ON THE EXISTENCE OF POSITIVE SOLUTIONS OF $Au = \lambda Bu$

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(Received 5th September 1972)

1. Introduction

It is well known that sufficient conditions for the existence of a positive vector $\boldsymbol{u}$ which satisfies the matrix equation $A\boldsymbol{u} = \lambda\boldsymbol{u}$ are that A should be non-negative and irreducible. This result, the qualitative part of the Perron-Frobenius theorem, has been proved in a variety of ways, one of the most attractive of which is that given by Alexandroff and Hopf in their treatise " Topologie ". The aim of this note is to show how their method can be adapted to deal with the generalised eigenvalue problem defined by $A\boldsymbol{u} = \lambda B\boldsymbol{u}$ where A and B are square matrices.

The simplest non-trivial conditions for a positive solution of the more general problem are that A and B should each be non-negative and irreducible and that $AB = BA$. In this case we have for each matrix separately a positive eigenvector, and it is easily shown that if A and B commute then these eigenvectors are in fact precisely the same vector and consequently this vector provides a solution for the general problem.

(The conditions on A and B separately are not sufficient to guarantee a positive solution; for example if $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 0{\cdot}1 & 1 \end{bmatrix}$ then the eigenvectors are complex.)

In a recent note (**1**) Mangasarian has given sufficient conditions for the existence of a positive solution of the problem when A and B are not necessarily square. We confine our attention to the case of square matrices, and prove an analogue of Perron's theorem. We also indicate how the proof can be adapted to obtain an analogue of Frobenius' theorem when an appropriate definition of reducibility is made. These analogues are concerned only with the qualitative properties of the eigenvalue and eigenvector (see Section 5, Remark 4).

2. Preliminaries

A, B, ... will denote real square matrices of some fixed but arbitrary order $n \times n$. A vector $\boldsymbol{x}$ is said to be *positive*, $\boldsymbol{x} > \boldsymbol{0}$, if $x_i > 0$; $\boldsymbol{x}$ is *non-negative*, $\boldsymbol{x} \geqq \boldsymbol{0}$, if $x_i \geqq 0$. (Here, as elsewhere unless otherwise stated, the range of the suffixes is 1, 2, ..., n.) A matrix is positive if all its components are positive.

https://doi.org/10.1017/S0013091500010051 Published online by Cambridge University Press

We define $f(\boldsymbol{x}) = \Sigma x_p$; then $f(A\boldsymbol{x}) = \Sigma(A\boldsymbol{x})_p = \Sigma x_q \sigma_q(A)$ where

$$\sigma_q(A) = \sum_p a_{pq};$$

let $\sigma(A, B) = \max \sigma_q(A)/\sigma_q(B)$.

Definition. If $A\boldsymbol{u} = \lambda B\boldsymbol{u}$ then the eigenvalue λ is *defective* if there exists a vector $\boldsymbol{v}$ such that $A\boldsymbol{v} = \lambda B\boldsymbol{v} + B\boldsymbol{u}$. If no such vector $\boldsymbol{v}$ exists then λ is *not defective.*

3. Perron

We prove here an analogue of Perron's theorem for the eigenvalue problem defined by $A\boldsymbol{u} = \lambda B\boldsymbol{u}$.

Theorem 1. *If $\sigma_q(A)$, $\sigma_q(B)>0$ and $a_{ij}>b_{ij}\sigma(A, B)$ for $i \neq j$ then there is a positive eigenvalue λ to which corresponds a unique positive eigenvector $\boldsymbol{u}$ such that $A\boldsymbol{u} = \lambda B\boldsymbol{u}$.*

Moreover if $\sigma_q(A^T)$, $\sigma_q(B^T)>0$ and $a_{ij}>b_{ij}\sigma(A^T, B^T)$ then this eigenvalue is not defective.

Proof. Let $S = \{\boldsymbol{x} \in \mathbf{R}^n\colon\ \boldsymbol{x} \geqq 0, f(\boldsymbol{x}) = 1\}$, so that S is a closed bounded convex set in $\mathbf{R}^n$. Define $T\colon\ \mathbf{R}^n \to \mathbf{R}^n$ by

$$T\boldsymbol{x} = \boldsymbol{x} + \alpha[A\boldsymbol{x}f(B\boldsymbol{x}) - B\boldsymbol{x}f(A\boldsymbol{x})]$$

where α is a scalar which satisfies

$$0<\alpha<\min 1/|\, b_{ii}\sigma_q(A) - a_{ii}\sigma_q(B)\,|. \tag{1}$$

Clearly $f(\boldsymbol{x}) = 1 \Rightarrow f(T\boldsymbol{x}) = 1$. Moreover

$$(T\boldsymbol{x})_i = x_i + \alpha[(A\boldsymbol{x})_i f(B\boldsymbol{x}) - (B\boldsymbol{x})_i f(A\boldsymbol{x})]$$

$$= x_i \sum_q x_q \left[1 + \alpha \begin{vmatrix} a_{ii} & b_{ii} \\ \sigma_q(A) & \sigma_q(B) \end{vmatrix}\right] + \alpha \sum_{j \neq i} x_j \sum_q x_q \begin{vmatrix} a_{ij} & b_{ij} \\ \sigma_q(A) & \sigma_q(B) \end{vmatrix}. \tag{2}$$

