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# DIVISIBILITY PROPERTIES OF GRADED DOMAINS

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**1. Introduction.** Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be an integral domain graded by an arbitrary torsionless grading monoid $\Gamma$. In this paper we consider to what extent conditions on the homogeneous elements or ideals of $R$ carry over to all elements or ideals of $R$. For example, in Section 3 we show that if each pair of nonzero homogeneous elements of $R$ has a GCD, then $R$ is a GCD-domain. This paper originated with the question of when a graded UFD (every homogeneous element is a product of principal primes) is a UFD. If $R$ is $\mathbf{Z}^+$ or $\mathbf{Z}$-graded, it is known that a graded UFD is actually a UFD, while in general this is not the case. In Section 3 we consider graded GCD-domains, in Section 4 graded UFD's, in Section 5 graded Krull domains, and in Section 6 graded $\pi$-domains. In each case we show that $R$ satisfies that divisibility property if and only if $R$ satisfies the corresponding graded divisibility property and $R_S$, the homogeneous quotient field of $R$, satisfies that divisibility property. In particular, if $R$ is $\mathbf{Z}^+$ or $\mathbf{Z}$-graded, each divisibility property is equivalent to its corresponding graded divisibility property. As an application, these results are used to determine when the semigroup ring $R[X; \Gamma]$ is a GCD-domain, a UFD, a Krull domain, or a $\pi$-domain.

**2. Graded integral domains.** In this section we include basic results about graded integral domains and homogeneous fractional ideals. All rings will be commutative integral domains, and all groups will be torsion-free abelian groups. General references for any undefined terminology are [**8**] and [**7**].

By a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$, we mean an integral domain graded by an arbitrary torsionless grading monoid $\Gamma$. That is, $\Gamma$ is a commutative cancellative monoid, written additively, and the quotient group $\langle \Gamma \rangle$ generated by $\Gamma$ is a torsion-free abelian group. A cancellative monoid $\Gamma$ is torsionless if and only if $\Gamma$ can be given a total order compatible with the monoid operation [**21**, p. 123], and this will be used throughout the paper. In this paper we will assume that all semigroups are torsionless grading monoids. We shall also often make the harmless assumption that each $R_\alpha$ is nonzero. A general reference on torsionless grading monoids and $\Gamma$-graded rings is [**21**].

One of the most important examples of a $\Gamma$-graded integral domain is the semigroup ring $R[X; \Gamma]$. Here $R[X; \Gamma] = R[\{X^g | g \in \Gamma\}]$ with

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$X^g X^h = X^{g+h}$. Note that the semigroup ring $R[X; \Gamma]$ is an integral domain if and only if $R$ is an integral domain and $\Gamma$ is a torsionless grading monoid. $R[X; \Gamma]$ is $\Gamma$-graded in the natural way with $\deg X^g = g$. For our next example, let $T$ be a subset of a torsionless grading monoid $\Gamma$ which generates $\Gamma$ as a monoid. Then the polynomial ring

$$A = R[\{X_g | g \in T\}]$$

is $\Gamma$-graded with

$$\deg X_{g_1}{}^{n_1} \ldots X_{g_r}{}^{n_r} = n_1 g_1 + \ldots + n_r g_r.$$

Note that $A = R[X; \Gamma']$, where $\Gamma'$ is the free monoid on $T$.

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a $\Gamma$-graded integral domain. Then $S = \{$nonzero homogeneous elements of $R\}$ is a multiplicatively closed set. Thus $R_S$ is a $G = \langle \Gamma \rangle$-graded quotient ring of $R$, where $R_S = \bigoplus_{\alpha \in G} (R_S)_\alpha$ with each $(R_S)_\alpha = \{a/b | a \in R_\beta, 0 \neq b \in R_\gamma, \text{ and } \beta - \gamma = \alpha\}$. In particular, $(R_S)_0$ is a field, and each nonzero homogeneous element of $R_S$ is a unit. We will often call $R_S$ the homogeneous quotient field of $R$. For future reference we include the following result.

PROPOSITION 2.1. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain and $S = \{$nonzero homogeneous elements of $R\}$. Then $R_S$ is a completely integrally closed* GCD-*domain.*

*Proof.* We have already noted that each nonzero homogeneous element of $R_S$ is a unit. The proposition thus follows from [**3**, Proposition 3.2 and Proposition 3.3].

If $R$ is $\mathbf{Z}$ or $\mathbf{Z}^+$-graded, it is well known that

$$R_S \approx (R_S)_0[t, t^{-1}] \approx (R_S)_0[X; \mathbf{Z}].$$

Also, if $R$ is an integral domain with quotient field $K$, then the homogeneous quotient field of $R[X; \Gamma]$ is just $K[X; \langle \Gamma \rangle]$. More generally, for an arbitrary graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$, $R_S$ is isomorphic to a twisted group ring $(R_S)_0{}^\gamma[X; \langle \Gamma \rangle]$. For more details, see [**4**].

Although $R_S$ is always a GCD-domain, it need not be a UFD. For example, let $R = K[X; \mathbf{Q}]$. Then $R_S = R$. But $R_S$ is not a UFD because $(1 - X) \subset (1 - X^{1/2}) \subset (1 - X^{1/4}) \subset \ldots$ is a strictly increasing chain of principal ideals. However, $R_S$ is a UFD if and only if $R_S$ is a Krull domain, if and only if $R_S$ satisfies the ascending chain condition on principal ideals [**3**, Corollary 3.4].

In [**9**, Theorem 7.13], Gilmer and Parker determined necessary and sufficient conditions for the group ring $R[X; G]$ to be a UFD. Matsuda [**14**, Proposition 3.3] used their result to determine when $R[X; G]$ is a Krull domain. We record these results for future reference.

PROPOSITION 2.2. (1) *The group ring* $R[X; G]$ *is a* UFD *if and only if* $R$ *is a* UFD *and* $G$ *satisfies the ascending chain condition on cyclic subgroups.*

(2) $R[X; G]$ *is a Krull domain if and only if* $R$ *is a Krull domain and* $G$ *satisfies the ascending chain condition on cyclic subgroups.*

We remark that $G$ satisfies the ascending chain condition on cyclic subgroups if and only if all rank one subgroups of $G$ are free [**3**, Lemma 2.5]. Gilmer and Parker, and Matsuda used the equivalent condition that every element of $G$ is of type $(0, 0, 0, \ldots)$.

Even if $\langle \Gamma \rangle$ does not satisfy the ascending chain condition on cyclic subgroups, one can easily construct $\Gamma$-graded integral domains $R$ such that $R_S$ is a UFD [**3**, p. 88]. However, as we have already seen, $K[X; \langle \Gamma \rangle]$ will not be a UFD. On the positive side, we have the following result.

PROPOSITION 2.3. ([**3**, Corollary 3.6]). *Let* $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ *be a graded integral domain and* $S = \{$*nonzero homogeneous elements of* $R\}$. *If* $\langle \Gamma \rangle$ *satisfies the ascending chain condition on cyclic subgroups, then* $R_S$ *is a* UFD.

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain and $S = \{$nonzero homogeneous elements of $R\}$. An overring $T$, with $R \subseteq T \subseteq R_S$, will be called a *homogeneous overring* if

$$T = \bigoplus_{\alpha \in \langle \Gamma \rangle} (T \cap (R_S)_\alpha).$$

Thus $T$ is a graded integral domain with $T_\alpha = T \cap (R_S)_\alpha$. We also define a fractional ideal $I$ of $R$ to be a *homogeneous fractional ideal* if there is a nonzero homogeneous element $r$ of $R$ such that $rI \subseteq R$ is homogeneous. In particular, a homogeneous fractional ideal is an $R$-submodule of the homogeneous quotient field $R_S$.

