HYDRODYNAMICS OF RELATIVISTIC PULSAR WIND

J. G. LOMINADZE
Abastumani Astrophysical Observatory

J. I. JAVAKHISHVILI
Tbilisi State University

E. G. TSIKARISHVILI
Abastumani Astrophysical Observatory

Pulsar pair-production theories imply that the plasma leaves the source already flowing relativistically. Such winds might also have relativistic injection temperatures. MHD models of relativistic pulsar winds were considered, and it was shown that in this case cold wind theories can be misleading.

On the other hand, due to synchrotron radiation losses, the temperature of collisionless, strongly magnetized, relativistic plasma quickly becomes anisotropic. In this case the plasma pressure is no longer a scalar and to describe the properties of stellar wind a hydrodynamical model of relativistic plasma with the anisotropic temperature distribution should be developed.

In this paper we get a closed set of relativistic hydrodynamical equations, which describes relativistic strongly magnetized, collisionless plasma with an anisotropic pressure tensor. Such a model was suggested by Chew, Goldberger and Low (1954) for a nonrelativistic plasma.

The microscopic state of the relativistic collisionless plasma can be described by means of the plasma particle's relativistic distribution function $\Phi_a(x_a^i, p_a^i)$, which satisfies the kinetic equations

$$[H\Phi_a] = \frac{\partial H}{\partial p_{ai}}\frac{\partial \Phi_a}{\partial x_a^i} - \frac{\partial H}{\partial x_{ai}}\frac{\partial \Phi_a}{\partial p_a^i} = 0 \tag{1}$$

here H is the Hamiltonian of the particle system, a denotes the particle species, $x_a^i = (ct, \boldsymbol{r}_a)$ denotes a 4-space vector, $p_a^i = (\varepsilon_a/c, \boldsymbol{p}_a)$ denotes the particle 4-momentum ($\varepsilon_a = c\sqrt{p_a^2 + m_a^2c^2}$), $i = 0,1,2,3$.

Of course, the following system of Hamilton equations is valid

$$\frac{\partial H}{\partial p_{ai}} = \frac{dx_a^i}{ds_a} = \frac{p_a^i}{m_a c}, \qquad \frac{\partial H}{\partial x_{ai}} = -\frac{dp_a^i}{ds_a} \tag{2}$$

where $ds_a = \sqrt{dx_a^i\, dx_{ai}} = c\gamma_a^{-1}dt$, and $\gamma_a = 1/\sqrt{1-u_a^2/c^2}$ is the relativistic factor.

Let us consider the plasma in the presence of electric and magnetic fields. The particle equation of motion is as follows (Landau and Lifshitz 1967)

$$\frac{dp_a^i}{ds_a} = \frac{e_a}{c}F^{ij}\frac{p_{aj}}{m_a c} \tag{3}$$

Using equations (2) and (3) one can reduce eq.(1) to the following form

$$p_a^i\frac{\partial \Phi_a}{\partial x_a^i} + \frac{e_a}{c}F^{ij}p_{aj}\frac{\partial \Phi_a}{\partial p_{ai}} = 0 \tag{4}$$

It is well known (Baranov and Krasnobaev 1986), that from the kinetic equations one can obtain the system of transport equations for the macroscopic plasma parameters (*e.g.* for the particle density n_a, hydrodynamical velocity $\boldsymbol{u}_a$, and for the mean thermal energy $m_a c^2 W_a$—taken in the particle rest frame). Hence for each plasma component the following definitions are introduced (Landau and Lifshitz 1986)

$$I_a^i = \int p_a'^i \Phi_a'\, d\Omega_a' = cn_a g^{i0}, \quad W_a = \frac{1}{n_a m_a c^2}\int (p_a'^0)^2 \Phi_a'\, d\Omega_a' - 1 \tag{5}$$

where prime means that all quantities under the integral are defined in the rest frame of the given plasma component, $d\Omega_a = d(\varepsilon_a/c)d^3p_a$ is the relativistic invariant, and g^{ij} is the ordinary fundamental tensor in the noncurved space

$$g^{ij} = g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -\delta^{\alpha\beta} \end{pmatrix} \tag{6}$$

