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Generic Partial Two-Point Sets Are Extendable

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Abstract. It is shown that under ZFC almost all planar compacta that meet every line in at most two points are subsets of sets that meet every line in exactly two points. This result was previously obtained by the author jointly with K. Kunen and J. van Mill under the assumption that Martin's Axiom is valid.

A planar set is called a *two-point set* if every line intersects the set in exactly two points and a *partial two-point set* if every line intersects the set in at most two points. We call a partial two-point set *extendable* if it is a subset of some two-point set. The existence of two-point sets is due to Mazurkiewicz [3]. His proof shows that every partial two-point set with cardinality less than $\mathfrak{c}$ is extendable. A circle is the standard example of a nonextendable partial two-point set. We are interested in the extendibility of compact partial two-point sets. The papers [1] and [2] give a number of results (both negative and positive) concerning this problem.

We denote the space of nonempty compacta in a metric space X equipped with the usual Hausdorff metric by $\mathcal{K}(X)$. $\mathcal{F}(X)$ denotes the dense subspace of $\mathcal{K}(X)$ consisting of the finite sets. Let X be a subset of the plane. The subset of $\mathcal{K}(X)$ consisting of partial two-point sets is denoted by $\mathcal{T}(X)$ and the subspace of $\mathcal{T}(X)$ consisting of extendable elements is denoted by $\mathcal{E}(X)$. If X is a partial two-point set then $\mathrm{L}(X)$ denotes the union of all lines in the plane that intersect X in two points.

We say that *generic* elements of a space Y have a certain property P if there exists a dense G_δ-subset G of Y such that every element of G has the property P. For instance, the statement 'generic compact partial two-point sets are extendable' means that $\mathcal{E}(\mathbf{R}^2)$ contains a dense G_δ-subset of $\mathcal{T}(\mathbf{R}^2)$.

Theorem 3.3 in [1] states that Martin's Axiom implies that every σ-compact partial two-point set A such that the Hausdorff 1-measure of $A \times A$ vanishes is extendable. The following result is an immediate corollary of that theorem (*cf.* [1, Corollary 3.5]).

Theorem 1 (MA) *If X is a subset of the plane then generic compact partial two-point subsets of X are extendable.*

It is also shown in [1, Proposition 3.7] that under ZFC generic subcompacta of a circle in the plane are extendable. The obvious question is whether Martin's Axiom is necessary in general. Expanding on an idea from [1] we prove:

Theorem 2 *Theorem 1 is valid in* ZFC.

Of particular interest is the case that X is a compact partial two-point set. The theorem then reads: generic subcompacta of X are extendable or, equivalently, the nonextendable elements of $\mathcal{K}(X)$ form a category I set.

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We introduce some terminology that will be useful in the proof. Let π_1 and π_2 be the projections of $\mathbf{R}^2$ onto the first respectively second coordinate. If Y is a subset of the plane then $\pi(Y)$ stands for the set $\pi_1(Y) \cup \pi_2(Y) \subset \mathbf{R}$. If Y is a subset of $\mathbf{R}$ then $\mathbf{Q}(Y)$ denotes the subfield of $\mathbf{R}$ that is generated by Y and $\mathcal{A}(Y)$ stands for the subfield of $\mathbf{R}$ consisting of all points that are algebraic in $\mathbf{Q}(Y)$. $\mathbf{Q}(\varnothing)$ stands for $\mathbf{Q}$ and $\mathcal{A}(\varnothing)$ is the set of algebraic real numbers. Let Y and Z be subsets of the real line. We say that Y is *free rel* Z if every point u in Y is not an element of $\mathcal{A}\big((Y \setminus \{u\}) \cup Z\big)$. Y is called *free* if Y is free rel $\varnothing$. A subset Z of Y is called a *basis for* Y if Z is free and $Y \subset \mathcal{A}(Z)$. Note that every free subset can be extended to a basis and that Y is free rel Z if and only if Y is free rel every element of $\mathcal{F}(Z)$.