PROPOSITION 2.4. *Let* $I$ *be a fractional ideal of the graded ring* $R$. *The following statements are equivalent.*

(1) $I$ *is a homogeneous fractional ideal.*

(2) $I = (1/s)J$, *where* $s \in S$ *and* $J \subseteq R$ *is a homogeneous ideal.*

(3) $I = \bigoplus_{\alpha \in \langle \Gamma \rangle} (I \cap R_S)_\alpha$.

*Proof.* (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. (3) $\Rightarrow$ (1). Since $I$ is a fractional ideal, there is a nonzero $r$ in $R$ so that $rI \subseteq R$. Let $r = r_{\alpha_1} + \ldots + r_{\alpha_n}$ with $\alpha_1 < \ldots < \alpha_n$, and each $r_{\alpha_i} \neq 0$. Let $i \in I$ be homogeneous. Since $ri \in R$, by comparing degrees it is clear that $r_{\alpha_1} i \in R$. But $I$ is generated by its homogeneous elements, so $r_{\alpha_1} I \subseteq R$. Thus $I$ is a homogeneous fractional ideal.

Let $R$ be an integral domain with quotient field $K$. Given fractional ideals $I$ and $J$, and an overring $T$ of $R$, we define

$$I :_T J = \{x \in T \mid xJ \subseteq I\},$$

which is also a fractional ideal. If $T = K$, the subscript $K$ will usually be omitted. We will also denote $R : I$ by $I^{-1}$ and $R : (R : I) = (I^{-1})^{-1}$ by $I_v$. We will say that $I$ is a *divisorial* or *v-ideal* if $I = I_v$. If $R$ is a graded integral domain and $I$ is a fractional ideal of $R$, then $I^*$ will denote the (homogeneous) fractional ideal generated by the homogeneous elements of $I$. Thus $I$ is homogeneous if and only if $I = I^*$. It is well known that if $P$ is a prime ideal, then $P^*$ is also a prime ideal [**21**, p. 124]. Thus a minimal prime ideal is homogeneous if and only if $P \cap S \neq \emptyset$, where $S = \{$nonzero homogeneous elements of $R\}$.

PROPOSITION 2.5. *Let $R$ be a graded integral domain and $S = \{$nonzero homogeneous elements of $R\}$. If $I$ and $J$ are homogeneous fractional ideals, then $I : J$ is also a homogeneous fractional ideal and $I : J = I :_{R_S} J$. Thus if $I$ is a homogeneous fractional ideal, $I_v$ is also homogeneous. Also, if $I$ is a v-ideal, then $I^* = 0$, or $I^*$ is a v-ideal.*

*Proof.* Let $x \in I : J$, so $x = a/b$ where $a, 0 \neq b \in R$. Choose a nonzero homogeneous element $j$ of $J$, so $j = c/d$ where $c, d \in S$. Then $(a/b)(c/d) \in I \subseteq R_S$. But then $x = a/b \in (d/c)R_S \subseteq R_S$, thus $I : J = I :_{R_S} J$. Since $I$ and $J$ are homogeneous $R$-submodules of $R_S$, so is $I :_{R_S} J$.

If $I$ is a homogeneous fractional ideal, then so is $I_v = R :_{R_S} (R :_{R_S} I)$. Finally, suppose that $I$ is a $v$-ideal. If $I^* \neq 0$, then $(I^*)_v$ is homogeneous. But $I^* \subseteq (I^*)_v \subseteq I_v = I$, so $I^* = (I^*)_v$.

**3. Graded GCD-domains.** In this section we prove that if each pair of nonzero homogeneous elements of a graded integral domain $R$ has a GCD, then $R$ is actually a GCD-domain. Formally, we make the following definition.

*Definition* 3.1. A graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a *graded* GCD-*domain* if each pair of nonzero homogeneous elements has a (necessarily homogeneous) GCD.

LEMMA 3.2. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then $R$ is a graded* GCD-*domain if and only if each pair of nonzero homogeneous elements of $R$ has a (necessarily homogeneous)* LCM.

*Proof.* Let $a$ and $b$ be nonzero homogeneous elements of $R$ with $d =$ GCD $\{a, b\}$ (necessarily homogeneous). We show that $aR \cap bR = (ab/d)R$. Let $x$ be a homogeneous element of $aR \cap bR$. Then $x = as = bt$ for some homogeneous $s, t \in R$. Thus $x/d = (a/d)s = (b/d)t$. But

$$\text{GCD}\,\{a/d, b/d\} = d/d = 1;$$

so $(b/d)|s$, and thus $s = (b/d)v$. Hence

$$x/d = (a/d)s = (a/d)(b/d)v,$$

so $x = (ab/d)v$. Thus $x \in (ab/d)R$. But $aR \cap bR$ is a homogeneous ideal, so $aR \cap bR = (ab/d)R$. Thus LCM $\{a, b\} = ab/d$. Conversely, it is well known that if two elements have a LCM, then they also have a GCD, namely GCD $\{a, b\} = ab/\text{LCM}\,\{a, b\}$ [**8**, p. 76].

Let $R$ be a graded GCD-domain and $f = a_{\alpha_1} + \ldots + a_{\alpha_n} \in R$, with each $0 \neq a_{\alpha_i} \in R_{\alpha_i}$. We say that $f$ is *primitive* if

$$\text{GCD}\,\{a_{\alpha_1}, \ldots, a_{\alpha_n}\} = 1.$$

Note that this definition differs from the usual definition used for a polynomial ring. For example, by our definition a primitive polynomial would necessarily have a nonzero constant term.

PROPOSITION 3.3. (Gauss's Lemma). *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded GCD-domain. If $f, g \in R$ are primitive, then $fg$ is also primitive.*

*Proof.* Our proof is modeled after [**9**, Proposition 4.6]. Let $f = a_{\alpha_1} + \ldots + a_{\alpha_n}$ and $g = b_{\beta_1} + \ldots + b_{\beta_m}$, where all the homogeneous terms are nonzero, and $\alpha_1 < \ldots < \alpha_n$ and $\beta_1 < \ldots < \beta_m$. We show that each nonzero, nonunit homogeneous element $d$ of $R$ fails to divide some homogeneous component of $fg$. We may assume that GCD $\{a_{\alpha_1}, d\} \neq 1$; for if

$$\text{GCD}\,\{a_{\alpha_1}, d\} = \text{GCD}\,\{b_{\beta_1}, d\} = 1,$$

then GCD $\{a_{\alpha_1}b_{\beta_1}, d\} = 1$. Let

$$d_j = \text{GCD}\,\{a_{\alpha_1}, \ldots, a_{\alpha_j}, d\}.$$

Then there is a smallest $i \geqq 2$ such that $d_i = 1$ because $f$ is primitive. Thus $d_{i-1}$ is a nonunit homogeneous divisor of $d$. It suffices to prove that $d_{i-1}$ fails to divide some homogeneous component of $fg$. So we may assume that $d = d_{i-1}$. Similarly, choose $k$ minimal so that

$$\text{GCD}\,\{b_{\beta_1}, \ldots, b_{\beta_k}, d\} = 1,$$

and then replace $d$ by

$$d' = \text{GCD}\,\{b_{\beta_1}, \ldots, b_{\beta_{k-1}}, d\}.$$

Thus we may assume that $d$ is a nonzero, nonunit divisor of $a_{\alpha_1}, \ldots, a_{\alpha_{i-1}}, b_{\beta_1}, \ldots, b_{\beta_{k-1}}$, and

$$\text{GCD}\,\{a_{\alpha_i}, d\} = \text{GCD}\,\{b_{\beta_k}, d\} = 1.$$

We show that $d$ does not divide the $\alpha_i + \beta_k$ component of $fg$. For the $\alpha_i + \beta_k$ component has the form $a_{\alpha_i}b_{\beta_k} + \ldots + a_\alpha b_\beta + \ldots$, where $\alpha < \alpha_i$ or $\beta < \beta_k$. Now $d$ divides each $a_\alpha b_\beta$, and hence if $d$ divides the $\alpha_i + \beta_k$ component, it divides $a_{\alpha_i}b_{\beta_k}$. But this is a contradiction because

$$\text{GCD}\,\{a_{\alpha_i}, d\} = \text{GCD}\,\{b_{\beta_k}, d\} = 1,$$

and hence GCD $\{a_{\alpha_i}b_{\beta_k}, d\} = 1$.

