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Detectability of Torus Topology

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Abstract. The global shape, or topology, of the universe is not constrained by the equations of General Relativity, which only describe the local universe. As a consequence, the boundaries of space are not fixed and topologies different from the trivial infinite Euclidean space are possible. The cosmic microwave background (CMB) is the most efficient tool to study topology and test alternative models. Multi-connected topologies, such as the 3-torus, are of great interest because they are anisotropic and allow us to test a possible violation of isotropy in CMB data. We show that the correlation function of the coefficients of the expansion of the temperature and polarization anisotropies in spherical harmonics encodes a topological signature. This signature can be used to distinguish an infinite space from a multi-connected space on sizes larger than the diameter of the last scattering surface (D_{LSS}). With the help of the Kullback-Leibler divergence, we set the size of the edge of the biggest distinguishable torus with CMB temperature fluctuations and E-modes of polarization to 1.15 D_{LSS}. CMB temperature fluctuations allow us to detect universes bigger than the observable universe, whereas E-modes are efficient to detect universes smaller than the observable universe.

Keywords. cosmic microwave background, methods: statistical, topology, Kullback-Leibler divergence, detectability

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1. Introduction

In the standard model of cosmology, the universe is described by a 6-parameter Λ-CDM model. The universe is assumed to be isotropic, homogeneous and Gaussian at very large scales. These properties are expected to be found in the CMB. The CMB is emitted from a sphere all around us, the last scattering surface (of diameter D_{LSS}). The last scattering surface is the limit of the observable universe. By studying the CMB, *via* its temperature and polarization anisotropies, we can extract information about the properties of our universe. This work used the `Healpix` package and `CAMB` software.

CMB. The most convenient way to work with the CMB is to project its temperature and polarization fluctuations on the spherical harmonic basis to get the $T_{\ell,m}$, $E_{\ell,m}$ and $B_{\ell,m}$ coefficients (see Hu & White 1997). These coefficients arise from initial conditions in the primordial universe to which a transfer function, describing the local perturbation (Sachs-Wolfe, ISW, Doppler), is applied and finally the eigenmodes of the Laplacian of the space. The E-modes are generated by scalar, vector and tensor perturbations whereas B-modes are generated by vector and tensor perturbations. In *Planck* results XXII (2013), the limit of the tensor-to-scalar ration r evaluated with the *Planck* temperature data is found to be $r < 0.11$ with a 95% C.L. More recently, the first detection of B-modes in the CMB by the BICEP2 collaboration set $r = 0.20^{+0.07}_{-0.05}$ (BICEP2 2014). As a consequence, even if B-modes are supposed to take part in the CMB fluctuations, their contribution is sub-dominant compared to E-modes. That is why in the remaining part of the study, we will only consider E-modes as in Riazuelo *et al.* (2006). Then, we use the covariance matrix ($C_{\ell m,\ell' m'}$) divided in 4 blocks $C^{XY}_{\ell m,\ell' m'} =< X_{\ell m} Y^*_{\ell' m'} >$, with $X, Y \in \{T, E\}$. For an isotropic space, each covariance matrix reduces to a pure diagonal matrix, $C^{XY}_{\ell m,\ell' m'} = \delta_{\ell\ell'}\delta_{mm'} C^{XY}_\ell$, where C^{XY}_ℓ is the power spectrum.

Topology. The topology is the *global* shape of the universe and it is not constrained by General Relativity, which is a set of differential equations that describes space *locally*, *via* the metric, but not the boundary conditions. That is why for the same metric, different topologies are possible. The standard model of cosmology assumes an isotropic infinite universe, but some anomalies, hints of the violation of the global isotropy, have been discovered in the WMAP data (Bielewicz *et al.* 2004). That is why multi-connected flat spaces have been considered (Riazuelo *et al.* 2004a, Riazuelo *et al.* 2004b) because they are anisotropic models of flat universes. A complete review of their characteristics can be found in Lachieze-Rey & Luminet (1995), Levin (2002).