For any positive α the second term is always non-negative if $\boldsymbol{x} \in S$, and if α satisfies (1) then the first term in (2) is also non-negative. Consequently $TS \subset S$, and since T is obviously continuous it follows from Brouwer's fixed point theorem that there exists a vector $\boldsymbol{u} \in S$ such that $\boldsymbol{u} = T\boldsymbol{u}$. Thus $\boldsymbol{u}$ satisfies $A\boldsymbol{u}f(B\boldsymbol{u}) = B\boldsymbol{u}f(A\boldsymbol{u})$. Since $\sigma_q(A)$, $\sigma_q(B)>0$ it follows that neither $f(A\boldsymbol{u})$ nor $f(B\boldsymbol{u})$ can vanish. Thus $\boldsymbol{u}$ is an eigenvector of the generalised problem; the corresponding eigenvalue is given by $f(A\boldsymbol{u})/f(B\boldsymbol{u})$. We now show that $\boldsymbol{u}>\boldsymbol{0}$. Suppose otherwise; then for at least one index k we would have $u_k = 0$ whilst the remaining components of $\boldsymbol{u}$ are non-negative and not all zero. It would follow from (2) with $\boldsymbol{x} = \boldsymbol{u}$ that

$$\sum_{j \neq k} u_j \sum_q u_q \begin{vmatrix} a_{kj} & b_{kj} \\ \sigma_q(A) & \sigma_q(B) \end{vmatrix} = 0, \tag{3}$$

which, since each of the determinants is positive and $\boldsymbol{u} \geqq \mathbf{0}$, $\boldsymbol{u} \neq \mathbf{0}$, cannot hold. Thus we conclude that $\boldsymbol{u} > \mathbf{0}$.

In a similar fashion we can show that $\boldsymbol{u}$ is the only eigenvector which corresponds to λ, for if there were another vector $\boldsymbol{v}$ then $\boldsymbol{z} = t\boldsymbol{u} - \boldsymbol{v}$ would also be an eigenvector for any scalar t. Choose $t = \max v_i/u_i$, then $\boldsymbol{z} \geqq \mathbf{0}$ and at least one component of $\boldsymbol{z}$ is zero. It follows as in the proof of the positivity of $\boldsymbol{u}$ that $\boldsymbol{z} = \mathbf{0}$. Consequently $\boldsymbol{u}$ is unique.

To prove that λ is not defective we use the last assumption stated in the theorem. This clearly implies that there is a unique positive vector $\boldsymbol{w}$ which satisfies $\boldsymbol{w}^{\mathrm{T}}A = \mu \boldsymbol{w}^{\mathrm{T}}B$. Furthermore $\boldsymbol{w}^{\mathrm{T}}A\boldsymbol{u} = \lambda \boldsymbol{w}^{\mathrm{T}}B\boldsymbol{u} = \mu \boldsymbol{w}^{\mathrm{T}}B\boldsymbol{u}$ and so $(\lambda - \mu)\boldsymbol{w}^{\mathrm{T}}B\boldsymbol{u} = 0$. However

$$\begin{aligned} \boldsymbol{w}^{\mathrm{T}}B\boldsymbol{u} &= \sum_p w_p \sum_q b_{pq}u_q \\ &= \sum_q u_q \sum_p b_{pq}w_p \\ &\geqq \sum_q u_q \sigma_q(B) \,.\, \min(w_p), \end{aligned}$$

and so $w^{\mathrm{T}}Bu \neq 0$, which implies that $\lambda = \mu$.

Suppose now that λ is defective, then there will exist $\boldsymbol{v}$ such that

$$A\boldsymbol{v} = \lambda B\boldsymbol{v} + B\boldsymbol{u},$$

and so $\boldsymbol{w}^{\mathrm{T}}A\boldsymbol{v} = \lambda \boldsymbol{w}^{\mathrm{T}}B\boldsymbol{v} + \boldsymbol{w}^{\mathrm{T}}B\boldsymbol{u}$. But $\boldsymbol{w}^{\mathrm{T}}A\boldsymbol{v} = \lambda \boldsymbol{w}^{\mathrm{T}}B\boldsymbol{v}$ hence $\boldsymbol{w}^{\mathrm{T}}B\boldsymbol{u} = 0$ which has been seen to be untrue. This contradiction shows that the hypothesis that λ is defective is false. In conclusion we note that

$$\lambda = f(A\boldsymbol{u})/f(B\boldsymbol{u}) = \Sigma\sigma_q(A)u_q/\Sigma\sigma_q(B)u_q,$$

and so $\sigma(B, A) \leqq \lambda \leqq \sigma(A, B)$, thus proving that λ is positive.