THEOREM 3.4. *If* $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ *is a graded* GCD-*domain, then* $R$ *is a* GCD-*domain.*

*Proof.* We will use [**9**, Theorem 3.1] to show that $R$ is a GCD-domain. We will follow their notation. Let $N = S$ be the set of nonzero homogeneous elements of $R$. By Proposition 3.3, it is clear that $T$, the set of elements of $R$ LCM prime to $N$, is precisely the set of primitive elements of $R$. Lemma 3.2 shows that (1) of [**9**, Theorem 3.1] holds, while it is clear that (2) holds since $R$ is a graded GCD-domain. By Proposition 2.1, $R_N$ is a GCD-domain, and hence by [**9**, Theorem 3.1], $R$ is a GCD-domain.

An alternate proof of Theorem 3.4 would be to use Northcott's Theorem [**20**] as in [**9**, Theorem 4.4].

If $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded GCD-domain, and hence a GCD-domain, then $R_0$ need not be a GCD-domain. As in [**3**, p. 96] or [**13**], let $R = K[X, Y, Z, W]$ with $\deg X = \deg Y = 1$ and $\deg Z = \deg W = -1$. Then $R$ is a $\mathbf{Z}$-graded UFD and $R_0 = K[XZ, XW, YZ, YW]$. However $R_0$ is not a GCD-domain (for example, $XZYZ$ and $XZYW$ are both homogeneous of deg 0, but GCD $\{XZYZ, XZYW\} = XYZ$ is homogeneous of deg 1, and hence not in $R_0$).

Following [**6**], we define an extension of rings $R \subset T$ to be *inert* if whenever $xy \in R$ for some nonzero $x, y \in T$, then $x = ru$ and $y = su^{-1}$ for some $r, s \in R$ and $u$ a unit of $T$.

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. It is easily proved that if $R$ is a GCD-domain and $R_0 \subset R$ is an inert extension, then $R_0$ is also a GCD-domain. Two important cases when $R_0 \subset R$ is an inert extension are when $R = R_0[X; \Gamma]$ is a semigroup ring, or when $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is $\Gamma$-graded with $\Gamma \cap -\Gamma = 0$ (that is, no nonzero element of $\Gamma$ has an inverse).

As an application of Theorem 3.4, we obtain the Gilmer and Parker [**9**, Theorem 6.1 and Theorem 6.4] characterization of when the semigroup ring $R[X; \Gamma]$ is a GCD-domain. A semigroup $\Gamma$ is a GCD-semigroup if for all $a, b \in \Gamma$, there is a $c \in \Gamma$ so that

$$(a + \Gamma) \cap (b + \Gamma) = c + \Gamma.$$

PROPOSITION 3.5. *The semigroup ring* $R[X; \Gamma]$ *is a* GCD-*domain if and only if* $R$ *is a* GCD-*domain and* $\Gamma$ *is a torsionless* GCD-*semigroup.*

*Proof.* ($\Rightarrow$). If $R[X; \Gamma]$ is a GCD-domain, then $R$ is also a GCD-domain by our earlier remarks on inert extensions. Let $a, b \in \Gamma$ and $X^c = \text{LCM}\{X^a, X^b\}$. Then clearly

$$c + \Gamma = (a + \Gamma) \cap (b + \Gamma),$$

so $\Gamma$ is a GCD-semigroup.

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($\Leftarrow$). By Theorem 3.4 we need only show that $R[X; \Gamma]$ is a graded GCD-domain. By Lemma 3.2 we need only show that $(aX^g) \cap (bX^h)$ is a principal ideal. Let

$$c = \mathrm{LCM}\,\{a, b\} \quad \text{and} \quad (g + \Gamma) \cap (h + \Gamma) = e + \Gamma.$$

Then $(aX^g) \cap (bX^h) = (cX^e)$. Thus $R[X; \Gamma]$ is a graded GCD-domain, and hence a GCD-domain.

Let $R$ be an integral domain with quotient field $K$. Let $U = U(R)$ be the group of units of $R$ and $K^* = K\backslash\{0\}$. The *group of divisibility* of $R$, $G(R)$, is $K^*/U$. $G(R)$ becomes a partially ordered abelian group with $xU \leqq yU \Leftrightarrow yx^{-1} \in R$. It is well known that $R$ is a GCD-domain if and only if $G(R)$ is a lattice ordered abelian group. If $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded integral domain, in a similar manner, we may define the *homogeneous group of divisibility* of $R$, $HG(R)$, to be $A/B$, where $A$ is the set of nonzero homogeneous elements of $R_S$, the homogeneous quotient field of $R$, and $B = S \cap U(R)$, the homogeneous units of $R$. In the natural way, $HG(R)$ is a partially ordered subgroup of $G(R)$.

It is clear that $R$ is a graded GCD-domain if and only if $HG(R)$ is a lattice ordered abelian group.

**4. Graded unique factorization domains.** In this section we study unique factorization in terms of homogeneous elements. Thus we are led to the following definition.

*Definition* 4.1. A graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a *graded* UFD if each nonzero, nonunit homogeneous element of $R$ is a product of (necessarily homogeneous) principal primes.

Unlike the case for graded GCD-domains, a graded UFD need not be a UFD. For example, as mentioned in Section 2, the group ring $K[X; \mathbf{Q}]$ is trivially a graded UFD since all of its nonzero homogeneous elements are units, but it is not a UFD. Our next result gives several other characterizations of a graded UFD. Each is just the graded analogue of a well-known equivalent condition for a UFD.

PROPOSITION 4.2. *The following statements are equivalent for a graded integral domain* $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$.

(1) *$R$ is a graded* UFD.

(2) *$R$ is a graded* GCD-*domain and $R$ satisfies the ascending chain condition on homogeneous principal ideals.*

(3) *$HG(R)$, the homogeneous group of divisibility of $R$, is order isomorphic to a direct sum of copies of* $\mathbf{Z}$ *with the usual product order.*

(4) *Each nonzero homogeneous prime ideal of $R$ contains a nonzero homogeneous principal prime ideal.*

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*Proof.* (1) $\Rightarrow$ (4) is clear. The proof of (1) $\Leftrightarrow$ (2) is similar to that of the ungraded case [**8**, Proposition 16.4], while the proof of (1) $\Leftrightarrow$ (3) is similar to [**19**, Theorem 4.3]. (4) $\Rightarrow$ (1). Let $N = \{$nonzero homogeneous elements of $R$ that are products of prime elements, or are units$\}$. Thus $N$ is a saturated multiplicative set with $N \subseteq S = \{$nonzero homogeneous elements of $R\}$, and $R_N$ has no nonzero homogeneous prime ideals. Thus each nonzero homogeneous element $x$ of $R$ is a unit. For if $x$ is not a unit, then it is contained in a nonzero prime ideal $P$. But then $P^*$ is a nonzero homogeneous prime ideal, a contradiction. Thus $R_N = R_S$, so $N = S$.

LEMMA 4.3. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded* UFD *and $p$ a nonzero homogeneous prime element of $R$. Then $\bigcap p^n R = 0$ and $R_{(p)}$ is a* DVR. *Also, for any infinite family $\{p_\alpha\}$ of nonassociate homogeneous prime elements of $R$, $\bigcap p_\alpha R = 0$.*

*Proof.* $I = \bigcap p^n R$ is a homogeneous prime ideal [**8**, Theorem 7.6]. If $I$ is nonzero, it contains a homogeneous prime $q$ by Proposition 4.2. But then $0 \neq (q) \subsetneq (p)$, a contradiction.