Multi-connected spaces are anisotropic and the covariance matrix of an isotropic space is purely block-diagonal. That is why it is very important to take into account the full covariance for any topological study. The restriction to the power spectrum is not enough because topological information will consequently be lost. As the CMB is emitted from the last scattering surface, one could imagine that any search of topology would not be able to detect a topology bigger than the observable universe. However the correlations between the modes inside the observable universe and outside should help us extracting information above the last scattering surface limit. That is why we looked how far we can see a topology beyond the observable universe, and we illustrate here this analysis with the example of the cubic 3-torus (see Fabre *et al.* 2013).

2. The Kullback-Leibler divergence $D_{\rm KL}$

Kullback-Leibler divergence. We would like to compare two theories that predict that the coefficients of the expansion of the temperature anisotropies in spherical harmonics, $a_{\ell m}$, are Gaussian and satisfy $\langle a_{\ell m} a^*_{\ell' m'}\rangle_1 = C^{(1)}_{\ell\ell' mm'} = C^{(1)}_\ell \delta_{\ell\ell'}\delta_{mm'}$ for the isotropic model and $\langle a_{\ell m} a^*_{\ell' m'}\rangle_2 = C^{(2)}_{\ell\ell' mm'}$ for the non-trivial topology, where the ensemble average are taken for each theory respectively. Such a comparison can be performed in terms of the Kullback-Leibler divergence (Kullback) for two probability distribution functions p_1 and p_2 defined by

$$D_{\rm KL}(p_1||p_2) = \int p_1(x) \ln\left[\frac{p_1(x)}{p_2(x)}\right] dx. \tag{2.1}$$

This divergence is the expectation value of $\ln(p_1/p_2)$ with the ensemble average related to p_1. Due to the Gibbs inequality, $D_{\rm KL}$ is always positive and equal to zero if and only if $p_1 = p_2$. In terms of information theory, $D_{\rm KL}(p_1||p_2)$ quantifies the amount of information lost when the data (p_1) is represented by the model (p_2). Comparing any multi-connected space (2) with the Euclidean trivial space (1) is interesting because the latter has a rotationally invariant covariance matrix. Consequently the Kullback-Leibler divergence does not depend on the relative orientation of the two spaces and thus quantifies how much information "separates" model 2 from model 1. Furthermore the flat Euclidean space is the most probable topology given previous studies. It is important to see if a deviation from it easily can be detected. When working with a 3-torus whose edge bigger than D_{LSS}, $D_{\rm KL}(1||2)$ is obviously close to zero. The main interest of this approach is that, unlike the circles-in-the-sky method (see Cornish *et al.* 1998), one can measure a

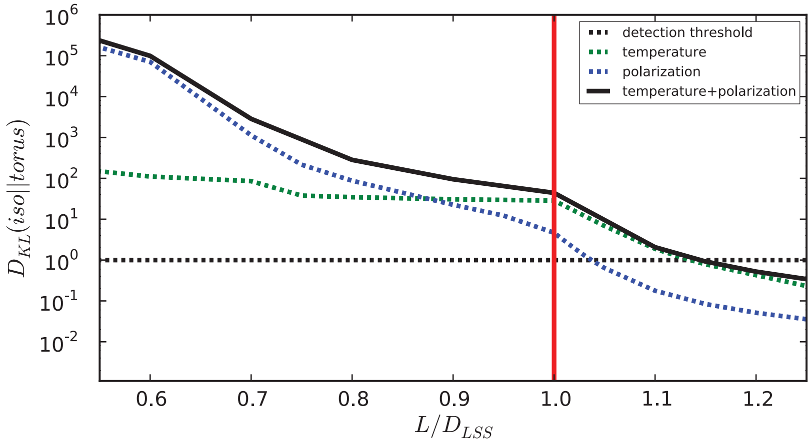

Figure 1. Kullback-Leibler divergence of a trivial infinite isotropic Euclidean space compared to a cubic 3-torus in function of the size L of the edge of the torus in units of D_{LSS}. The black line is the threshold of detection above which the divergence should be in order to detect a 3-torus from a Euclidean space. The green dotted line is the contribution of the temperature fluctuations only, the blue dotted line is the contribution of the E-modes polarization fluctuations only and the black line is taken into account both temperature and polarization fluctuations.

distance even for spaces with a size larger than D_{LSS}. Furthermore, the Kullback-Leibler divergence is not affected by instrumental noise or galactic masking and the small scales only improve the detection of universes smaller than the observable universe, which is very interesting computationally speaking (see Fabre *et al.* 2013 for more details).

