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INVEXITY AT A POINT : GENERALISATIONS AND CLASSIFICATION

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This paper uses Clarke's generalised directional derivative to describe several types of invexity, pseudoinvexity and quasiinvexity at a point of a nonlinear function. Direct implications of the relations existing between the various types of invexity and generalised invexity are presented, as well as a block diagram of these implications. In particular, similar results in the class of quasiconvex functions are obtained.

1. INTRODUCTION

The notion of "invexity of a function" was introduced into optimisation theory by Hanson [7] and the name of "invex function" was given by Craven [2]. Let $X \subseteq \mathbb{R}^n$ be an open and nonempty set, and let $f\colon X \to \mathbb{R}$.

DEFINITION 1.1: (*Global invexity.*) The differentiable function f is called *invex on* X if a vector function $\eta\colon X \times X \to \mathbb{R}^n$ exists such that

$$\forall x, u \in X\colon f(x) - f(u) \geqslant \eta^t(x, u)\nabla f(u)$$

where $\nabla f(u)$ denotes the gradient vector.

If u is fixed then we obtain invexity at the point u. The study of these aspects was initiated by Craven [2] and more directly presented in the papers of Craven and Glover [4], Kaur and Kaul [10].

For the subdifferentiable case, Craven [3] introduced a "generalised invexity condition" on X for the function f, which was justified by Giorgi and Mititelu [5] as a natural definition of invexity in this case. This new definition is based on Clarke's generalised directional derivative for Lipschitzian functions.

Thus, for the function f Lipschitzian on X, Clarke defined the generalised directional derivative of f at a point $x \in X$ in the direction $v \in \mathbb{R}^n$ by

$$f^0(x; v) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{f(x' + \lambda v) - f(x')}{\lambda}.$$

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Also, he defined the subdifferential (or generalised gradient) of the function f at a point x by the unique, nonempty, convex and compact set

$$\partial f(x) = \{\xi \in \mathbb{R}^n \mid f^0(x;v) \geqslant \xi^t v,\ \forall v \in \mathbb{R}^n\}.$$

The elements of $\partial f(x)$ are called subgradients. Starting from Definition 1.1, Giorgi and Mititelu [6] consider that the Lipschitzian function f is invex at u if a vector function $\eta\colon X \times X \to \mathbb{R}^n$ exists such that

$$\forall x \in X:\ f(x) - f(u) \geqslant \eta^t(x,\, u)\xi,\ \forall \xi \in \partial f(u)$$

or, equivalently,

$$\forall x \in X:\ f(x) - f(u) \geqslant \max_{\xi \in \partial f(u)} \eta^t(x,\, u)\xi$$

or once more

$$\forall x \in X:\ f(x) - f(u) \geqslant f^0(u; \eta(x,\, u)) \tag{1}$$

since it is well-known [1] that

$$f^0(u; \eta(x,\, u)) = \max_{\xi \in \partial f(u)} \eta^t(x,\, u)\xi. \tag{2}$$

For x and u arbitrarily in X, we notice that the inequality (1) is the "generalised invexity condition" as presented by Craven [3].

Mititelu [11] showed recently that instead of Lipschitzian functions we can consider a more general class, namely, that of arbitrary nonlinear functions for which f^0 and ∂f may be defined in a similar manner, and for which the relation (2) exists when $f^0(x;\cdot)$ is finite. Thus, we introduce

DEFINITION 1.2: (*Invexity at a point.*) The nonlinear function f is said to be *invex at* $u \in X$ if a vector function $\eta\colon X \times X \to \mathbb{R}^n$ exists such that

$$\forall x \in X:\ f(x) - f(u) \geqslant f^0(u; \eta(x,\, u)).$$

Based on Definition 1.2, several types of invexity, pseudoinvexity and quasiinvexity at a point of a nonlinear function will be pointed out in this paper. These types are presented together using the model of Vial [13], Jeyakumar [8] and Preda [12]. Direct implications of the relations existing between the various types of invexity and generalised invexity are presented, as well as a block diagram of these implications. In particular, the definitions of the various types of convexity, pseudoconvexity and quasiconvexity at u are obtained as well as other similar properties.

2. Invexity at a point and some of its generalisations

In this section we consider new classes of functions, called ρ-invex, ρ-pseudoinvex and ρ-quasiinvex at a point. The implications between these functions are also presented.