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In the nonrelativistic limit one can obtain for an isotropic plasma that $W_a \to \frac{3}{2}T_a/(m_a c^2)$, where T_a is the temperature of the given plasma component in its proper frame. However, in relativistic case the problem of temperature definition is complicated (Javakhishvili and Tsintsadze 1973). It will be much more reasonable to consider W_a as one of the macroscopic parameters, describing together with parameters n_a and $\boldsymbol{u}_a$ the plasma hydrodynamical motion. The transition to the laboratory frame may be made by means of the Lorentz transformation for p_a^i: $p_a'^i = L_a^{ij} p_{aj}$, where the Lorentz transformation matrix has the form (Landau and Lifshitz 1967)

$$L_a^{ij} = \begin{pmatrix} \gamma_a & \gamma_a \boldsymbol{u}_a/c \\ -\gamma_a \boldsymbol{u}_a/c & -s_a^{\alpha\beta} \end{pmatrix} \tag{7}$$

In eq.(7) $s^{\alpha\beta}$ is obtained in 3-space as

$$s_a^{\alpha\beta} = \delta^{\alpha\beta} + (\gamma_a - 1)\frac{u^\alpha u^\beta}{u_a^2}, \quad \alpha = 1,2,3$$

Multiplying eq.(4) by 1 and p_a^i and integrating over $d\Omega_a$, using equations (5) and (7) the continuity and energy-momentum equations may be obtained

$$\frac{\partial J_a^i}{\partial x_a^i} = 0, \quad \frac{\partial T_a^{ij}}{\partial x_a^j} = \frac{e_a}{c} F^{ij} J_{aj}, \quad T_a^{ij} = L_a^{mi} L_a^{nj} \pi_{amn} \tag{8}$$

where $J_a^i = L_a^{ij} I_{aj}$; $J_a^i = (c\gamma_a n_a, n_a \gamma_a \boldsymbol{u}_a)$ is the current 4-vector, and T_a^{ij} is the energy-momentum tensor. The symmetrical tensor π_a^{ij} is defined as

$$\pi_a^{ij} = \int p'^i p'^j \Phi_a' \, d\Omega_a', \tag{9}$$

Obviously $\pi_a^{00} = m_a c^2 n_a (W_a + 1)$.

The system (8) is not closed, because eq.(9) introduces some new undefined macroscopic values $\pi_a^{0\alpha}$ and $\pi_a^{\alpha\beta}$. To close the system one should connect $\pi_a^{0\alpha}$ and $\pi_a^{\alpha\beta}$ with the macroscopic parameters $n_a, \boldsymbol{u}_a, W_a$ and its derivatives. It should be done in the case when all quantities characterizing the plasma are slowly changing at distances of the order of the mean free path. It is shown below that this problem can be solved in case when the collisionless plasma is strongly magnetized, which leads to the anisotropy of thermal parameters describing plasma.

As is clear from (8), it is necessary to redefine $\pi_a^{0\alpha}$ and $\pi_a^{\alpha\beta}$. It can be obtained *e.g.* by the construction of the additional transport equation for π_a^{ij}. This leads to the necessity of considering the equations for the third-order moments of the kinetic equation. For this purpose eq.(4) should be multiplied by $P_a^i P_a^j$ and integrated over $d\Omega_a$. This leads to

$$\frac{\partial M_a^{ijk}}{\partial x_a^k} = \frac{e_a}{c}\left(F^{im} T_{am}^j - T_a^{im} F_m^j\right), \tag{10}$$

