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LIMIT CYCLES CLOSE TO INFINITY OF CERTAIN NON-LINEAR DIFFERENTIAL EQUATIONS

BY

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ABSTRACT. Through successive radial perturbations of a certain planar Hamiltonian polynomial vector field of degree $2K + 1$, we obtain a least K limit cycles containing $(2K + 1)^2$ singularities.

1. **Introduction and statement of result.** One of the problems posed by Tian Jinghuang in [4] is: Find for polynomial vector fields of degree 3 the maximum number $f(3)$ of foci; he believes that $f(3)$ is at least three. Furthermore, this question can be asked for polynomial vector fields of degree n, with n any positive integer.

Another interesting problem is to find, for polynomial vector fields of degree n, the maximum number $S(n)$ of isolated singular points contained in the bounded region determined by a limit cycle. The answer to this problem for $n = 2$ is $S(2) = 1$ [2].

In this paper, when n is odd, we prove that $f(n) \geqq (n^2 + 1)/2$ (therefore $f(3) \geqq 5$) and $S(n) = n^2$. Moreover, we give a polynomial vector field of degree $2k + 1$, with at least k limit cycles containing $((2k + 1)^2 + 1)/2$ hyperbolic foci and $((2k + 1)^2 - 1)/2$ hyperbolic saddle points. More precisely, we have

THEOREM. *For any positive odd integer $n = 2k+1$, let X_n be the Hamiltonian vector field of degree $n, X_n(x,y) = (P,Q)$, where*

$$P(x,y) = -y\prod_{i=1}^{k}(i^2 - y^2) \text{ and } Q(x,y) = x\prod_{i=1}^{k}(i^2 - x^2),$$

then for every $R > 0$ large enough and $\epsilon > 0$, there exist constants $\epsilon_0, \epsilon_1, \dots, \epsilon_k$ with $|\epsilon_i| < \epsilon$ such that the polynomial vector field of degree n

$$X_{n,\epsilon_0,\epsilon_1,\dots,\epsilon_k}(x,y) = X_n(x,y) + \sum_{i=0}^{k}\epsilon_i(x^2+y^2)^i(x,y)$$

has at least k concentric limit cycles. These limit cycles are outside the ball centered at the origin with radius R and they contain $(n^2+1)/2$ hyperbolic foci and $(n^2-1)/2$ hyperbolic saddle points.

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2. **Preliminaries.** Before providing the proof, we informally introduce the Poincaré compactification and a test for existence of limit cycles in an annular region (Poincaré Bendixson Theorem).

For a polynomial vector field $X = (P, Q)$ of degree n in the plane, the Poincaré compactification $\pi(X)$ is an analytical vector field defined on the unit sphere $S^2 = \{(x, y, z)/x^2 + y^2 + z^2 = 1\}$. On the upper ($z > 0$) and lower ($z < 0$) hemisphere, $\pi(X)$ is the central projection of X multiplied by the factor z^{n-1}, whereas, its action on the equator $S^1 = \{(x, y, z) \in S^2/z = 0\}$ (which is left invariant) reflects the behaviour of X at infinity.

The expression of $\pi(X)$ in polar coordinates (θ, ρ) defined by the covering map from $\mathbf{R}\times(-1, 1)$ onto $S^2\backslash\{(0, 0, \pm 1)\}$ given by $(\theta, \rho) \longrightarrow (x, y, z) = (1+\rho^2)^{1/2}(\cos\theta, \sin\theta, \rho)$ is:

$$(1+\rho^2)^{(1-n)/2}\left[\left(\sum \rho^i A_{n-i}(\theta)\right)\frac{\partial}{\partial\theta} - \rho\left(\sum \rho^i R_{n-i}(\theta)\right)\frac{\partial}{\partial\rho}\right], \quad i = 0, \ldots, n$$

where

$$A_k(\theta) = -P_k(\cos\theta,\ \sin\theta)\sin\theta + Q_k(\cos\theta,\ \sin\theta)\cos\theta$$

$$R_k(\theta) = P_k(\cos\theta,\ \sin\theta)\cos\theta + Q_k(\cos\theta,\ \sin\theta)\sin\theta$$

Here $P = \sum P_k,\ Q = \sum Q_k, k = 0,\ \ldots\ , n$ with $P_k,\ Q_k$ homogeneous polynomials of degree k.