Definition 2.1: (*Invexities at a point.*) The function f is said to be *ρ-invex at the point $u \in X$* (abbreviated as ρI), if vector functions $\eta, \theta \colon X \times X \to \mathbb{R}^n$ and some real number ρ exist such that

$$(\rho\mathrm{I}) \qquad \forall x \in X\colon\ f(x) - f(u) \geqslant f^0(u; \eta(x, u)) + \rho \left\|\theta(x, u)\right\|^2 .$$

If

(1a) $\rho > 0$, then the function f is called *strongly invex at u* (SgI);
(1b) $\rho = 0$, then the function f is called *invex at u* (I);
(1c) $\rho < 0$, then the function f is called *weakly invex at u* (WI);
(1d) $\forall x \in X,\ x \neq u\colon\ f(x) - f(u) > f^0(u; \eta(x, u))$, then the function f is called *strictly invex at u* (SI).

In the case of differentiable functions we recover the definition of ρ-invexity at a point as given by Preda [**12**] (He used as a model the definition of the global ρ-invexity given by Jeyakumar [**8**].)

Theorem 1. *For the function f the following implications hold at u:*

(a) *Strongly invex (SgI) and ($x \neq u \Rightarrow \theta(x, u) \neq 0$) $\Rightarrow$ Strictly invex (SI);*
(b) *Strictly invex (SI) $\Rightarrow$ Invex (I) $\Rightarrow$ Weakly invex (WI).*

Proof: (a) For $\rho > 0$ and $\theta(x, u) \neq 0$ ($x \neq u$) we have

$$f(x) - f(u) - f^0(u; \eta(x, u)) \geqslant \rho \left\|\theta(x, u)\right\|^2 > 0,$$

that is, f is strictly invex at u.

(b) Obvious. ∎

Definition 2.2: (*Pseudoinvexities at a point.*) The function f is said to be *ρ-pseudoinvex at u* (ρPI), if there exist vector functions $\eta, \theta \colon X \times X \to \mathbb{R}^n$ and some real number ρ such that

$$(\rho\mathrm{PI}) \qquad \forall x \in X\colon\ f^0(u; \eta(x, u)) + \rho \left\|\theta(x, u)\right\|^2 \geqslant 0 \Rightarrow f(x) \geqslant f(u).$$

If

(2a) $\rho > 0$, then the function f is called *strongly pseudoinvex at u* (SgPI);
(2b) $\rho = 0$, then the function f is called *pseudoinvex at u* (PI);
(2c) $\rho < 0$, then the function f is called *weakly pseudoinvex at u* (WPI);
(2d) $\forall x \in X,\ x \neq u\colon\ f^0(u; \eta(x, u)) \geqslant 0 \Rightarrow f(x) > f(u)$, then the function f is called *strictly pseudoinvex at u* (SPI).

For the differentiable case, definitions (2a), (2b) and (2c) coincide with those given by Preda [12].

THEOREM 2. *For the function f the following implications hold at u*

(a) *Strongly pseudoinvex (SgPI) and injective ⇒ Strictly pseudoinvex (SPI).*
(b) *Strictly pseudoinvex (SPI) ⇒ Pseudoinvex (PI) ⇒ Weakly pseudoinvex (WPI).*

PROOF: (a) Suppose that f is strongly pseudoinvex at u. By virtue of Definition 2.2 (written in a equivalent form) we have

$$\forall x \in X,\, x \neq u \colon f(x) < f(u) \Rightarrow f^0(u;\eta(x,u)) + \rho \|\theta(x,u)\|^2 < 0.$$

But $\rho > 0$ implies $-\rho \|\theta(x,u)\|^2 \leqslant 0$ and then

$$f^0(u;\eta(x,y)) < -\rho \|\theta(x,u)\|^2 \leqslant 0.$$

Hence

$$\forall x \in X,\, x \neq u \colon f(x) < f(u) \Rightarrow f^0(u;\eta(x,u)) < 0.$$

It follows that

$$\forall x \in X,\, x \neq u \colon f^0(u;\eta(x,u)) \geqslant 0 \Rightarrow f(x) \geqslant f(u). \tag{3}$$

Since f is an injective function, it follows from (3) that

$$\forall x \in X,\, x \neq u \colon f^0(u;\eta(x,u)) \geqslant 0 \Rightarrow f(x) > f(u).$$

Thus, f is (SPI) at u.