$$\text{where} \quad M_a^{ijk} = L^{ei} L^{mj} L^{nk} N_{aemn} \tag{11}$$

and N_a^{ijk} is defined as

$$N_a^{ijk} = \int p_a'^i p_a'^j p_a'^k \Phi_a' \, d\Omega_a'. \tag{12}$$

The N_a^{ijk} and π_a^{ij} tensors can be defined as

$$\pi_a^{ij} = \begin{pmatrix} m_a c^2 n_a (W_a + 1), & \boldsymbol{q}_a/c \\ \boldsymbol{q}_a/c, & \pi_a^{\alpha\beta} \end{pmatrix}, \quad N_a^{ij0} = m_a c \begin{pmatrix} m_a c^2 n_a V_a, & 2\boldsymbol{q}_a/c \\ 2\boldsymbol{q}_a/c, & \mu_a^{\alpha\beta} \end{pmatrix}$$

$$N_a^{\alpha\beta\gamma} = m_a \eta_a^{\alpha\beta\gamma} \tag{13}$$

where it is convenient to define $n_a, W_a, V_a, \pi_a^{\alpha\beta}, \mu_a^{\alpha\beta}, \boldsymbol{q}_a, \boldsymbol{g}_a$ and $\eta_a^{\alpha\beta\gamma}$ using three dimensional distribution functions $f_a(\boldsymbol{r}_a, \boldsymbol{p}_a, t)$ in the following way (Javakhishvili and Tsintsadze 1973)

$$n_a = \int f_a' \, d\boldsymbol{p}_a', \quad W_a = \frac{1}{m_a c^2 n_a}\int \varepsilon_a' f_a' \, d\boldsymbol{p}' - 1, \quad V_a = \frac{1}{m_a^2 c^4 n_a}\int \varepsilon_a'^2 f_a' \, d\boldsymbol{p}_a',$$

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$$\pi_a^{\alpha\beta} = c^2 \int \frac{p_a'^\alpha p_a'^\beta}{\varepsilon_a'} f_a' \, d\boldsymbol{p}_a', \quad \mu_a^{\alpha\beta} = \frac{1}{m_a} \int p_a'^\alpha p_a'^\beta f_a' \, d\boldsymbol{p}_a', \quad \boldsymbol{q}_a = c^2 \int \boldsymbol{p}_a' f_a' \, d\boldsymbol{p}_a'$$

$$\boldsymbol{g}_a = \frac{1}{2m_a} \int \varepsilon_a' \boldsymbol{p}_a' f_a' \, d\boldsymbol{p}_a', \quad c^2 \int \frac{\boldsymbol{p}_a'}{\varepsilon_a'} f_a' \, d\boldsymbol{p}_a' = 0, \quad \eta_a^{\alpha\beta\gamma} = \frac{c^2}{m_a} \int \frac{p_a'^\alpha p_a'^\beta p_a'^\gamma}{\varepsilon_a'} f_a' \, d\boldsymbol{p}' \tag{14}$$

Here all integrals are taken in the plasma rest frame; $\pi_a^{\alpha\beta}$ is the viscous stress tensor and $\boldsymbol{q}_a$ is the heat flux density (Javakhishvili and Tsintsadze 1973). It can be seen that in the nonrelativistic limit ($\varepsilon' = mc^2$) $\mu_a^{\alpha\beta} \to \pi_a^{\alpha\beta}$ and the vector $\boldsymbol{g}_a$ reduces to the $\boldsymbol{q}_a$. Therefore the tensor $\mu_a^{\alpha\beta}$ can be called the modified stress tensor and $\boldsymbol{g}$ is the modified heat flux vector. Note that $Sp(\mu_a^{\alpha\beta}) = \mu_a^{\alpha\alpha} = n_a m_a c^2 (V_a - 1)$. To close the set of equations one should *e.g.* neglect the third order moment $\eta_a^{\alpha\beta\gamma}$.