The following notations will be used in the proofs of some technical lemmas. Let n be a fixed natural number. We let $p_{n1}, p_{n2}, \dots$ enumerate all nonzero polynomials in n variables with integer coefficients. Let the polynomial $q_{nm}(x_1, \dots, x_{n-1})$ be the leading coefficient of $p_{nm}(x, x_1, \dots, x_{n-1})$ if we regard p_{nm} as a polynomial in the first variable x. Let $\{P_{ni} : i \in \mathbf{N}\}$ be the set of closures of the elements of some countable basis for the topology of $\mathbf{R}^{n-1}$. If $n = 1$ then $\mathbf{R}^0$ (in general Y^0) and every P_{1i} is the singleton $\{\varepsilon\}$ where ε is the 0-tuple or empty string. Finally, let $(M_1, N_1), (M_2, N_2), \dots$ be a sequence of pairs of disjoint closed subsets of $\mathbf{R}$ such that for each $x \in \mathbf{R}$ and neighbourhood U of x there is a $k \in \mathbf{N}$ with $x \in M_k \subset \mathbf{R} \setminus N_k \subset U$.

Lemma 3 *If A is a countable subset of* $\mathbf{R}$ *then* $\mathcal{G} = \{D \in \mathcal{K}(\mathbf{R}) : D$ *is free rel* $A\}$ *is a dense* G_δ*-subset of* $\mathcal{K}(\mathbf{R})$.

Proof First we show that $\mathcal{G}$ is dense in $\mathcal{K}(\mathbf{R})$. Let $F = \{x_1, \dots, x_n\}$ be a finite set in $\mathbf{R}$ and let $\varepsilon > 0$. We define by induction sets $C_0, \dots, C_n$ with $|C_i| = i$. Put $C_0 = \varnothing$ and let $0 \le i < n$. Since $A \cup C_i$ is countable so is $\mathcal{A}(A \cup C_i)$. So we have no problem finding a point y_{i+1} in $(x_{i+1}, x_{i+1} + \varepsilon) \setminus \mathcal{A}(A \cup C_i)$. Put $C_{i+1} = C_i \cup \{y_{i+1}\}$. Note that C_n is ε-close to F in the Hausdorff metric and that $C_n \in \mathcal{G}$. So the closure of $\mathcal{G}$ contains all finite sets and hence it is $\mathcal{K}(\mathbf{R})$.

Let $A_1, A_2, \dots$ enumerate the finite subsets of A. Given natural numbers n, m, k, i, and j we put $O_{nmkij} = \mathcal{K}(\mathbf{R})$ if $0 \in q_{nm}(P_{ni})$ and if $0 \notin q_{nm}(P_{ni})$ we define

$$O_{nmkij} = \Big\{ D \in \mathcal{K}(\mathbf{R}) : 0 \notin p_{nm}\Big((D \cap M_k) \times \Big(\big((D \cap N_k) \cup A_j\big)^{n-1} \cap P_{ni} \Big) \Big) \Big\}.$$

We verify that O_{nmkij} is open in $\mathcal{K}(\mathbf{R})$. If $O_{nmkij} \ne \mathcal{K}(\mathbf{R})$ then we have

$$O_{nmkij} = \Big\{ D \in \mathcal{K}(\mathbf{R}) : p_{nm}\Big(\Big(D \times \big((D \cap N_k) \cup A_j\big)^{n-1} \Big) \cap (M_k \times P_{ni}) \Big) \subset \mathbf{R} \setminus \{0\} \Big\}.$$

Defining the open subset $U = p_{nm}^{-1}(\mathbf{R} \setminus \{0\}) \cup \big(\mathbf{R}^n \setminus (M_k \times P_{ni})\big)$ of $\mathbf{R}^n$ we find

$$O_{nmkij} = \{ D \in \mathcal{K}(\mathbf{R}) : D \times \big((D \cap N_k) \cup A_j\big)^{n-1} \subset U \}.$$

Let $\{U_0^\beta \times \cdots \times U_{n-1}^\beta : \beta \in B\}$ be the collection of all open subsets of U that have the form $V_0 \times \cdots \times V_{n-1}$. Since $D \times \big((D \cap N_k) \cup A_j\big)^{n-1}$ is a product of compacta it is a subset of

U if and only if it is a subset of some $U_0^\beta \times \cdots \times U_{n-1}^\beta$. So we have

$$O_{nmkij} = \bigcup_{\beta \in B} \Big(\mathcal{K}(U_0^\beta) \cap \bigcap_{l=1}^{n-1} \{D \in \mathcal{K}(\mathbf{R}) : (D \cap N_k) \cup A_j \subset U_l^\beta\} \Big).$$