(b) (SPI) ⇒ (PI). Obvious.

(PI) ⇒ (WPI). We have $\rho < 0$ and

$$\forall x \in X \colon f^0(u;\eta(x,u)) \geqslant 0 \Rightarrow f(x) \geqslant f(u).$$

Then, for $-\rho \|\theta(x,u)\|^2 \geqslant 0$, we have

$$\forall x \in X \colon f^0(u;\eta(x,u)) \geqslant -\rho \|\theta(x,u)\|^2 \geqslant 0 \Rightarrow f(x) \geqslant f(u),$$

that is, f is (WPI). ∎

THEOREM 3. *If f is ρ-invex at $u \in X$, then f is ρ-pseudoinvex at u. Moreover, if f is strictly invex at u, then f is strictly pseudoinvex at u.*

PROOF: In the relation (ρI) we apply

$$f^0(u;\eta(x,u)) + \rho \|\theta(x,u)\|^2 \geqslant 0$$

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and we obtain $f(x) \geqslant f(u)$, that is, f is (ρPI) at u. Also, in the definition of (SI) we apply $f^0(u;\eta(x, u)) \geqslant 0$ and we obtain $f(x) > f(u)$, that is, f is (SPI) at u. □

DEFINITION 2.3: (*Quasiinvexities at a point.*) The function f is said to be *ρ-quasiinvex at $u \in X$* (ρQI), if there exist vector functions η, θ: $X \times X \to \mathbb{R}^n$ and some real number ρ such that

$$(\rho\text{QI}) \qquad \forall x \in X: f(x) \leqslant f(u) \Rightarrow f^0(u;\eta(x, u)) + \rho \|\theta(x, u)\|^2 \leqslant 0.$$

If

(3a) $\rho > 0$, then the function f is called *strongly quasiinvex at u* (SgQI);
(3b) $\rho = 0$, then the function f is called *quasiinvex at u* (QI);
(3c) $\rho < 0$, then the function f is called *weakly quasiinvex at u* (WQI);
(3d) $\forall x \in X$, $x \neq u$: $f(x) \leqslant f(u) \Rightarrow f^0(u;\eta(x, u)) < 0$, then the function f is called *strictly quasiinvex at u* (SQI);
(3e) $\forall x \in X$, $x \neq u$: $f(x) < f(u) \Rightarrow f^0(u;\eta(x, u)) < 0$, then the function f is called *semistrictly quasiinvex at u* (SSQI).

In the differentiable case, the definition of ρ-quasiinvexity at u (without the cases (3d) and (3e)) was given by Preda [**12**]. The cases (3d) and (3e) were introduced by Giorgi and Mititelu [**6**].

The following lemma will be used in proving direct implications between the various types of quasiinvexity.

LEMMA. *Suppose that $f^0(u;\cdot)$ is finite on X. If f is lower semicontinuous (l.s.c.) and $\eta(\cdot, u)$ is bounded on X, then a number $\lambda_0 > 0$ exists such that*

$$\forall x \in X: f^0(u;\eta(x, u)) > 0 \Rightarrow f(u + \lambda\eta(x, u)) > f(u), \ \forall \lambda \in (0, \lambda_0).$$

PROOF: $f^0(u;\eta(x, u)) > 0$ yields

$$\limsup_{\substack{y \to u \\ \lambda \downarrow 0}} \frac{f(y + \lambda\eta(x, u)) - f(y)}{\lambda} > 0.$$

Then, there exists a neighbourhood V of u and a number $\lambda_0 > 0$, sufficiently small, such that for any $y \in V$ and any $\lambda \in (0, \lambda_0)$ we have

$$\frac{f(y + \lambda\eta(x, u)) - f(y)}{\lambda} > 0, \ \forall y \in V, \ \forall \lambda \in (0, \lambda_0),$$

or once more

$$f(y + \lambda\eta(x, u)) > f(y), \ \forall y \in V, \ \forall \in (0, \lambda_0).$$

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In particular, for $y = u$, since f is l.s.c. and $\eta(\cdot, u)$ is bounded on X, we have

$$f(u + \lambda\eta(x, u)) > f(u), \ \forall\lambda \in (0, \lambda_0).$$

□

Now, we establish direct implications of the relations existing between the various types of quasiinvexity at a point.