Clearly $R_{(p)}$ is thus a DVR. If $J = \bigcap p_\alpha R$ is nonzero, then it contains a nonzero homogeneous element which is divisible by infinitely many nonassociate homogeneous primes, a contradiction.

THEOREM 4.4. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain and $S = \{$nonzero homogeneous elements of $R\}$. Then $R$ is a* UFD *if and only if $R$ is a graded* UFD *and $R_S$ is a* UFD.

*Proof.* If $R$ is a UFD, then certainly $R$ is a graded UFD and $R_S$ is a UFD. Conversely, assume that $R$ is a graded UFD and $R_S$ is a UFD. Let $P$ be a nonzero prime ideal of $R$. If $P \cap S \neq \emptyset$, then $P$ contains a nonzero homogeneous principal prime ideal. If $P \cap S = \emptyset$, then $P_S$ is a proper prime ideal of the UFD $R_S$. Thus there is a nonzero $p \in P$ so that $(p)_S$ is prime in $R_S$. By Lemma 4.3, we may assume that no homogeneous prime of $R$ divides $p$. But then $(p)$ is a prime ideal of $R$. For if $p|ab$, then we may assume $a \in (p)_S$, and hence $sa = rp$ for some $s \in S$ and $r \in R$. But $s$ is a product of homogeneous primes, none of which divide $p$, so $s|r$. Thus $p|a$, so $a \in (p)$. Thus each nonzero prime ideal of $R$ contains a principal prime ideal, so $R$ is a UFD [**11**, Theorem 5].

COROLLARY 4.5. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded* UFD. *If $\langle \Gamma \rangle$ satisfies the ascending chain condition on cyclic subgroups, then $R$ is a* UFD.

*Proof.* By Proposition 2.3, if $\langle \Gamma \rangle$ satisfies the ascending chain condition on cyclic subgroups, then $R_S$ is a UFD.

As a special case we obtain the following theorem of Anderson and Matijevic [**2**, Theorem 5].

COROLLARY 4.6. *Let $R$ be a $\mathbf{Z}^+$ or $\mathbf{Z}$-graded* UFD. *Then $R$ is a* UFD.

There are several alternate proofs of Theorem 4.4. If $R$ is a graded UFD, it may be shown that $R = R_S \cap (\cap R_{(p_\alpha)})$, where $\{p_\alpha\}$ is the set of nonassociate homogeneous primes of $R$, and this intersection is locally finite. By Lemma 4.3, each $R_{(p_\alpha)}$ is a DVR. Thus, if $R_S$ is a UFD, then $R$ is a Krull domain. But $R$ is also a graded GCD-domain, and hence a GCD-domain by Theorem 3.4. Hence $R$ is a UFD. This approach will be used in the next section in our study of graded Krull domains. Another proof would be to use [**9**, Theorem 3.2]. (Note that property $(\Delta)$ of [**9**, Theorem 3.4] is just our Lemma 4.3.)

As with graded GCD-domains, the same example of Section 3 shows that $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ may be a graded UFD, yet $R_0$ need not be a UFD. Again, it is also easy to show that if $R_0 \subset R$ is an inert extension, then $R_0$ is a UFD whenever $R$ is a graded UFD.

Finally, we will use Theorem 4.4 to give another proof of the Gilmer and Parker characterization [**9**, Theorem 7.17] of when the semigroup ring is a UFD. A monoid $\Gamma$ is a *unique factorization semigroup* (UFS) if each nonzero, nonunit element of $\Gamma$ is a sum of prime elements (in additive notation, $p$ is prime if and only if $a + b \in p + \Gamma$ implies $a \in p + \Gamma$ or $b \in p + \Gamma$). As expected, $p \in \Gamma$ is prime if and only if $X^p$ is prime in $R[X; \Gamma]$ ([**9**, Lemma 7.16]).

PROPOSITION 4.7. *The semigroup ring $R[X; \Gamma]$ is a* UFD *if and only if $R$ is a* UFD, *$\Gamma$ is a* UFS, *and the maximal subgroup of $\Gamma$, $H = \Gamma \cap -\Gamma$, satisfies the ascending chain condition on cyclic subgroups.*

*Proof.* ($\Rightarrow$). Suppose that $R[X; \Gamma]$ is a UFD. By earlier remarks about inert extensions, $R$ is a UFD. Now $R[X; \langle\Gamma\rangle]$ is a localization of $R[X; \Gamma]$ and hence a UFD. Thus $\langle\Gamma\rangle$, and hence its subgroup $H$, satisfies the ascending chain condition on cyclic subgroups by Proposition 2.2. Clearly $\Gamma$ is a UFS, with primes $\{p_\alpha\}$, such that $\{X^{p_\alpha}\}$ are primes in $R[X; \Gamma]$.

($\Leftarrow$). By Theorem 4.4 we need only show that $R[X; \Gamma]$ is a graded UFD and $K[X; \langle\Gamma\rangle]$ is a UFD, here $K$ is the quotient field of $R$. But $K[X; \langle\Gamma\rangle]$ is a UFD by Proposition 2.2 since $\langle\Gamma\rangle \approx H \oplus (\oplus \mathbf{Z}_\alpha)$ satisfies the ascending chain condition on cyclic subgroups whenever $H$ does [**9**, Lemma 7.15]. To show that $R[X; \Gamma]$ is a graded UFD we need only show that each nonzero, nonunit homogeneous element $aX^g$ is a product of primes. But this is clear because $a$ is a product of primes in $R$ and $g = u + n_1p_1 + \ldots + n_rp_r$ for some $u \in H$ and primes $p_1, \ldots, p_r \in \Gamma$. So $aX^g = aX^u(X^{p_1})^{n_1} \ldots (X^{p_r})^{n_r}$ is a product of primes in $R[X; \Gamma]$.

Let $\Gamma$ be a UFS with $\{p_\alpha\}$ a set of nonassociate primes and let $H = \Gamma \cap -\Gamma$ be the set of invertible elements of $\Gamma$, that is, the maximal subgroup of $\Gamma$. Then $\Gamma \approx H \oplus (\oplus \mathbf{Z}_\alpha^+)$ [**9**, Proof of Lemma 7.15]. Thus

$$R[X; \Gamma] \approx R[X; H][\{X_\alpha\}].$$

For emphasis we restate this as a corollary.

COROLLARY 4.8. *The semigroup ring* $R[X; \Gamma]$ *is a* UFD *if and only if it has the form* $R[X; G][\{X_\alpha\}]$, *where* $R$ *is a* UFD *and* $G$ *satisfies the ascending chain condition on cyclic subgroups.*

**5. Graded Krull domains.** Since an integral domain $R$ is a Krull domain if and only if $R$ is completely integrally closed and satisfies the ascending chain condition on integral $v$-ideals, we are led to make the following definition.

*Definition* 5.1. A graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a *graded Krull domain* if it is completely integrally closed with respect to homogeneous elements (if $a/b$, with $a, b \in S$, is almost integral over $R$, then $a/b \in R$) and satisfies the ascending chain condition on homogeneous integral $v$-ideals.

We first show that if $R$ is completely integrally closed with respect to homogeneous elements, then $R$ is actually completely integrally closed.

PROPOSITION 5.2. *Let* $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ *be a graded integral domain and* $S = \{$*nonzero homogeneous elements of* $R\}$. *Then the following statements are equivalent.*

(1) $R$ *is completely integrally closed.*
(2) $R$ *is completely integrally closed with respect to homogeneous elements.*
(3) $I : I = R$ *for each nonzero homogeneous fractional ideal* $I$.
(4) $R$ *is completely integrally closed in* $R_S$.