The set $\mathcal{K}(U_0^\beta)$ is obviously open in $\mathcal{K}(\mathbf{R})$. Since for $1 \le l < n$,

$$\{D \in \mathcal{K}(\mathbf{R}) : (D \cap N_k) \cup A_j \subset U_l^\beta\} = \begin{cases} \mathcal{K}\big(U_l^\beta \cup (\mathbf{R} \setminus N_k)\big), & \text{if } A_j \subset U_l^\beta \\ \varnothing, & \text{if } A_j \not\subset U_l^\beta \end{cases}$$

it is also an open subset of $\mathcal{K}(\mathbf{R})$. We may conclude that O_{nmkij} is open.

It is left to show that

$$\mathcal{G}' = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} O_{nmkij}$$

equals $\mathcal{G}$. Let $D \in \mathcal{K}(\mathbf{R}) \setminus \mathcal{G}'$. Then there exist natural numbers n, m, k, i, and j such that $D \notin O_{nmkij}$. So we have $0 \notin q_{nm}(P_{ni})$ and

$$0 \in p_{nm}\bigg((D \cap M_k) \times \Big(\big((D \cap N_k) \cup A_j \big)^{n-1} \cap P_{ni} \Big) \bigg)$$

which means that there exists an $a \in D \cap M_k$ and points $a_1, \ldots, a_{n-1}$ in $(D \cap N_k) \cup A_j$ such that $p_{nm}(a, a_1, \ldots, a_{n-1}) = 0$ and $q_{nm}(a_1, \ldots, a_{n-1}) \neq 0$. So $p_{nm}(x, a_1, \ldots, a_{n-1})$ is not the zero function and we may conclude that $a \in \mathcal{A}\big((D \cap N_k) \cup A_j\big) \subset \mathcal{A}\big((D \setminus \{a\}) \cup A\big)$ and hence D is not free rel A.

Let $D \in \mathcal{K}(\mathbf{R}) \setminus \mathcal{G}$. Then D is not free rel A and we can find an $a \in D \cap \mathcal{A}\big((D \setminus \{a\}) \cup A\big)$. So there exists a polynomial $p(x, x_1, \ldots, x_{n-1})$ and points $a_1, \ldots, a_{n-1}$ in $(D \setminus \{a\}) \cup A$ such that $p(x, a_1, \ldots, a_{n-1})$ is not identically zero but $p(a, a_1, \ldots, a_{n-1}) = 0$. Without loss of generality we may assume that the coefficient of the highest power of x in $p(x, x_1, \ldots, x_{n-1})$ is nonzero when we substitute a_i for x_i, $1 \le i \le n-1$. Consequently, we can find a natural number m such that $p = p_{nm}$ and $q_{nm}(a_1, \ldots, a_{n-1}) \neq 0$. Select a $j \in \mathbf{N}$ such that $A_j = A \cap \{a_1, \ldots, a_{n-1}\}$. We can also find a $k \in \mathbf{N}$ such that $a \in M_k$ and $\{a_1, \ldots, a_{n-1}\} \setminus A_j \subset N_k$. Finally, since $q_{nm}(a_1, \ldots, a_{n-1}) \neq 0$ we can find a neighbourhood P_{ni} of $(a_1, \ldots, a_{n-1})$ in $\mathbf{R}^{n-1}$ with the property $0 \notin q_{nm}(P_{ni})$. Observe that

$$a \in D \cap M_k \quad \text{and} \quad (a_1, \ldots, a_{n-1}) \in \big((D \cap N_k) \cup A_j \big)^{n-1} \cap P_{ni}$$

and hence

$$0 = p_{nm}(a, a_1, \ldots, a_{n-1}) \in p_{nm}\bigg((D \cap M_k) \times \Big(\big((D \cap N_k) \cup A_j \big)^{n-1} \cap P_{ni} \Big) \bigg).$$

So D is not in O_{nmkij} and we may conclude that $D \notin \mathcal{G}'$.

We need another technical lemma:

Lemma 4 *If D is a compact subset of $\mathbf{R}$ then*

$$\mathcal{G}_D = \{K \in \mathcal{K}(\mathbf{R}^2) : D \text{ is free rel } \pi(K)\}$$

is a G_δ-subset of $\mathcal{K}(\mathbf{R}^2)$.