Theorem 4. *For the function f the following implications hold at u:*

(a) *Strongly quasiinvex (SgQI) and ($x \neq u \Rightarrow \theta(x, u) \neq 0$) $\Rightarrow$ Strictly quasiinvex (SQI),*
(b) *Strictly quasiinvex (SQI) $\Rightarrow$ Semistrictly quasiinvex (SSQI),*
(c) *Semistrictly quasiinvex (SSQI) and lower semicontinuous on X and $\eta(\cdot; u)$ bounded on $X \Rightarrow$ Quasiinvex (QI),*
(d) *Quasiinvex (QI) $\Rightarrow$ Weakly quasiinvex (WQI).*

Proof: (a) For $\rho > 0$ and ($x \neq u \Rightarrow \theta(x, u) \neq 0$) we have $-\rho \left\|\theta(x, u)\right\|^2 < 0$ and then

$$\forall x \in X, \, x \neq u \colon f(x) \leqslant f(u) \Rightarrow f^0(u; \eta(x, u)) \leqslant -\rho \left\|\theta(x, u)\right\|^2 < 0.$$

It follows from this implication that the function f is (SQI) at u.

(b) Obvious.

(c) We must show that f is (QI) at u, that is,

$$\forall x \in X \colon f(x) \leqslant f(u) \Rightarrow f^0(u; \eta(x, u)) \leqslant 0. \tag{4}$$

Since f is (SSQI) at u it follows that

$$\forall x \in X, \, x \neq u \colon f(x) < f(u) \Rightarrow f^0(u; \eta(x, u)) < 0 \leqslant 0.$$

Hence, (4) is true.

We now have to prove that

$$\forall x \in X \colon f(x) = f(u) \Rightarrow f^0(u; \eta(x, u)) \leqslant 0. \tag{5}$$

Assume by *reductio ad absurdum*, that (5) is not true. Then,

$$\exists t \in X \colon f(t) = f(u) \text{ and } f^0(u; \eta(t, u)) > 0. \tag{6}$$

According to the previous lemma, a number $\lambda_0 > 0$ exists such that

$$f(u + \lambda\eta(t, u)) > f(u), \ \forall\lambda \in (0, \lambda_0). \tag{7}$$

Consider $\overline{\lambda} \in (0, \lambda_0)$ and $\overline{x} = u + \overline{\lambda}\eta(t, u)$. Then (7) becomes $f(\overline{x}) > f(u)$. Denote

$$f(\overline{x}) - f(u) = a\ (> 0).$$

Because f is lower semicontinuous at x it follows that for any $\varepsilon > 0$, there is $\delta_\varepsilon > 0$ such that for any $x \in X$ for which $\|x - \overline{x}\| < \delta_\varepsilon$ one has $f(x) > f(\overline{x}) - \varepsilon$. In particular, for $x = u$ one gets that $\|u - \overline{x}\| < \delta_\varepsilon$ implies $f(u) > f(\overline{x}) - \varepsilon$. Choosing $\varepsilon = a$ it follows that $f(u) > f(u)$, which is contradictory.

In this proof we supposed that $\|u - \overline{x}\| < \delta_\varepsilon$ which is equivalent to $\overline{\lambda}\,\|\eta(t, u)\| < \delta_\varepsilon$ or $\|\eta(t, u)\| < \delta_\varepsilon/\lambda$. From this it follows that the function $\eta(\cdot, u)$ must be bounded on X.

(d) $\rho < 0$ mean that $0 \leqslant -\rho\,\|\theta(x, u)\|^2$ and (QI) yields

$$\forall x \in X:\ f(x) \leqslant f(u) \Rightarrow f^0(u; \eta(x, u)) \leqslant 0 \leqslant -\rho\,\|\theta(x, u)\|^2.$$

Therefore,

$$\forall x \in X:\ f(x) \leqslant f(u) \Rightarrow f^0(u; \eta(x, u)) + \rho\,\|\theta(x, u)\|^2 \leqslant 0,$$

that is, f is (WQI) at u. □

The implications between the various types of pseudoinvexity and quasiinvexity at a point are established through the following theorem.

