak UA). *Suppose U is a countably complete ultrafilter on an ordinal. Then $U \cap HOD \in HOD$.*

Since HOD satisfies the Axiom of Choice, it follows that if U is a countably complete ultrafilter on a set $X \in \mathrm{HOD}$, then $U \cap \mathrm{HOD} \in \mathrm{HOD}$: just fix an OD bijection $f : X \to \mathrm{Ord}$, and note that $f_*(U) \cap \mathrm{HOD} \in \mathrm{HOD}$ by Lemma 5.14, and so $U \cap \mathrm{HOD} \in \mathrm{HOD}$.

We will use this to show that HOD has the κ-approximation and cover properties at the least strongly compact cardinal. This in turn yields that HOD is locally definable from parameters.

THEOREM 5.15 (Weak UA). *Assume there is an extendible cardinal and let κ be a strongly compact cardinal. Then* HOD *has the κ-approximation and cover properties. If κ is supercompact, then* HOD *is a weak extender model for the supercompactness of κ.*

PROOF. By the strongly compact version of the HOD dichotomy theorem [7], since HOD computes sufficiently large successor cardinals, HOD has the κ-cover property. By Lemma 5.14, every countably complete ultrafilter on an ordinal is amenable to HOD. Therefore by Theorem 4.13, HOD has the κ-approximation and cover properties. The second part of the theorem is similar to Theorem 4.14. ⊣

Given Theorem 5.15, one obtains Theorem 5.13 simply by counting quantifiers.

PROOF OF THEOREM 5.13. Let κ be the least strongly compact cardinal, so $\kappa < \delta$. Let $H = \mathrm{HOD} \cap H(\kappa^+)$. By Hamkins's pseudoground model definability theorem [2], HOD is uniformly definable from H in $H(\gamma)$ for any strong limit cardinal $\gamma > \kappa$.[2] Therefore the statement that V is a generic extension of HOD is Σ_3 in the parameter H, and so it reflects to V_δ. Then taking a generic $G \in V_\delta$ such that $V_\delta = (\mathrm{HOD} \cap V_\delta)[G]$, the correctness of V_δ implies that in fact, $V = \mathrm{HOD}[G]$. ⊣

Now repeating the proofs, we can state slightly nicer theorems:

THEOREM 5.16 (Weak UA). *If κ is an extendible cardinal, the class of κ-complete ultrafilters on ordinals is well-ordered by the weak Ketonen order. In particular, every κ-complete ultrafilter is ordinal definable.*

DEFINITION 5.17. We say the Ultrapower Axiom holds for a pair of ultrapower embeddings $j_0 : V \to M_0$ and $j_1 : V \to M_1$ if there is an inner model N admitting internal ultrapower embeddings $k_0 : M_0 \to N$ and $k_1 : M_1 \to N$ such that $k_0 \circ j_0 = k_1 \circ j_1$.

[2]This follows from the proof of the theorem, which does not require that κ^+ be correctly computed by HOD.

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THEOREM 5.18 (Weak UA). *Suppose κ is an extendible cardinal. Then in* HOD, *the Ultrapower Axiom holds for any pair of ultrapower embeddings with critical point at least κ.*

PROOF. Fix ultrapower embeddings $j_0 : \mathrm{HOD} \to M_0$ and $j_1 : \mathrm{HOD} \to M_1$ with critical point at least κ. Since V is a forcing extension of HOD for a forcing in V_κ, these ultrapower embeddings lift to $j_0^* : V \to M_0^*$ and $j_1^* : V \to M_1^*$. Applying Lemma 5.14, every countably complete ultrafilter is amenable to HOD, so by elementarity, every countably complete ultrafilter of M_0^* (resp. M_1^*) is amenable to M_0 (resp. M_1). In particular, any internal ultrapower embedding of M_0^* (resp. M_1^*) restricts to a close embedding of M_0 (resp. M_1).