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Proof The proof is very similar to the proof of Lemma 3. Given natural numbers n, m, k, and i we define the following open subset of $\mathcal{K}(\mathbf{R}^2)$: if $0 \in q_{nm}(P_{ni})$ then $O_{nmki} = \mathcal{K}(\mathbf{R}^2)$ and if $0 \notin q_{nm}(P_{ni})$ then

$$O_{nmki} = \left\{ K \in \mathcal{K}(\mathbf{R}^2) : 0 \notin p_{nm}\left((D \cap M_k) \times \left(\left((D \cap N_k) \cup \pi(K) \right)^{n-1} \cap P_{ni} \right) \right) \right\}.$$

The proof that $\mathcal{G}_D$ is the intersection of all the O_{nmki}'s is virtually identical to the argument in the preceding lemma.

Proof of Theorem 2 Let X be a subset of the plane. Select a countable dense subset $\{C_i : i \in \mathbf{N}\}$ of $\mathcal{T}(X)$ and find for every $i \in \mathbf{N}$ a countable dense subset A_i of C_i. Consider the countable set $A = \bigcup_{i=1}^{\infty} A_i$. Note that $\mathcal{F}(A) \cap \mathcal{T}(X)$ is dense in $\mathcal{T}(X)$. Since every dense G_δ in $\mathcal{K}(\mathbf{R})$ contains Cantor sets Lemma 3 allows us to select a Cantor set $D \subset \mathbf{R}$ that is free rel $\pi(A)$. With Lemma 4 we define the following G_δ-subset of $\mathcal{T}(X)$:

$$\mathcal{O} = \mathcal{G}_D \cap \mathcal{T}(X).$$

Since D is free rel $\pi(A)$ we have that $\mathcal{F}(A) \cap \mathcal{T}(X) \subset \mathcal{O}$ so $\mathcal{O}$ is a dense G_δ in $\mathcal{T}(X)$.

We now show that every element of $\mathcal{O}$ is extendable. Let $K \in \mathcal{O}$. Then K is a compact partial two-point set such that D is free rel $\pi(K)$. Select a basis Z for $\pi(K)$. Since D is free rel $\pi(K)$ the sets Z and D are disjoint with a union that is free. Extend $Z \cup D$ to a basis B for $\mathbf{R}$. Let $\{\ell_\alpha : \alpha < \mathfrak{c}\}$ enumerate the lines in the plane. We shall construct by transfinite induction a nondecreasing sequence $(E_\alpha)_{\alpha \le \mathfrak{c}}$ of subsets of $\mathbf{R}^2 \setminus K$ with induction hypotheses:

(1)$|E_\alpha| \le |\alpha| + \omega$,
(2)$K \cup E_\alpha$ is a partial two-point set.

Put $E_0 = \varnothing$ and if $\lambda \le \mathfrak{c}$ is a limit ordinal then $E_\lambda = \bigcup_{\alpha<\lambda} E_\alpha$. Let α be a fixed ordinal $< \mathfrak{c}$ and consider E_α and ℓ_α. Let $n_\alpha \le 2$ be the number of points in $(K \cup E_\alpha) \cap \ell_\alpha$. If $n_\alpha = 2$ then we put $E_{\alpha+1} = E_\alpha$. Assume that $n_\alpha \le 1$ and that ℓ_α is the graph of $ax + by = c$. Since $|E_\alpha| \le |\alpha| + \omega$ we can find a $B' \subset B$ such that $|B'| \le |\alpha| + \omega$ and $\pi(E_\alpha) \cup \{a, b, c\} \subset \mathcal{A}(B')$. Since $|B'| < \mathfrak{c} = |D|$ we can find two points u and v in ℓ_α such that at least one of their coordinates is in $D \setminus B'$.