*Proof.* (1) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (3). Let $I$ be a nonzero homogeneous fractional ideal. By Proposition 2.5, $J = I : I$ is also a homogeneous fractional ideal. Let $x \in J$ be homogeneous. Then $x$ is almost integral over $R$, and hence by hypothesis in $R$. Thus $I : I = R$. (3) $\Rightarrow$ (2). Let $x$ be homogeneous and almost integral over $R$. Then $R[x]$ is a homogeneous fractional ideal. Thus $x \in R[x] : R[x] = R$. (2) $\Rightarrow$ (4). Let $x = r/s \in R_S$ be almost integral over $R$. Then

$$R[r/s] \subseteq (a_1/s_1)R + \ldots + (a_n/s_n)R \subseteq R_S,$$

where $a_i \in R$ and $s_i \in S$. Thus $R[r/s] \subseteq (1/t)R$, where $t = s_1 \ldots s_n \in S$. Let $r = r_{\alpha_1} + \ldots + r_{\alpha_n}$, where each $r_{\alpha_i}$ is homogeneous and $\alpha_1 < \ldots < \alpha_n$. Since $(1/t)R$ is homogeneous, $(r_{\alpha_1}/s)^n \in (1/t)R$ for $n \geqq 1$, and hence $R[r_{\alpha_1}/s] \subseteq (1/t)R$. Hence $r/s - r_{\alpha_1}/s = (r_{\alpha_2} + \ldots + r_{\alpha_n})/s$ is almost integral over $R$. Similarly, each $r_{\alpha_i}/s$ is almost integral over $R$ and hence each $r_{\alpha_i}/s \in R$. Thus $r/s \in R$. (4) $\Rightarrow$ (2). Let $x = a/b$, with $a, b \in S$, be almost integral over $R$. Then $R[x]$ is a homogeneous fractional ideal, so $R[x] \subseteq (1/r)R \subseteq R_S$, for some homogeneous $r \in R$. Thus $x \in R_S$ is almost integral over $R$ as an element of $R_S$. Hence $x \in R$. (2) $\Rightarrow$ (1). Let $x = a/b$, with $a, b \in R$, be almost integral over $R$. Since $R_S$ is completely integrally closed, $x \in R_S$. Thus we may assume that $b \in S$. Let

$0 \neq t \in R$ such that $t(a/b)^n \in R$ for all $n \geqq 1$. Let $a = a_{\alpha_1} + \ldots + a_{\alpha_m}$ and $t = t_{\beta_1} + \ldots + t_{\beta_k}$, where each term is nonzero and homogeneous and $\alpha_1 < \ldots < \alpha_m$ and $\beta_1 < \ldots < \beta_k$. Then $t_{\beta_1}(a_{\alpha_1}/b)^n$ is the term of least degree in $t(a/b)^n$. Thus

$$t_{\beta_1}(a_{\alpha_1}/b)^n \in R \quad \text{for } n \geqq 1,$$

so by hypothesis $a_{\alpha_1}/b \in R$. In a similar manner each $a_{\alpha_i}/b \in R$, and hence $a/b \in R$.

In [**3**, p. 103], a torsionless grading monoid $\Gamma$ was said to satisfy property (*) if whenever $g \in \Gamma$, $h \in \langle \Gamma \rangle$, and $g + nh \in \Gamma$ for all $n \geqq 1$, then $h \in \Gamma$. As a corollary of Proposition 5.2, we obtain the following result of Anderson [**3**, Proposition 7.9] and Matsuda [**17**, Proposition 4.16].

COROLLARY 5.3. *The semigroup ring $R[X; \Gamma]$ is completely integrally closed if and only if $R$ is completely integrally closed and $\Gamma$ satisfies* (*).

While we will make no use of our next result, we state it because it, and its proof, are very similar to Proposition 5.2.

PROPOSITION 5.4. *For a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$, the following statements are equivalent.*

(1) *$R$ is integrally closed.*

(2) *$R$ is integrally closed with respect to homogeneous elements.*

(3) *$I : I = R$ for each finitely generated nonzero homogeneous fractional ideal $I$.*

(4) *$R$ is integrally closed in $R_S$.*

We have seen that a graded Krull domain is completely integrally closed. The next several propositions show that the homogeneous rank one prime ideals of a graded Krull domain behave very much like the rank one prime ideals of a Krull domain.

PROPOSITION 5.5. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded Krull domain. The set of maximal proper homogeneous $v$-ideals of $R$ coincides with the set $X_H(R) = X_H$ of homogeneous rank one prime ideals. If $P \in X_H$, then $R_P$ is a* DVR. *Hence maximal homogeneous $v$-ideals are maximal prime $v$-ideals. Also, if $P \in X_H$, then $P^{(n)}$ is homogeneous.*

*Proof.* The proof of [**7**, Proposition 3.5] adapted to the homogeneous case shows that a maximal homogeneous $v$-ideal $P$ is prime. Then [**7**, Theorem 3.10] shows that $R_P$ is a DVR. Thus a maximal homogeneous $v$-ideal is a rank one prime ideal. Conversely, let $P$ be a homogeneous rank one prime ideal. Let $f$ be a nonzero homogeneous element of $P$. Enlarge $(f)$ to a maximal homogeneous $v$-ideal $Q$ contained in $P$. It suffices to show that $Q$ is prime. For then $P = Q$, and then $P$ is a $v$-ideal, and hence necessarily a maximal homogeneous $v$-ideal. (For if $P$ is not maximal, it can be enlarged to a maximal $v$-ideal which is a rank one prime ideal.) So suppose that $ab \in Q$, where $a$ and $b$ are homogeneous and $b \notin Q$. Then $Q :_R (a) \supsetneq Q$ is a homogeneous $v$-ideal, thus $Q : (a) \not\subseteq P$ (we will now omit the subscript $R$). Choose a homogeneous $h \in Q : (a) \backslash P$. Now $Q : (h) \supseteq Q$ is a homogeneous $v$-ideal and $(Q : (h))h \subseteq Q \subseteq P$. Thus $Q : (h) \subseteq P$ because $h \notin P$. So $Q = Q : (h)$, and hence $a \in Q : (h) = Q$, so $Q$ is prime.

Let $P$ be a homogeneous rank one prime ideal. Then $P^{(n)}$ is $P$-primary. Since $P^n \subseteq P^{(n)}$, $P^{(n)*}$ is a nonzero $P$-primary ideal [**21**, p. 125]. Thus $P^{(n)*} = P^{(n+k)}$ for some $k \geqq 0$. But then $P^n \subseteq P^{(n)*} = P^{(n+k)}$, so

$$P_P^{(n)} = P_P^n = P_P^{(n+k)} = P_P^{n+k}.$$

Thus $k = 0$, so $P^{(n)}$ is a $P$-primary homogeneous ideal.

