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ABELIAN THEOREMS FOR HARDY TRANSFORMATIONS

BY

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ABSTRACT. Initial and final value theorems for Hardy transformations $\int_0^\infty f(x)C_\nu(xy)\,dx$ and $\int_0^\infty f(x)F_\nu(xy)\,dx$ of a suitably chosen function $f(x)$ under a certain set of conditions on ν and p where

(1) $$C_\nu(x)=\cos p\pi\, J\nu(x)+\sin p\pi\, Y_\nu(x)$$

$J_\nu(x)$ and $Y_\nu(x)$ being Bessel functions of the first and second kind, and

(2) $$F_\nu(x)=2^{2-\nu-2p}s_{\nu+2p-1,\nu}(x)/\{\Gamma(p)\Gamma(\nu+p)\}$$

$s_{u,\nu}(x)$ being Lommel's function, are proved.

1. Introduction. Applications of Abelian theorems in solving boundary value problems are well known. Abelian theorems for Laplace transforms are given by Widder [8] and that for the Hankel transform are given by Zemanian [9]. Abelian theorems for Y- and H-transforms are not available in Literature. In the following we give Abelian theorems for the Hardy transforms which incorporate Y- and H- transforms as special cases.

THEOREM 1. *Let* $3/2<\eta<2-|\text{Re}^\nu|$ *where* ν *is complex. Let* $f(x)$ *be a measurable function on the interval* $(0,\infty)$ *such that* $x^\eta f(x)$ *is Lebesgue integrable on every interval of the form* (x,∞), $X>0$.

Assume that

(3) $$\lim_{x\to 0+} x^\eta f(x)=\lambda,$$

where λ *is complex in general and define the Hardy transformation* $F(y)$ *of* $f(x)$ *by*

(4) $$F(y)=\int_0^\infty f(x)\times C_\nu(xy)\,dx$$

where $C_\nu(x)$ *is the function defined by* (1).

Then

(5) $$\lim_{y\to\infty} y^{2-\eta}F(y)=\lambda G(\nu,\eta)$$

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where

(6) $$G(\nu,\eta)=\frac{\cos(p\pi)\Gamma\left(\frac{2+\nu-\eta}{2}\right)}{2^{\eta-1}\Gamma\left(\frac{\nu+\eta}{2}\right)}\quad\frac{\sin p\pi\cot\left[\frac{(\nu+\eta)\pi}{2}\right]\Gamma\left(1+\frac{\nu-\eta}{2}\right)}{2^{\eta-1}\Gamma\left(\frac{\nu+\eta}{2}\right)}$$

p being a complex number.

Proof. From (3), $f(x)=0[x^{-\eta}]$, $x\to 0+$. Since,

$$C_\nu(x)=\begin{matrix}0\,[x^{-|\nu|}], & x\to 0+\\ 0\,[x^{-1/2}], & x\to\infty\end{matrix}$$

the Hardy transform $F(y)$ of $f(x)$ exists.

Now form [2; pp. 326, 329]

(7) $$\int_0^\infty t^{1-\eta}C_\nu(t)\,dt=G(\nu,\eta);\qquad 1<\eta<2-|\mathrm{Re}^\nu|.$$

using the transformation $t=xy$, $y>o$ in (6) we have

(8) $$\int_0^\infty y^{2-\eta}x^{1-\eta}C_\nu(xy)\,dx=G(\nu,\eta)$$

Therefore, in view of (4) and (7) we have

(9) $$\begin{aligned}|y^{2-\eta}F(y)-\lambda G(\nu,\eta)|&\le y\int_0^\infty |x^\eta f(x)-\lambda||(xy)^{1-\eta}C_\nu(xy)|\,dx\\ &=y\int_0^\delta |x^\eta f(x)-\lambda||(xy)^{1-\eta}C_\nu(xy)|\,dx\\ &\quad+y\int_\delta^\infty |x^\eta f(x)-\lambda||(xy)^{1-\eta}C_\nu(xy)|\,dx,\ \delta>0\\ &\leqq \sup_{0<t\le\delta}|t^\eta f(t)-\lambda|\int_0^\delta |x^{1-\eta}C_\nu(x)|\,dx\\ &\quad+y^{3/2-\eta}\int_\delta^\infty |x^{1/2}f(x)-\lambda x^{-\eta+1/2}||\sqrt{(xy)}C_\nu(xy)|\,dx\end{aligned}$$