Applying the Weak Ultrapower Axiom, fix an inner model N^* and elementary embeddings $k_0^* : M_0^* \to N^*$ and $k_1^* : M_1^* \to N^*$. Letting $k_0 = k_0^* \restriction M_0$ and $k_1 = k_1^* \restriction M_1$, the amenability of countably complete ultrafilters to HOD implies k_0 and k_1 are close to M_0 and M_1. Also Theorem 3.4 implies $k_0 \circ j_0 = k_1 \circ j_1$. Let $X = H^N(k_0[M_0] \cup k_1[M_1])$, let H be the transitive collapse of X, let $h : H \to N$ be the inverse of the transitive collapse embedding, and let $i_0 : M_0 \to H$ and $i_1 : M_1 \to H$ be given by $i_0 = h^{-1} \circ k_0$ and $i_1 = h^{-1} \circ k_1$. It is then easy to show that i_0 and i_1 are internal ultrapower embeddings of M_0 and M_1. ⊣

PROPOSITION 5.19. *Suppose κ is supercompact and the Ultrapower Axiom holds for embeddings with critical point at least κ. Then for all cardinals $\lambda \geq \kappa$, $2^\lambda = \lambda^+$.*

PROOF. This follows from the proof of the main theorem of [6]. ⊣

THEOREM 5.20 (Weak UA). *If κ is extendible, then for all cardinals $\lambda \geq \kappa$, $2^\lambda = \lambda^+$.*

PROOF. By Theorem 5.18 and Proposition 5.19, in HOD, the Generalized Continuum Hypothesis holds at all cardinals greater than or equal to the least extendible cardinal. By Theorem 5.13, V is a generic extension of HOD for a forcing of size less than the least extendible cardinal, and so the Generalized Continuum Hypothesis holds in V at all cardinals greater than or equal to the least extendible cardinal. ⊣

A uniform ultrafilter U on a cardinal λ is *Dodd sound* if the function $E : P(\lambda) \to M_U$ defined by $E(A) = j_U(A) \cap [\mathrm{id}]_U$ belongs to M_U. At least in the context of GCH, one can think of Dodd soundness as a generalization of supercompactness: if $2^{<\lambda} = \lambda$ and $\mathcal{U}$ is a normal fine ultrafilter on $P(\lambda)$, then there is a unique Dodd sound ultrafilter Rudin–Keisler equivalent to $\mathcal{U}$. (Not every Dodd sound ultrafilter is equivalent to a normal fine ultrafilter.)

PROPOSITION 5.21. *Suppose κ is a cardinal such that the Ultrapower Axiom holds for embeddings with critical point at least κ. Then the Mitchell order is linear on κ-complete Dodd sound ultrafilters.*

SKETCH. A similar theorem is proved in [4, Theorem 4.3.29] under the stronger assumption of full UA; the point is that the proof only needs a comparison of the two Dodd sound ultrafilters one is trying to show are comparable in the Mitchell order, and so if we are considering κ-complete Dodd sound ultrafilters, UA for embeddings with critical point at least κ suffices for the argument. ⊣

Let $\mathcal{N}_\kappa(\lambda)$ be the set of κ-complete normal fine ultrafilters on $P_{\mathrm{bd}}(\lambda)$, and let $\mathcal{N}_\kappa = \bigcup_{\lambda\in\mathrm{Card}} \mathcal{N}_\kappa(\lambda)$.

THEOREM 5.22 (Weak UA). *If κ is extendible, then the Mitchell order is linear on κ-complete Dodd sound ultrafilters. In particular, the Mitchell order is linear on $\mathcal{N}_\kappa$.*

PROOF. Theorem 5.18 and Proposition 5.21 yield the linearity of the Mitchell order on κ-complete Dodd sound ultrafilters in HOD. By Theorem 5.13, V is a generic extension of HOD for a forcing of size less than the least extendible cardinal, which implies the linearity of the Mitchell order on κ-complete Dodd sound ultrafilters in V. The fact that the linearity of the Mitchell order on Dodd sound ultrafilters implies the linearity of the Mitchell order on normal fine ultrafilters is a result from [3]. The result there only applies to normal fine ultrafilters on $P_{\mathrm{bd}}(\lambda)$ if $2^{<\lambda} = \lambda$, which is true by Theorem 5.20. ⊣

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