Detection threshold. Let us introduce the Bayes factor $B_{12} = \frac{P_1(d|M_1)}{P_2(d|M_2)}$. If $B_{12} > 1$ (resp. $B_{12} < 1$) it represents the increase (resp. decrease) of the credence in favour of model 1 (M_1) versus model 2 (M_2) given the observed data (see Trotta 2008). It gives the factor by which the relatives odds between the two models have changed after taking into account the data. The data are the $a_{\ell m}$ in this experiment. If we take into account formula (2.1), we have $D_{\mathrm{KL}}(1||2) = \langle \ln(B_{12}) \rangle_1$. There is thus a direct link between the Kullback divergence and the Bayes factor. The Jeffrey scale, usually used to interpret the Bayes factor, is not modified if we consider $\langle \ln(B_{12}) \rangle_1$ instead of $\ln(B_{12})$. As a consequence we obtain the same levels of significance, with a threshold of detectability (represented in black dotted line in Fig.1) for $D_{\mathrm{KL}} = 1$. This threshold of detectability quantifies the level at which we can distinguish a torus topology from the isotropic model. If $D_{\mathrm{KL}} < 1$, the result is inconclusive and the torus topology cannot be distinguished from a Euclidean space.

3. Results and discussion

Results. The three D_{KL} curves decrease as the size of the 3-torus increase which is logical since we are getting close to an infinite isotropic universe. There is also a change of slope at $L = D_{LSS}$ (red vertical line in Fig. 1), especially for the temperature curve. It is because we are crossing the last scattering surface, *i.e.* the limit of the observable universe, and losing an important amount of information. It appears that the biggest distinguishable 3-torus has an edge of size 1.15 D_{LSS} given by the temperature fluctuations contribution only. Similar results can be found in Ben-David *et al.* (2012) with other topologies. With the E-modes polarization contribution only, the biggest distinguishable torus is only of size 1.04 D_{LSS}. The CMB temperature fluctuations are the

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best to constrain big universes and CMB E-modes polarization fluctuations are very efficient to constrain universes smaller than the last scattering surface. There are similar results in Riazuelo *et al.* (2006), Bielewicz *et al.* (2011) where it was noticed that the circle-in-the-sky method (only adapted to universes smaller than D_{LSS}) is more efficient with polarization data. Nonetheless, it was only about the efficiency of the circle-in-the-sky method itself and not about the influence of the size of the space. Using of both temperature and polarization data will improve the detectability around 0.9 D_{LSS}.

Discussion. One could object that the B-modes were not taken into account in our study but in addition to our explanation in Section 1, we can cite Kunz *et al.* 2008: gravitational waves only add a noise-like like contribution and lower the detectability of topologies. Multi-connected topologies have not been detected yet in CMB data. On the one side, in the *Planck* paper on cosmology (*Planck* results XXVI 2013), with CMB temperature data only, no back-to-back pairs of circles of correlation was found. Nevertheless it was possible to constrain the lower spatial dimension to $L_{min} = 0.94\ D_{LSS}$ at the 99% C.L. On the other side, the likelihood maximization analysis performed was unable to make any decisive detection: there is only a faint hint of a 3-torus universe bigger than the observable universe, but without any statistical significance. That is why we can consider that the *Planck* E-modes data will only improve constraints on small universes, pushing the actual *Planck* constraint on the lower boundary of the size of the universe L_{min} =0.94 D_{LSS} closer to D_{LSS}. Furthermore, although the isotropy anomalies arising from the WMAP data are also found in the *Planck* data, they seem to be fainter.

4. Conclusion

We present a test of cosmology beyond the standard model, with an exploration of a possible violation of the isotropy of the CMB with the help of a multi-connected topology, the cubic 3-torus. The size L_* of the edge of the biggest distinguishable 3-torus was found to be equal to 1.15 D_{LSS}. The CMB temperature fluctuations are the best tool to study topologies bigger that the observable universe whereas E-modes are more efficient to study smaller universes. As a consequence the joined contribution of temperature and E-modes fluctuations will not be helpful to detect a multi-connected universe of size bigger than D_{LSS}, but only to better constrain small universes.

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