Since $\int_0^\infty |x^{1-\eta}C_\nu(x)|\,dx$ is convergent in view of (3) for $\varepsilon>0$ we can choose a positive δ such that

$$|t^\eta f(t)-\lambda|<\frac{\varepsilon}{2\int_0^\infty |x^{1-\eta}C_\nu(x)|\,dx}$$

Fix δ this way. Therefore in view of (8)

(10) $$|y^{2-\eta}F(y)-\lambda G(\nu,\eta)|<\frac{\varepsilon}{2}+y^{(3/2)-\eta}\int_0^\infty |x^{1/2}f(x)-\lambda x^{-\eta+1/2}||\sqrt{xy}C_\nu(xy)|\,dx.$$

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Let

$$\sup_{x>0} |\sqrt{x} C_\nu(x)| = K$$

and

$$\int_0^\infty |x^{1/2} f(x) - \lambda x^{-\eta+1/2}| \, dx = C.$$

Then, from (9)

$$|y^{2-\eta} F(y) - \lambda G(\nu, \eta)| < \frac{\varepsilon}{2} + y^{3/2-\eta} KC. \tag{11}$$

We can now choose $N>0$ sufficiently large such that

$$y^{3/2-\eta} KC < \frac{\varepsilon}{2} \quad \text{for all} \quad y > N. \tag{12}$$

Now from (10) and (11) we have

$$|y^{2-\eta} F(y) - \lambda G(\nu, \eta)| < \varepsilon \quad \text{for all} \quad y > N.$$

Since ε is arbitrary our result is proved.

THEOREM 2. *Let ν and p be complex numbers and η a real number satisfying $3/2 < \eta < 2 - |\text{Re}^\nu|$. Assume that $f(x)$ is a measurable function in $(0, \infty)$ such that $x^{1-|\text{Re}^\nu|} f(x)$ is Lebesgue integrable on every interval of the form $0 < x < X$ $(X < \infty)$ and that there exists a complex number λ such that*

$$\lim_{x\to\infty} x^\eta f(x) = \lambda. \tag{13}$$

Then with $F(y)$ and $G(\nu, \eta)$ as defined by (4) and (6) respectively

$$\lim_{y\to 0+} y^{2-\eta} F(y) = \lambda G(\nu, \eta).$$

Proof. Since $C_\nu(x) = 0(x^{-|\nu|})$ as $x \to 0+$ and $C_\nu(x) = 0(x^{-1/2})$ as $x \to \infty$ our conditions on $f(x)$ insure that the transform $F(y)$ of $f(x)$ as defined by (4) exists for $y > 0$.

Now,

$$|y^{2-\eta} F(y) - \lambda G(\nu, \eta)| \le y \int_0^\infty |x^\eta f(x) - \lambda| |(xy)^{1-\eta} C_\nu(xy)| \, dx$$

$$\le y \int_0^X |x^\eta f(x) - \lambda| \, |(xy)^{1-\eta} C_\nu(xy)| \, dx$$

$$+ y \int_X^\infty |x^\eta f(x) - \lambda| \, |(xy)^{1-\eta} C_\nu(xy)| \, dx.$$

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Exploiting (13) for $\varepsilon>0$ we can find $X>0$ such that

$$|f(x)x^{\eta}-\lambda|<\frac{\varepsilon}{\int_0^{\infty} t^{1-\eta}\,|C_\nu(t)|\,dt} \quad \text{for all} \quad x>X.$$

Therefore,

$$|y^{2-\eta}F(y)-\lambda G(\nu,\eta)|<y\int_0^X |x^{\eta}f(x)-\lambda|(xy)^{1-\eta}C_\nu(xy)\,dx+\varepsilon$$

or

$$(14)\quad |y^{2-\eta}F(y)-\lambda G(\nu,\eta)|<\varepsilon+y^{3/2-\eta}\int_0^X |f(x)-\lambda x^{-\eta}|\,|C_\nu(xy)\sqrt{xy}\sqrt{x}\,dx.$$