Theorem 5. *For function f the following implications hold at u:*

(a) *Strongly pseudoinvex (SgPI) $\Rightarrow$ Strongly quasiinvex (SgQI),*
(b) *Strictly pseudoinvex (SPI) $\Rightarrow$ Strictly quasiinvex (SQI),*
(c) *Weakly pseudoinvex (WPI) $\Rightarrow$ Weakly quasiinvex (WQI),*
(d) *Pseudoinvex (PI) $\Rightarrow$ Semistrictly quasiinvex (SSQI).*

Proof: (a) Equivalently, f is (SgQI) at u $(\rho > o)$ when

$$\forall x \in X:\ f^0(u; \eta(x, u)) + \rho\,\|\theta(x, u)\|^2 > 0 \Rightarrow f(x) > f(u). \tag{8}$$

But since f is (SgPI) at u, the relation (ρPI) holds with $\rho > 0$. But this latter implication (ρPI) is stronger than implication (8).

(b) If f is (SPI) at u, then Definition 2.2 written in a equivalent form yields

$$\forall x \in X; x \neq u:\ f(x) \leqslant f(u) \Rightarrow f^0(u; \eta(x, u)) < 0.$$

Thus, f is (SQI) at u.

(c) Can be proved in a similar manner to (a).

(d) If f is (PI) at u, then Definition 2.2 written in a equivalent form yields

$$\forall x \in X:\ f(x) < f(u) \Rightarrow f^0(u; \eta(x, u)) < 0.$$

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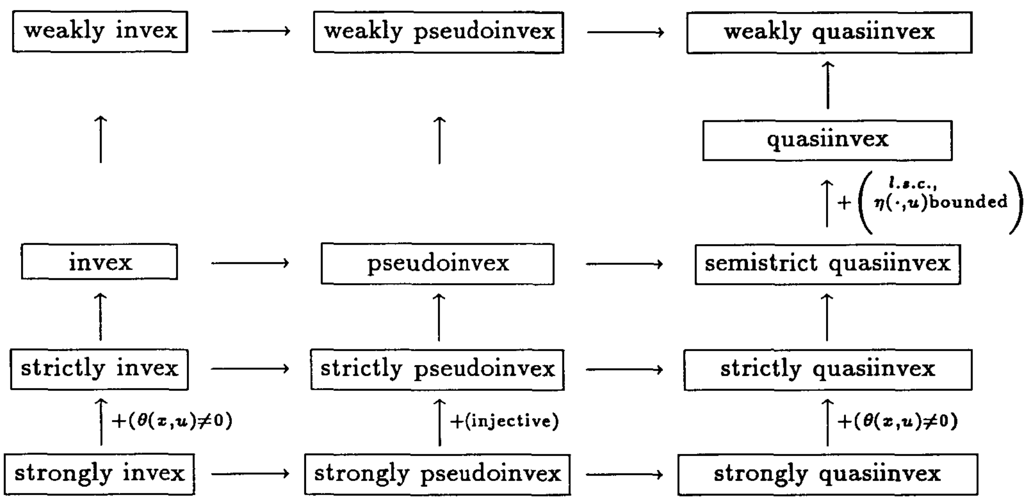
Hence $x \neq u$ and then f is (SSQI) at u. ∎

Direct implications of the relations existing between the various types of quasiinvexity at a point, according Theorems 1-5, are given in the following block Diagram



Block Diagram 1

3. Types of quasiconvexity at a point

In the particular case of $\eta(x, u) = x - u$ and $\theta(x, u) = x - u$ we obtain the types of convexities, pseudoconvexities and quasiconvexities at a point as follows:

Definition 3.1: (*Convexities at a point.*) The function f is said to be *ρ-convex at* $u \in X$ (abbreviated ρC), if there exists some real number ρ such that

$$(\rho\text{C}) \qquad \forall x \in X: f(x) - f(u) \geqslant f^0(u; x-u) + \rho \|x-u\|^2 .$$

If

(1′a) $\rho > 0$, then the function f is called *strongly convex at* u (SgC);
(1′b) $\rho = 0$, then the function f is called *convex at* u (C);
(1′c) $\rho < 0$, then the function f is called *weakly convex at* u (WC);
(1′d) $\forall x \in X,\ x \neq u: f(x) - f(u) > f^0(u; x-u)$, then the function f is called *strictly convex at* u (SC).