In order to prove that u and v are not in $\mathrm{L}(K \cup E_\alpha)$ assume that for instance $u = (x, y) \in \mathrm{L}(K \cup E_\alpha)$. There exist two distinct points $(x_1, y_1), (x_2, y_2) \in K \cup E_\alpha$ such that u is the unique point of intersection of ℓ_α and the line through (x_1, y_1) and (x_2, y_2). Consequently, x and y are elements of the field $\mathbf{Q}(\{x_1, x_2, y_1, y_2, a, b, c\})$ and hence elements of the field $\mathbf{Q}\big(\pi(K \cup E_\alpha) \cup \{a, b, c\}\big)$. Since $\mathcal{A}(Z \cup B')$ is a field that contains both $\pi(K)$ and $\pi(E_\alpha) \cup \{a, b, c\}$ we have that it also contains x and y. We may assume without loss of generality that $x \in D \setminus B'$. Since x is an element of the free set B we have that $x \notin \mathcal{A}(B \setminus \{x\})$. Since $Z \cup B' \subset B \setminus \{x\}$ we have that $x \notin \mathcal{A}(Z \cup B')$, contradicting a result obtained above. So

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we may conclude that neither u nor v are elements of $\mathrm{L}(K \cup E_\alpha)$. This property guarantees that if we define

$$E_{\alpha+1} = \begin{cases} E_\alpha \cup \{u\}, & \text{if } n_\alpha = 1 \\ E_\alpha \cup \{u, v\}, & \text{if } n_\alpha = 0 \end{cases}$$

then $K \cup E_{\alpha+1}$ is a partial two-point set which intersects ℓ_α in exactly two points. The induction is now complete. It is obvious that $K \cup E_{\mathfrak{c}}$ is a two-point set.

Since every dense G_δ-subset of for instance $\mathcal{K}$(circle) contains Cantor sets we have the following immediate consequence of Theorem 2 (which was already established in [1]):

Corollary 5 *There exist two-point sets that contain Cantor sets.*

On the other hand we have:

Proposition 6 *There exist two-point sets that contain only countable compacta.*

Proof Let $\{C_\alpha : \alpha < \mathfrak{c}\}$ and $\{\ell_\alpha : \alpha < \mathfrak{c}\}$ enumerate all uncountable compacta respectively all lines in the plane. We shall construct by induction a nondecreasing sequence $\{E_\alpha : \alpha < \mathfrak{c}\}$ of subsets of the plane and a sequence $\{x_\alpha : \alpha < \mathfrak{c}\}$ of points in the plane with induction hypotheses:

(1) E_α is a partial two-point set,
(2) $|E_{\alpha+1} \cap \ell_\alpha| = 2$,
(3) $|E_\alpha| \leq |\alpha| + \omega$,
(4) for every $\beta \leq \alpha$, $x_\beta \in C_\beta \setminus E_\alpha$.

Put $E_0 = \varnothing$ and for every limit ordinal $\lambda \leq \mathfrak{c}$, $E_\lambda = \bigcup_{\beta<\lambda} E_\beta$. It is an obvious consequence of the induction hypotheses that $E_{\mathfrak{c}}$ is a two-point set. In addition, hypothesis (4) states that every C_β has an element x_β that is not contained in any E_α for $\beta \leq \alpha < \mathfrak{c}$. Since E_α is a nondecreasing sequence we have $x_\beta \notin E_{\mathfrak{c}}$ and hence $E_{\mathfrak{c}}$ contains none of the C_β's.

It remains to perform the successor step of the induction. Assume that E_α has been constructed. Since $|E_\alpha| \leq |\alpha| + \omega < \mathfrak{c} = |C_\alpha|$ we can find an $x_\alpha \in C_\alpha \setminus E_\alpha$. If ℓ_α intersects E_α in two points then we put $E_{\alpha+1} = E_\alpha$. Let $|\ell_\alpha \cap E_\alpha| \leq 1$. Then every line that is determined by two points in E_α intersects ℓ_α in at most one point and hence we have

$$|\mathrm{L}(E_\alpha) \cap \ell_\alpha| \leq |E_\alpha|^2 \leq (|\alpha| + \omega)^2 = |\alpha| + \omega < \mathfrak{c}.$$

Since also $\{x_\beta : \beta \leq \alpha\}$ has less than $\mathfrak{c}$ points we can find two distinct elements u and v in $\ell_\alpha \setminus \big(\mathrm{L}(E_\alpha) \cup \{x_\beta : \beta \leq \alpha\}\big)$. Let $E_{\alpha+1}$ be $E_\alpha \cup \{u\}$ or $E_\alpha \cup \{u, v\}$ as needed. It is obvious that the induction hypotheses are satisfied.

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