It is obvious that the kinetic equation for the function f_a should be solved. This equation in the relativistic theory has the form (Belyaev and Budker 1956)

$$\frac{\partial f_a}{\partial t} + \frac{c^2}{\varepsilon_a} \boldsymbol{p}_a \frac{\partial f_a}{\partial \boldsymbol{r}_a} + e_a \left\{ \boldsymbol{E} + \frac{c}{\varepsilon_a} [\boldsymbol{p}_a \boldsymbol{B}] \right\} \frac{\partial f_a}{\partial \boldsymbol{p}_a} = 0 \tag{15}$$

where f_a is the relativistic invariant (Belyaev and Budker 1956) $d\boldsymbol{p}_a/\varepsilon_a$. One should transform eq.(15) to the plasma rest frame. If $R_a/L \ll 1$, where R_a is the Larmor radius of the particles, and L is the characteristic spatial scale of the problem, one can look for the solution of eq.(15) in a form of power series over a small parameter R_a/L. Consequently the kinetic equation for f_a' reduces to the infinite set of equations

$$\left\{ \boldsymbol{E}' + \frac{c}{\varepsilon_a'} [\boldsymbol{p}_a' \boldsymbol{B}'] \right\} \frac{\partial f_a'^{(0)}}{\partial \boldsymbol{p}_a'} = 0$$

$$\frac{\partial f_a'^{(0)}}{\partial t} + \frac{c^2}{\varepsilon'} \boldsymbol{p}_a' \frac{\partial f_a'^{(0)}}{\partial \boldsymbol{r}_a} + e_a \left\{ \boldsymbol{E}' + \frac{c^2}{\varepsilon_a'} [\boldsymbol{p}_a' \boldsymbol{B}'] \right\} \frac{\partial f_a'^{(1)}}{\partial \boldsymbol{p}_a'} = 0 \tag{16}$$

If one limits by finding only the functions $f_a'^{(0)}$, then the solution can be obtained easily by assuming $\boldsymbol{E}' = 0$.

$$f_a'^{(0)} = f_a'^{(0)}(t, r_a, p_{a\perp}'^2, \boldsymbol{p}_a' \cdot \boldsymbol{B}'), \tag{17}$$

where $p_{a\perp}'^2$ is transverse with respect to the $\boldsymbol{B}'$ component of the particle momentum.

Using the distribution function eq.(17) it can be shown from eq.(14) that $\boldsymbol{q}_a = \boldsymbol{g}_a = 0$ and

$$\pi_a^{\alpha\beta} = \mathbf{P}_{a\parallel} b'^\alpha b'^\beta + \mathbf{P}_{a\perp} (\delta^{\alpha\beta} - b'^\alpha b'^\beta), \quad \mu_a^{\alpha\beta} = \mu_{a\parallel} b'^\alpha b'^\beta + \mu_{a\perp} (\delta^{\alpha\beta} - b'^\alpha b'^\beta) \tag{18}$$

where $\mathbf{P}_{a\parallel}$, $\mu_{a\parallel}$ and $\mathbf{P}_{a\perp}$, $\mu_{a\perp}$ are the parallel and transverse components of the pressure and modified pressure, respectively, which are defined as

$$\mathbf{P}_{a\parallel} = c^2 \int \frac{p_{a\parallel}'^2}{\varepsilon_a'} f_a'^{(0)} \, d^3 p_a', \quad \mathbf{P}_{a\perp} = c^2 \int \frac{p_{a\perp}'^2}{2\varepsilon_a'} f_a'^{(0)} \, d^3 p_a' \tag{19}$$

$$\mu_{a\parallel} = \frac{1}{m_a} \int p_{a\parallel}'^2 f_a'^{(0)} \, d^3 p_a', \quad \mu_{a\perp} = \frac{1}{2m_a} \int p_{a\perp}'^2 f_a'^{(0)} \, d^3 p_a'$$