Definition 3.2: (*Pseudoconvexities at a point.*) The function f is said to be *ρ-pseudoconvex at* $u \in X$ (ρPC), if there exists some real number ρ such that

$$(\rho\text{PC}) \qquad \forall x \in X: f^0(u; x-u) + \rho \|x-u\|^2 \geqslant 0 \Rightarrow f(x) \geqslant f(u).$$

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If

(2′a) $\rho > 0$, then the function f is called *strongly pseudoconvex at u* (SgPC);
(2′b) $\rho = 0$, then the function f is called *pseudoconvex at u* (PC);
(2′c) $\rho < 0$, then the function f is called *weakly pseudoconvex at u* (WPC);
(2′d) $\forall x \in X,\ x \neq u\colon f^0(u; x-u) \geqslant 0 \Rightarrow f(x) > f(u)$, then f is called *strictly pseudoconvex at u* (SPC).

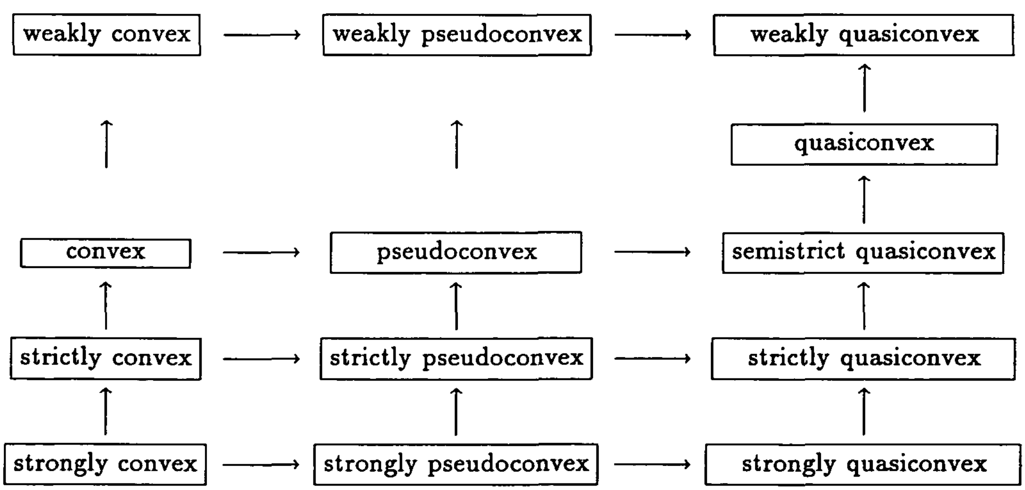
DEFINITION 3.3: (*Quasiconvexities at a point.*) The function f is said to be *ρ-quasiconvex at $u \in X$* (ρQC), if there exists some real number ρ such that

$$(\rho\text{QC}) \qquad \forall x \in X\colon f(x) \leqslant f(u) \Rightarrow f^0(u; x-u) + \rho \|x-u\|^2 \leqslant 0.$$

If

(3′a) $\rho > 0$, then the function f is called *strongly quasiconvex at u* (SgQC);
(3′b) $\rho = 0$, then the function f is called *quasiconvex at u* (QC);
(3′c) $\rho < 0$, then the function f is called *weakly quasiconvex at u* (WQC);
(3′d) $\forall x \in X,\ x \neq u\colon f(x) \leqslant f(u) \Rightarrow f^0(u; x-u) < 0$, then the function f is called *strictly quasiconvex at u* (SQC);
(3′e) $\forall x \in X,\ x \neq u\colon f(x) < f(u) \Rightarrow f^0(u; x-u) < 0$, then the function f is called *semistrictly quasiconvex at u* (SSQC).

Similarly, we can define global convexity, global pseudoconvexity and global quasiconvexity, which in the differentiable case were formulated by Jeyakumar [**9**]. However, strict quasiconvex, and similarly strict pseudoconvex, semistrict quasiconvex and strict convex were not treated by Jeyakumanr.



Block Diagram 2

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The implications which exist between the types of convexity and generalised convexity at a point, based on f^0, are given in Block Diagram 2. Moreover, any type of convexity at a point, simple or generalised, implies the corresponding type of invexity at a point (for instance, weakly convex $\Rightarrow$ weakly invex, convex $\Rightarrow$ invex et cetera).

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