4. Frobenius

The extension by Frobenius of the usual form of Perron's theorem is made by the weakening of the condition of the positivity of the matrix to the requirement that it should be non-negative and irreducible.

Definition. The matrices A and B are *mutually reducible* (f) if there is a permutation matrix Π such that is $C = \Pi A \Pi^{\mathrm{T}}$ and $D = \Pi B \Pi^{\mathrm{T}}$ then an index r exists so that

$$\begin{vmatrix} c_{ij} & d_{ij} \\ \sigma_q(C) & \sigma_q(D) \end{vmatrix} = 0 \text{ for } 1 \leqq i \leqq r, \quad r+1 \leqq j \leqq n.$$

If no such matrix Π exists then A and B are *mutually irreducible* (f).

Alternatively A and B are mutually reducible (f) if there exist two disjoint sets of integers I and J where $I \cup J = \{1, 2, \ldots, n\}$ such that

$$\begin{vmatrix} a_{ij} & b_{ij} \\ \sigma_q(A) & \sigma_q(B) \end{vmatrix} = 0 \text{ for } i \in I \text{ and } j \in J.$$

When B is the unit matrix this becomes the usual definition of reducibility.

Theorem 2. *Let $\sigma_q(A)$, $\sigma_q(B)>0$, $a_{ij} \geqq b_{ij}\sigma(A, B)$ for $i \neq j$ and let A and B be mutually irreducible (f). Then there exists a positive eigenvalue λ to which corresponds a unique positive eigenvector $\boldsymbol{u}$ such that $A\boldsymbol{u} = \lambda B\boldsymbol{u}$.*

If $\sigma_q(A^T)$, $\sigma_q(B^T)>0$, $a_{ij} \geqq b_{ij}\sigma(A^T, B^T)$ and A^T and B^T are mutually irreducible (f) then the eigenvalue λ is not defective.

Proof. The proof of the existence of a non-negative eigenvector $\boldsymbol{u}$ which satisfies $A\boldsymbol{u} = \lambda B\boldsymbol{u}$ follows precisely the same lines as that of the corresponding one in the previous theorem. In order to show that this vector is in fact positive we again suppose the contrary. Let Π be a permutation matrix such that if $\boldsymbol{v} = \Pi\boldsymbol{u}$ then $v_1 = v_2 = \ldots = v_r = 0$ whilst the remaining components of $\boldsymbol{v}$ are strictly positive. Let $C = \Pi A \Pi^T$ and $D = \Pi B \Pi^T$ then

$$C\boldsymbol{v} = \lambda D\boldsymbol{v},$$

which can be written

$$C\boldsymbol{v}f(D\boldsymbol{v}) = D\boldsymbol{v}f(C\boldsymbol{v}).$$

It follows that

$$\sum_{j=r+1}^{n} c_{ij}v_j f(D\boldsymbol{v}) = \sum_{j=r+1}^{n} d_{ij}v_j f(C\boldsymbol{v}), \quad i = 1, 2, \ldots, r.$$

This equation can also be written in the form

$$\sum_{j=r+1}^{n} v_j \sum_{q=r+1}^{n} v_q \begin{vmatrix} c_{ij} & d_{ij} \\ \sigma_q(C) & \sigma_q(D) \end{vmatrix} = 0, \quad i = 1, 2, \ldots, r. \tag{4}$$

However $v_{r+1}, v_{r+2}, \ldots, v_n$ are strictly positive, and since A and B are mutually irreducible (f) the determinants in (4) are strictly positive. Hence (4) cannot hold and so the supposition that $\boldsymbol{u}$ has zero components is false.

The proofs of the remaining propositions in Theorem 2 follow in a similar fashion to those of Theorem 1.

5. Remarks

We shall confine the remarks to the analogue of Perron's theorem.

1. When $B = I$ the conditions become $\sigma_q(A)>0$, $a_{ij}>0$, $i \neq j$. These apparently extend the usual condition for the truth of Perron's theorem. However, on closer examination this will be found to be false since we can add a multiple of the identity matrix to any matrix with positive off diagonal elements to obtain a matrix with all its elements positive which has the same eigenvectors as the original matrix.

2. The choice of f was made for simplicity of presentation; in fact any positive linear functional on $\mathbf{R}^n$ can be used so long as the ancilliary conditions on A and B are satisfied with that functional.

3. The roles of A and B can be reversed to give similar results for the eigenvalue problem $\lambda A\boldsymbol{u} = B\boldsymbol{u}$.

4. The Perron-Frobenius theorem contains the quantitative statement that the positive eigenvalue is equal to the spectral radius. The following example

shows that such a result does not necessarily hold with the type of conditions which were considered here. If

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then the eigensystem is -3, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and 1, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Acknowledgement

I am grateful to Professor C. W. Clenshaw of the University of Lancaster for his useful comments.

REFERENCE

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https://doi.org/10.1017/S0013091500010051 Published online by Cambridge University Press