PROPOSITION 5.6. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded Krull domain and let $b$ be a nonzero homogeneous element of $R$. Then $b$ is contained in only a finite number of rank one (necessarily homogeneous) prime ideals $P_1, \ldots, P_s$, and $(b) = P_1^{(n_1)} \cap \ldots \cap P_s^{(n_s)}$ for some positive integers $n_1, \ldots, n_s$.*

*Proof.* Let $P$ be a prime ideal minimal over $(b)$. Then $(b) \subseteq P^* \subseteq P$, so $P$ is homogeneous. As in Proposition 5.5, $(b)$ may be enlarged to a prime homogeneous $v$-ideal contained in $P$. Thus $P$ is a $v$-ideal of rank one. It follows, as in [**7**, Theorem 3.12], that the number of such primes is finite. Let these primes be $P_1, \ldots P_s$. Let $(b)_{p_i} = P_{iP_i}^{n_i}$, so $(b) \subseteq P_i^{(n_i)}$, and hence

$$(b) \subseteq P_1^{(n_1)} \cap \ldots \cap P_s^{(n_s)}.$$

Suppose that $(b) \subsetneq P_1^{(n_1)} \cap \ldots \cap P_s^{(n_s)}$, so we may choose a homogeneous

$$x \in P_1^{(n_1)} \cap \ldots \cap P_s^{(n_s)} \backslash (b).$$

Now $(x)_{P_i} \subseteq P_{iP_i}^{n_i}$. Since $x \notin (b)$, $(b) :_R (x)$ is a proper homogeneous $v$-ideal. Hence $(b) :_R (x) \subseteq Q$, where $Q$ is a homogeneous rank one prime $v$-ideal. Since $(b) \subseteq (b) :_R (x)$, we may assume that $P_1 = Q$. Thus $(b) :_R (x) \subseteq P_1$, so that

$$(b)_{P_1} :_{R_{P_1}} (x)_{P_1} = ((b) :_R (x))_{P_1} \subseteq P_{1_{P_1}}.$$

But $(b)_{P_1} = P_{1P_1}^{n_1}$ and $(x)_{P_1} \subseteq P_{1P_1}^{n_1}$, so

$$(b)_{P_1} :_{R_{P_1}} (x)_{P_1} = R_{P_1},$$

a contradiction. Thus we must have $(b) :_R (x) = R$, or $x \in (b)$.

PROPOSITION 5.7. *Let $b$ be a homogeneous element of a graded Krull domain $R$. If $a$ is a nonzero (not necessarily homogeneous) element of $R$, then $(b) :_R (a)$ is a homogeneous $v$-ideal.*

*Proof.* By Proposition 5.6, $(b) = P_1^{(n_1)} \cap \ldots \cap P_s^{(n_s)}$, where $P_1, \ldots, P_s$ are the homogeneous rank one prime ideals that contain $(b)$. Again, we will omit the subscript $R$ from $I :_R J$. Now

$$\begin{aligned}(b) : (a) &= (P_1^{(n_1)} \cap \ldots \cap P_s^{(n_s)}) : (a) \\ &= (P_1^{(n_1)} : (a)) \cap \ldots \cap (P_s^{(n_s)} : (a)).\end{aligned}$$

Each $P_i^{(n_i)} : (a)$ is either $R$ or $P_i$-primary. In the second case

$$P_i^{(n_i)} : (a) = P_i^{(n_i+K)} \quad \text{for some } k \geqq 0.$$

In either case, each $P_i^{(n_i)} : (a)$ is homogeneous, and thus so is their intersection.

We can now prove the main theorem of this section. Recall that $X_H = X_H(R)$ is the set of homogeneous rank one prime ideals of $R$.

THEOREM 5.8. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded Krull domain and $S = \{$nonzero homogeneous elements of $R\}$. Then $R = R_S \cap_{P \in X_H} (\cap R_P)$, and the intersection is locally finite. Thus $R$ is a Krull domain if and only if $R$ is a graded Krull domain and $R_S$ is a Krull domain.*

*Proof.* Clearly the second statement follows from the first. It is clear that $R \subseteq R_S \cap (\cap R_P)$. Let $x \in R_S \cap (\cap R_P)$, with $x = a/b$, where $a \in R$ and $b \in S$. Suppose that $x \notin R$. Then $(b) :_R (a) \subseteq P$ for some homogeneous rank one prime $v$-ideal $P$. But $a/b \in R_P$, so $a/b = r/t$, where $t \notin P$. Thus $t(a/b) \in R$, so $t \in (b) :_R (a) \subseteq P$. This contradiction shows that $x \in R$, so that $R = R_S \cap (\cap R_P)$. Let $x$ be a nonzero element of $R$. Then $x$ is a unit in $R_P$ unless $x \in P$ (here $P \in X_H$). If $x$ is contained in infinitely many $P_i \in X_H$, then $\cap P_i$ is a nonzero homogeneous ideal. But then $\cap P_i$ contains a nonzero homogeneous element $x$ which is contained in infinitely many rank one prime ideals, which contradicts Proposition 5.6.

Unlike the earlier cases, if $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded Krull domain, then $R_0$ is actually a Krull domain.

COROLLARY 5.9. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded Krull domain. Then $R_0$ is a Krull domain.*

*Proof.* Let $K$ be the quotient field of $R_0$. Then by Theorem 5.8, $R_0 = R \cap K = K \cap (\cap R_P)$ is a Krull domain.

COROLLARY 5.10. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded Krull domain. If $\langle \Gamma \rangle$ satisfies the ascending chain condition on cyclic subgroups, then $R$ is a Krull domain. In particular, if $R$ is $\mathbf{Z}^+$ or $\mathbf{Z}$-graded, then $R$ is a Krull domain.*

*Proof.* Apply Theorem 5.8 and Proposition 2.3.

We next use Theorem 5.8 to derive necessary and sufficient conditions for the semigroup ring $R[X; \Gamma]$ to be a Krull domain, thus giving another proof of a recent result of Chouinard [**5**, Theorem 1]. Let $\Gamma$ be a semigroup with $G = \langle \Gamma \rangle$. For $X \subseteq G$, we define

$$\Gamma : X = \{g \in G | \, x + g \in \Gamma \quad \text{for all } x \in X\}.$$

An ideal $I$ of $\Gamma$ is a $v$-ideal if $\Gamma : (\Gamma : I) = I$. Let $K$ be the quotient field of $R$. It is clear that there is a one-to-one correspondence between the $v$-ideals of $\Gamma$ and the homogeneous $v$-ideals of $K[X; \Gamma]$ given by

$$I \leftrightarrow (\{x^g | \, g \in I\}).$$

Thus $K[X; \Gamma]$ satisfies the ascending chain condition on homogeneous $v$-ideals if and only if $\Gamma$ satisfies the ascending chain condition on $v$-ideals. Following Chouinard [**5**, Proposition 2], we define $\Gamma$ to be a *Krull semigroup* if $\Gamma$ satisfies

(*) $\quad (g \in \Gamma, h \in \langle \Gamma \rangle, \text{ and } g + nh \in \Gamma \text{ for all } n \geqq 1 \text{ implies } h \in \Gamma)$

and $\Gamma$ satisfies the ascending chain condition on $v$-ideals.

PROPOSITION 5.11. *The semigroup ring $R[X; \Gamma]$ is a Krull domain if and only if $R$ is a Krull domain, $\Gamma$ is a torsionless Krull semigroup, and $\langle \Gamma \rangle$ satisfies the ascending chain condition on cyclic subgroups.*

*Proof.* ($\Rightarrow$) Suppose that $R[X; \Gamma]$ is a Krull domain. By Corollary 5.9, $R$ is a Krull domain (with quotient field $K$). Also, the localization $K[X; \Gamma]$ is a Krull domain, so by our earlier remarks, $\Gamma$ satisfies the ascending chain condition on $v$-ideals. By Corollary 5.3, $\Gamma$ satisfies (*), so $\Gamma$ is a Krull semigroup. Since the localization $R[X; \langle \Gamma \rangle]$ is also a Krull domain, by Proposition 2.2, $\langle \Gamma \rangle$ satisfies the ascending chain condition on cyclic subgroups.

($\Leftarrow$). By Proposition 2.2, $R[X; \langle \Gamma \rangle]$ is a Krull domain. Since

$$R[X; \Gamma] = R[X; \langle \Gamma \rangle] \cap K[X; \Gamma],$$

we need only show that $K[X; \Gamma]$ is a Krull domain. And by Theorem 5.8 we need only show that $K[X; \Gamma]$ is a graded Krull domain. Since $\Gamma$ is a Krull semigroup, $\Gamma$ satisfies (*), and thus $K[X; \Gamma]$ is completely integrally closed by Corollary 5.3. Earlier remarks show that $K[X; \Gamma]$ also satisfies the ascending chain condition on homogeneous $v$-ideals. Thus $K[X; \Gamma]$ is a graded Krull domain, and hence a Krull domain.