In the case considered ($\boldsymbol{q} = 0$, $\boldsymbol{g} = 0$) and from eq.(18)) one can obtain

$$\pi_a^{ij} = \left[m_a c^2 n_a (W_a + 1) + \mathbf{P}_{a\perp} \right] g^{i0} g^{j0} - \mathbf{P}_{a\perp} g^{ij} + (\mathbf{P}_{a\parallel} - \mathbf{P}_{a\perp}) g^{i\alpha} g^{j\beta} b_\alpha' b_\beta',$$

$$\frac{N^{ij0}}{m_a c} = (m_a c^2 n_a V_a + \mu_{a\perp}) g^{i0} g^{j0} - \mu_{a\perp} gij + (\mu_{a\parallel} - \mu_{a\perp}) g^{i\alpha} g^{j\beta} b_\alpha' b_\beta', \tag{20}$$

$$N_a^{\alpha\beta\gamma} = 0$$

Using the definitions (8), (11), and (20) and the Lorentz transformation matrix one can obtain

$$T_a^{ij} = \left[m_a c^2 n_a (W_a + 1) + \mathbf{P}_{a\perp} \right] u_a^i u_a^j - \mathbf{P}_{a\perp} (g^{ij} + \Lambda_a^{\alpha i} \Lambda_a^{\beta j} b_\alpha b_\beta) + \mathbf{P}_{a\parallel} \Lambda_a^{\alpha i} \Lambda_a^{\beta j} b_\alpha b_\beta,$$

$$\frac{M_a^{ijk}}{m_a c} = (m_a c^2 n_a V_a + 3\mu_{a\perp}) u_a^i u_a^j u_a^k +$$

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$$+(\mu_{a\parallel}-\mu_{a\perp})[u_a^i\Lambda_a^{\alpha j}\Lambda_a^{\beta k}+u_a^j\Lambda_a^{\alpha i}\Lambda_a^{\beta k}+u_a^k\Lambda_a^{\alpha i}\Lambda_a^{\beta j}]-\mu_{a\perp}[u_a^i g^{jk}+u_a^j g^{ik}+u_a^k g^{ij}] \quad (21)$$

where $u_a^i(\gamma_a, \gamma_a \boldsymbol{u}/c)$ is the 4-velocity of hydrodynamical motion and

$$\Lambda_a^{ij} \equiv \frac{g^{ij}-u_a^i u_a^j}{\sqrt{1+\gamma_a^2(\boldsymbol{u}_a\cdot\boldsymbol{b})^2/c^2}} \quad (22)$$

Hence for hydrodynamical description of the relativistic collisionless plasma in a strong magnetic field, the set of macroscopic parameters n_a, W_a, $\mathbf{P}_{a\parallel}$, $\mathbf{P}_{a\perp}$, $\mu_{a\parallel}$ and $\mu_{a\perp}$ should be introduced, *i.e.* the number of parameters in the relativistic theory increases.

Let us now write the closed equation system for the set of macroscopic parameters, describing the relativistic strongly magnetized plasma. We have yet the continuity equation. From the second relation of eq.(8) at $i=\alpha$, using eq.(20) one can obtain the equation of motion. Multiplying this by u_{ai} and summing up over i one obtains the equation for W_a (in fact it is a thermal balance equation)

$$n_a m_a c^2\frac{d}{ds_a}\left(W_a+\frac{\mathbf{P}_{a\perp}}{m_a c^2 n_a}\right)-\frac{d\mathbf{P}_{a\perp}}{ds_a}-(\mathbf{P}_{a\perp}-\mathbf{P}_{a\parallel})\Lambda_a^{\alpha i}\Lambda_a^{\beta j}b_\alpha b_\beta\frac{\partial u_{ai}}{\partial x_a^j}=0 \quad (23)$$

$$\text{where}\quad \frac{d}{ds_a}=\left(u_a^i\frac{\partial}{\partial x_a^i}\right)=\frac{\gamma_a}{c}\left(\frac{\partial}{\partial t}+u_a^\alpha\frac{\partial}{\partial x_a^\alpha}\right)$$

is the hydrodynamical derivative. In obtaining eq.(23) we used that $F^{ij}u_{ai}u_{aj}\equiv 0$. Multiplying eq.(10) by u_{aj} one obtains the modified equation for energy-momentum

$$u_{aj}\frac{\partial M_a^{ijk}}{\partial x_a^k}=\frac{e_a}{c}(F^{im}T_{am}^j-F_m^j T_a^{ij})u_{aj} \quad (24)$$