There exists a positive number A_ν such that

$$(15)\quad |\sqrt{t}\,C_\nu(t)|\le A_\nu t^{-|\mathrm{Re}\,\nu|+1/2} \quad \text{for all} \quad t>0.$$

Therefore exploiting (14) and (15) we have

$$\begin{aligned}|y^{2-\eta}F(y)-\lambda G(\nu,\eta)|&<\varepsilon+A_\nu y^{2-\eta-|\mathrm{Re}\,\nu|}\int_0^X |f(x)-\lambda x^{-\eta}|\,x^{-|\mathrm{Re}\,\nu|+1}\,dx\\ &<\varepsilon+A_\nu y^{2-\eta-|\mathrm{Re}\,\nu|}\int_0^X |f(x)-\lambda x^{-\eta}|\,x^{-|\mathrm{Re}\,\nu|+1}\,dx.\end{aligned}$$

Letting $y\to 0+$ we have

$$\overline{\lim_{y\to 0+}}\,|y^{2-\eta}F(y)-\lambda G(\nu,\eta)|\le\varepsilon.$$

Since ε is arbitrary our lemma is proved.

THEOREM 3. *Let $\sigma+2<\eta<\mathrm{Re}(\nu+2p+2)$ where $\sigma=\max(-\frac{1}{2},\mathrm{Re}(\nu+2p-2))$ ν and p being complex numbers. Let $f(x)$ be a measurable function on $0<x<\infty$ such that $x^{\sigma+1}f(x)$ is Lebesgue integrable on every interval of the form $X<x<\infty$ $(X>0)$ and that there exists a complex number such that $\lim_{x\to 0+}x^{\eta}f(x)=\lambda$.*

Let the F_ν-transform of $f(x)$ be defined by

$$(16)\quad F(y)=\int_0^{\infty} f(x)F_\nu(xy)x\,dx \quad \text{for each} \quad y>0,$$

where

$$F_\nu(x)=\frac{2^{2-\nu-2p}s_{\nu+2p-1,\nu}(x)}{\{\Gamma(p)\Gamma(\nu+p)\}}$$

$s_{\mu,\nu}(x)$ being Lommel's function. Then

$$\lim_{y\to\infty} y^{2-\eta}F(y)=\lambda H(\nu,\eta)$$

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where

$$(17)\quad H(\nu,\eta)=\frac{\Gamma\left(\frac{\eta-\nu-2p}{2}\right)\Gamma\left(1-\frac{\eta-\nu-2p}{2}\right)}{2^{\eta-1}\Gamma\left(\frac{\eta-\nu}{2}\right)\Gamma\left(\frac{\eta+\nu}{2}\right)},\quad \frac{1}{2}<\eta<2+\mathrm{Re}(\nu+2p).$$

Proof. The proof follows quite readily by using the fact that

$$\int_0^\infty t^{1-\eta}F_\nu(t)\,dt=H(\nu,\eta);\quad \eta<2+\mathrm{Re}(\nu+2p),$$

[3, p. 385]

and the technique used in the proof of Theorem 1.

THEOREM 4. *Let* $\sigma+2<\eta<\mathrm{Re}(\nu+2p)+2$ *where* $\sigma=\max(-\frac{1}{2},-\mathrm{Re}(\nu+2p+2))$ ν *and* p *being complex constant. Let* $f(x)$ *be a measurable function on* $0<x<\infty$ *such that* $x^{\nu+2p+1}f(x)$ *is Lebesgue integrable over any interval of the form* $0<x<X$ $(X<\infty)$ *and that* $\lim_{x\to\infty}x^\eta f(x)=\lambda$, λ *being a complex number in general.*

Then with $F(y)$ *and* $H(\nu,\eta)$ *as defined by* (16) *and* (17) *respectively,*

$$\lim_{y\to 0+} y^{2-\eta}F(y)=\lambda\,H(\nu,\eta).$$

Proof. The proof follows quite readily by using the fact that

$$\int_0^\infty t^{1-\eta}F_\nu(t)\,dt=H(\nu,\eta),\ \eta<2+\mathrm{Re}(\nu+2p),$$

and the technique used in the proof of Theorem 2.

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