In Corollary 4.8 we saw that a semigroup ring is a UFD if and only if it has the form $R[X; G] [\{X_\alpha\}]$, where $R$ is a UFD and $G$ satisfies the ascending chain condition on cyclic subgroups. Chouinard [**5**, Proposition 1] has shown that the semigroup ring $R[X; \Gamma]$ is a Krull domain if and only if it has the form $R[X; G][Y; S]$, where $G$ satisfies the ascending chain condition on cyclic subgroups and $S$ is a submonoid of a free group $F = \oplus \mathbf{Z}_\alpha$ such that $S = \langle S \rangle \cap F_+$. Thus $R[X; G]\,[Y; S]$ may be regarded as a subring of $R[X; G]\,[\{X_\alpha\}]$ generated by monomials over $R[X; G]$.

THEOREM 5.12. *Let* $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ *be a graded integral domain and* $S = \{$*nonzero homogeneous element of* $R\}$. *The following statements are equivalent.*

(1) $R$ *is a graded Krull domain.*

(2) $R = R_S \cap (\bigcap_{P \in X_H} R_P)$, *the intersection is locally finite, and each* $R_P$ *is a* DVR.

(3) $R = R_S \cap (\bigcap_\alpha V_\alpha)$, *the intersection is locally finite, and each* $V_\alpha$ *is a* DVR.

*Proof.* (1) $\Rightarrow$ (2) follows from Theorem 5.8, while (2) $\Rightarrow$ (3) is obvious. (3) $\Rightarrow$ (1) is a modification of [**7**, Theorem 3.6]. Since $R_S$ and each $V_\alpha$ is completely integrally closed, so is their intersection $R$. We may assume that each $R \subseteq V_\alpha \subseteq K$, where $K$ is the quotient field of $R$. If $I$ is a homogeneous fractional ideal of $R$, then

$$R : I = R_S \cap (\cap\, (V_\alpha : IV_\alpha)).$$

Thus for a homogeneous $v$-ideal $I$,

$$I = R_S \cap (\cap\, (V_\alpha : (R : I)V_\alpha)).$$

Let $0 \neq I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ be an ascending chain of integral homogeneous $v$-ideals. Then for each $\alpha$, $V_\alpha : (R : I_1)V_\alpha \subseteq V_\alpha : (R : I_2)V_\alpha \subseteq \dots$ is an ascending chain of ideals of $V_\alpha$. Since the intersection is locally finite,

$$V_\alpha : (R : I_1)V_\alpha = V_\alpha \quad \text{for almost all } \alpha.$$

Since each $V_\alpha$ is noetherian, there is an $n$ such that

$$V_\alpha : (R : I_n)V_\alpha = V_\alpha : (R : I_{n+1})V_\alpha = \dots \quad \text{for all } \alpha.$$

Thus

$$\begin{aligned} I_m &= R_S \cap (\cap\, (V_\alpha : (R : I_m)V_\alpha)) \\ &= R_S \cap (\cap\, (V_\alpha : (R : I_n))V_\alpha) = I_n \quad \text{for all } m \geqq n. \end{aligned}$$

Thus $R$ satisfies the ascending chain condition on homogeneous $v$-ideals and hence is a graded Krull domain.

We will define a graded integral domain $R$ to be a *graded* DVR if $R$ is a graded UFD with a single homogeneous principal prime. Note that a graded DVR is a graded Krull domain. More generally, one can define $R$ to be a *graded valuation ring* if for each homogeneous $x$ in $R_S$, the homogeneous quotient field of $R$, either $x$ or $x^{-1} \in R$. Thus $R$ is a graded valuation ring if and only if $HG(R)$, its homogeneous group of divisibility, is totally ordered. Graded valuation rings were introduced by Johnson [**10**].

LEMMA 5.13. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded Krull domain, $P$ a homogeneous rank one prime ideal and $S = \{$nonzero homogeneous elements of $R\}$. Then $R_S \cap R_P$ is a homogeneous overring of $R$ which is a graded* DVR.

*Proof.* By Theorem 5.8, $R = R_S \cap (\bigcap_{P \in X_H} R_P)$. Let $T = S \backslash P$. (If $T = \emptyset$, then $R = R_S \cap R_P$ and we are trivially finished.) Since the intersection is locally finite,

$$R_T = (R_S)_T \cap (\cap (R_P)_T) = R_S \cap R_P.$$

But clearly $R_T$ is a homogeneous overring of $R$. By Theorem 5.12, $R_T$ is a graded Krull domain with $P_T$ as its unique homogeneous rank one prime ideal. Since every homogeneous element is contained in a homogeneous rank one prime ideal, $P_T$ is the unique nonzero homogeneous prime ideal of $R_T$. By Proposition 5.6, every homogeneous ideal has the form $P_T^{(n)}$ for some $n$. Thus $P_T$ must be principal, and it easily follows that $R_T$ is a graded UFD with a single homogeneous principal prime $P_T$.

COROLLARY 5.14. *Let $R = R_S \cap (\bigcap_{P \in X_H} R_P)$ be a graded Krull domain. Then for $Y \subseteq X_H$, the subintersection $R_Y = R_S \cap (\bigcap_{P \in Y} R_P)$ is a graded Krull domain.*

*Proof.* $R_Y = R_S \cap (\bigcap_{P \in Y} R_P) = \bigcap_{P \in Y} (R_S \cap R_P)$, so $R_Y$ is a homogeneous overring of $R$. It follows from Theorem 5.12 that $R$ is a graded Krull domain.

In particular, if $T \subseteq S$ is a multiplicatively closed set, then $R_T = R_Y$, where $Y = \{P \in X_H | P \cap T = \emptyset\}$. We say that the intersection $R = \cap R_\alpha$ is *homogeneously locally finite* if each homogeneous element of $R$ is a unit in almost all $\alpha$.

THEOREM 5.15. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then $R$ is a graded Krull domain if and only if $R = \cap R_\alpha$, where the intersection is homogeneously locally finite and each $V_\alpha$ is a homogeneous overring of $R$ which is a graded* DVR.

*Proof.* If $R$ is a graded Krull domain, this follows from Theorem 5.12 and Lemma 5.13. Conversely, if $R = \cap V_\alpha$ is a homogeneously locally finite intersection of graded DVR homogeneous overrings, then as in the proof of Theorem 5.12, we see that $R$ is actually a graded Krull domain.

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded Krull domain. Since $R$ is completely integrally closed, $D(R)$, the set of nonzero fractional $v$-ideals of $R$ with the usual $v$-operation is an abelian group [**7**, Proposition 3.4]. We can also form the subgroup $HD(R)$ of homogeneous $v$-ideals. It is not hard to show that $HD(R)$ is a free abelian group generated by the maximal homogeneous integral $v$-ideals. We can then define the *homogeneous divisor class group* of $R$ to be $\mathrm{HCl}\,(R) = \mathrm{HD}\,(R)/\mathrm{HPrin}\,(R)$, where $\mathrm{HPrin}\,(R)$ is the subgroup of homogeneous principal ideals. There is a natural embedding HCl $(R) \to$ Cl $(R)$, which is an isomorphism whenever $R$ is a Krull domain [**3**, Theorem 4.2].

**6. Graded $\pi$-domains.** An integral domain $R$ is a *$\pi$-domain* if every principal ideal is a product of prime ideals. Several other equivalent conditions for $R$ to be a $\pi$-domain may be found in [**1**] and [**2**]. For example, $R$ is a $\pi$-domain if and only if $R$ is a Krull domain and every $v$-ideal of $R$ is invertible.