Excluding from eq.(10) the terms containing du_a^i/ds_a and dV_a/ds_a leads to

$$u_a^i u_a^j u_{am}u_{an}\frac{\partial M_a^{*mnk}}{\partial x_a^k}-u_a^i u_{am}\frac{\partial M_a^{*jmk}}{\partial x_a^k}-u_a^j u_{am}\frac{\partial M_a^{*imk}}{\partial x_a^k}+\frac{\partial M_a^{*ijk}}{\partial x_a^k}=R_a^{ij}-R_a^{jm}u_a^i u_{am}-R_a^{im}u_a^j u_{am} \quad (25)$$

where $M_a^{*ijk}\equiv M_a^{ijk}-m_a c(m_a c^2 n_a V_a+3\mu_{a\perp}u_a^i u_a^j u_a^k)$, and R_a^{ij} denotes the right-hand side of eq.(10). Let us introduce the projection operators

$$D_{aij}^{(l)}=\Lambda_{a\alpha i}\Lambda_{a\beta j}b^\alpha b^\beta,\quad D_{aij}^{(tr)}=g_{ij}+D_{aij}^{(l)} \quad (26)$$

Multiplying eq.(25) by $D_{ij}^{(l)}$ and $D_{ij}^{(tr)}$ and summing over the indices i, j one can obtain the equations for the macroscopic parameters $\mu_\parallel$ and $\mu_\perp$

$$\frac{d\mu_{a\parallel}}{ds_a}-\frac{\mu_{a\parallel}}{n_a}\frac{dn_a}{ds_a}-2\mu_{a\parallel}\Lambda_a^{\alpha i}\Lambda_a^{\beta j}b_\alpha b_\beta\frac{\partial u_{ai}}{\partial x_a^j}=0 \quad (27)$$

$$\frac{d\mu_{a\perp}}{ds_a}-\frac{2\mu_{a\perp}}{n_a}\frac{dn_a}{ds_a}+\mu_{a\perp}\Lambda_a^{\alpha i}\Lambda_a^{\beta j}b_\alpha b_\beta\frac{\partial u_{ai}}{\partial x_a^j}=0$$

Hence the equations (8) [the second part of eq.(8) at $i=0$ should be substituted with eq.(23)], eq.(24) [for $i=\alpha$] and eq.(27) then constitute a closed system of hydrodynamical equations for the relativistic collisionless plasma in the strong magnetic field. Eq.(24) connects the ordinary ($\mathbf{P}_{a\parallel}$ and $\mathbf{P}_{a\perp}$) and the "modified" ($\mu_{a\parallel},\mu_{a\perp}$) macroscopic parameters of plasma. It is clear that this set of equations is the relativistic generalization of the Chew, Goldberger and Low theory for the nonrelativistic strongly magnetized plasma.

The relativistic case considered is caused by two reasons. First, the fact that the hydrodynamical velocity of plasma motion can be of the order of speed of light, and second, that the mean energy of plasma particles' thermal motion can be more than or of the order of the particle's rest energy. In the nonrelativistic limit ($u_a \ll c$, $T_a \ll m_a c^2$) the system of hydrodynamical equations obtained reduces to a well known closed set of hydrodynamical equations (Baranov and Krasnobaev 1986). Indeed, in this case $W_a \to 0$, $V_a \to 1$, $\gamma_a \to 1$, $\Lambda^{\alpha i}\to g^{\alpha i}$, $\mu_{a\parallel}\to\mathbf{P}_{a\parallel}$, $\mu_{a\perp}\to\mathbf{P}_{a\perp}$, and the set of macroscopic parameters is reduced. In this limit eq.(27) coincides with the equations for the parallel and transverse pressure for the nonrelativistic case, and eq.(23) reduces to eq.(27) and becomes unnecessary. A detailed examination of these results will be published subsequently.