*Definition* 6.1. A graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is a *graded $\pi$-domain* if each principal homogeneous ideal is a product of (necessarily homogeneous invertible) prime ideals.

The example $R = K[X, Y, Z, W]$ with deg $X$ = deg $Y = 1$ and deg $Z$ = deg $W = -1$ of Section 3 shows that if $R$ is a graded $\pi$-domain, $R_0$ need not be a $\pi$-domain. However, if $R_0 \subseteq R$ is an inert extension, then $R_0$ is a $\pi$-domain wherever $R$ is a graded $\pi$-domain. For if $x$ is a homogeneous element of $R_0$, then $xR = P_1 \ldots P_n$ for some prime ideals of $R$. But then $xR_0 = Q_1 \ldots Q_n$, where each $Q_i = P_i \cap R_0$. For $\pi$-domains, we have the analogue of Theorem 5.8 on Krull domains.

THEOREM 6.2. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain and $S$ = {nonzero homogeneous elements of $R$}. Then $R$ is a $\pi$-domain if and only if $R$ is a graded $\pi$-domain and $R_S$ is a $\pi$-domain.*

*Proof.* ($\Rightarrow$) is clear. So conversely suppose that $R$ is a graded $\pi$-domain and $R_S$ is a $\pi$-domain, or equivalently a UFD [**3**, Corollary 3.4]. We first show that $R$ is a graded Krull domain, and hence a Krull domain by Theorem 5.8. It is clear that each homogeneous rank one prime ideal $P$ of $R$ is invertible, and thus each $R_P$ is a DVR. Also, the intersection $R_S \cap (\bigcap_{P\in X_H} R_P)$ is locally finite. Thus by Theorem 5.12, to show that $R$ is a graded Krull domain we need only show that $R = R_S \cap (\bigcap R_P)$. Let $x = r/s \in R_S \cap (\bigcap R_P)$, where $r \in R$ and $s \in S$. Then as in Proposition 5.7, $(s) :_R (r)$ is a homogeneous $v$-ideal. Thus we may assume that $r$ is actually homogeneous. If $sR = P_1 \ldots P_n$ and $rR = Q_1 \ldots Q_m$, then each $P_i$ is equal to some $Q_j$. For $r/s \in R_{P_i}$ implies $r/s = x/t$ for some $x \in R$ and $t \notin P_i$. Thus $rt = xs \in P_i$, and hence $r \in P_i$. So $Q_1 \ldots Q_m \subset P_i$ and hence $P_i$ is equal to some $Q_j$. Thus

$$(s) :_R (r) = ((P_1 \ldots P_n) :_K (Q_1 \ldots Q_m)) \cap R = (R :_K I) \cap R = R,$$

where

$$I = (Q_1 \ldots Q_m)(P_1 \ldots P_n)^{-1}.$$

Thus $1 \in (s) :_R (r)$, so $x = r/s \in R$. Thus $R$ is a graded Krull domain, and hence a Krull domain. As mentioned earlier, the natural embedding

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of the homogeneous divisor class group HCl $(R)$ into Cl $(R)$, the divisor class group, is an isomorphism [**3**, Theorem 4.2]. Thus Cl $(R)$ is generated by the homogeneous rank one prime ideals of $R$, which are invertible. Hence every divisorial ideal of $R$ is invertible, so $R$ is a $\pi$-domain.

COROLLARY 6.3. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded $\pi$-domain. If $\langle \Gamma \rangle$ satisfies the ascending chain condition on cyclic subgroups, then $R$ is a $\pi$-domain.*

As a special case of Corollary 6.3, we have the following theorem of Anderson and Matijevic [**2**, Theorem 6].

COROLLARY 6.4. *A $\mathbf{Z}^+$ or $\mathbf{Z}$-graded $\pi$-domain is a $\pi$-domain.*

Finally, we obtain a characterization of when the semigroup ring $R[X; \Gamma]$ is a $\pi$-domain. Special cases of this have been considered by Anderson [**1**] and Matsuda [**16**, § 12].

PROPOSITION 6.5. *The semigroup ring $R[X; \Gamma]$ is a $\pi$-domain if and only if $R$ is a $\pi$-domain, $\Gamma$ is a* UFS, *and $\langle \Gamma \rangle$ satisfies the ascending chain condition on cyclic subgroups.*

*Proof.* Suppose that $R[X; \Gamma]$ is a $\pi$-domain. Since $R \subset R[X; \Gamma]$ is an inert extension, $R$ is a $\pi$-domain (with quotient field $K$). The localization $K[X; \Gamma]$ is a Krull domain, so by Proposition 5.11 $\langle \Gamma \rangle$ satisfies the ascending chain condition on cyclic subgroups. By the remarks after Proposition 5.11, $\Gamma \approx G \times S$, where $G = \Gamma \cap -\Gamma$, so $K[X; \Gamma] \approx A[Y; S]$, which is a subring of some $A[\{X_\alpha\}]$ generated by monomials over $A = K[X; G]$. By [**12**, p. 57] each homogeneous invertible ideal of $K[X; \Gamma]$ is extended from the UFD $K[X; G]$, and hence is principal. But thus $K[X; \Gamma]$ is a graded UFD, and hence a UFD. So $\Gamma$ is a UFS by Proposition 4.7. Conversely, we need only show that $R[X; \Gamma]$ is a graded $\pi$-domain. By the remarks before Corollary 4.8,

$$R[X; \Gamma] \approx R[X; G][\{X_\alpha\}],$$

where $G = \Gamma \cap -\Gamma$. Thus $R[X; \Gamma]$ is a $\pi$-domain if and only if $R[X; G]$ is a $\pi$-domain [**1**, p. 200]. If $aX^g \in R[X; G]$, then

$$aX^g R[X; G] = P_1 \ldots P_n R[X; G],$$

where $aR = P_1 \ldots P_n$. But each $P_i R[X; G]$ is an invertible prime ideal of $R[X; G]$. Thus $R[X; G]$ is a $\pi$-domain, and hence so is $R[X; \Gamma]$.

Thus a semigroup ring $R[X; \Gamma]$ is a $\pi$-domain if and only if it has the form $R[X; G][\{X_\alpha\}]$, where $R$ is a $\pi$-domain and $G$ satisfies the ascending chain condition on cyclic subgroups. So by Corollary 4.8, the only difference between whether $R[X; \Gamma]$ is a UFD or a $\pi$-domain is whether $R$ is a UFD or a $\pi$-domain.

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain and $S = \{$nonzero homogeneous elements of $R\}$. We have defined graded GCD-domain, graded UFD, graded completely integrally closed, graded Krull domain, and graded $\pi$-domain. We showed that $R$ satisfies each particular property if and only if $R$ satisfies the corresponding graded property and the homogeneous quotient field $R_S$ satisfies that property. Since $R_S$ is a completely integrally closed GCD-domain, in each of these two cases, the graded and non-graded properties are equivalent.

Let $R$ be an integral domain with quotient field $K$. For the semigroup ring $R[X; \Gamma]$ and $S = R \backslash \{0\}$, we have seen that $R[X; \Gamma]$ satisfies any of these five properties if and only if $R$ and $R[X; \Gamma]_S = K[X; \Gamma]$ satisfy that property. Thus one is led to ask if $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ satisfies one of the divisibility properties if and only if $R_0$ and $R_S$ satisfies that property, where here $S = R_0 \backslash \{0\}$. The answer is no. For let $R = \mathbf{Z} + X\mathbf{Q}[X]$, where deg $X = 1$. Then $R$ is $\mathbf{Z}^+$-graded with $R_0 = \mathbf{Z}$ and $R_S = \mathbf{Q}[X]$. Thus $R_0$ and $R_S$ both satisfy any of the five divisibility properties, but it is easy to see that $R$ satisfies none of them.

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