When $A_n(\theta)$ does not vanish, the equator S^1 is a periodic orbit of $\pi(X)$ which is hyperbolic if

$$I = \int_0^{2\pi} R_n(\theta)A_n^{-1}(\theta)d\theta \neq 0$$

and when $I \cdot A_n(\theta)$ is positive (resp. negative), the equator S^1 is an attracting (resp. repelling) limit cycle of $\pi(X)$. That is, the trajectories of $\pi(X)$ spiral into the limit cycle from both sides as $t \longrightarrow +\infty$ (resp. $t \longrightarrow -\infty$).

We will use the following proposition (which follows from the Poincaré Bendixson Theorem) for detecting attracting or repelling limit cycles.

PROPOSITION. *Let X be an analytical vector field defined in a neighbourhood of an annular region G bounded by two cycles without contact c_1 and c_2. If G contains no singular points and all paths crossing c_1 and c_2 enter G (resp. leave G) with increasing t, then there must be an attracting (resp. repelling) limit cycle in G.*

3. **Proof of Theorem.** The points $(\pm i, \pm j)$, $i,\ j = 0,\ \ldots\ , k$, are the singular points of X_n, and all of them are simple (i.e. det $DX_n(\pm i,\ \pm j) \neq 0$). Moreover, these points are centers when $i + j$ is even and hyperbolic saddle points when $i + j$ is odd.

The phase portrait of X_n when $n = 3$ is shown in Fig. 1.

Let $\epsilon > 0$, and $R > \sqrt{2k^2}$ such that

$$(-1)^k\left[x^2\prod_{j=1}^{k}(j^2 - x^2) + y^2\prod_{j=1}^{k}(j^2 - y^2)\right] > 0$$

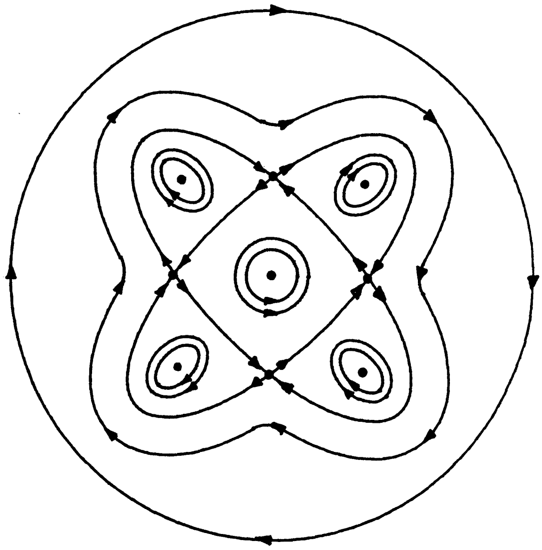

FIGURE 1.

for every (x, y) with $x^2 + y^2 > R^2$.

Let $\epsilon_0 > 0$, $\epsilon_0 < \epsilon$ be small enough, such that the field $X_{n,\epsilon_0,\dots,0}$ has n^2 singular points, $(n^2+1)/2$ hyperbolic foci (since divergence of this vector field is constant, equal to $2\epsilon_0$) and $(n^2-1)/2$ hyperbolic saddle points; all of them contained in the ball B centered at the origin with radius R. Let $\delta_1 = \delta_1(\epsilon_0) > 0$, $\delta_1 < \epsilon$ be such that for every $u_1, \dots, u_k$ with $|u_i| \leqq \delta_1, i = 1, \dots, k$ the field $X_{n,\epsilon_0,u_1,\dots,u_k}$ has n^2 singular points, $(n^2+1)/2$ hyperbolic foci and $(n^2-1)/2$ hyperbolic saddle points; all of them contained in the ball B.

Let $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$ be concentric periodic orbits of X_n contained in $\mathbf{R}^2 \backslash B$ such that if

$$r_i^- = \inf\{(x^2+y^2)^{1/2}/(x,y) \in \gamma_i\}$$
$$r_i^+ = \sup\{(x^2+y^2)^{1/2}/(x,y) \in \gamma_i\}$$

then $R < r_i^- \leqq r_i^+ < r_{i+1}^-, i = 0, 1, \dots, k-2$ and $\epsilon_0/r_0^- \binom{k}{j} < \delta_1$ for every $j = 0, \dots, k-1$.

For $i = 1, \dots, k-1$, we choose s_i such that $r_{i-1}^+ < s_i < r_i^+$. Let

$$s_k = \frac{\epsilon_0}{s_1 s_2 \dots, s_{k-1}\delta} \quad \text{and } g(r) = \delta \prod_{i=1}^{k}(s_i - r).$$

Then $g(r) = \epsilon_0 + \epsilon_1 r + \cdots + \epsilon_k r^k$; $|\epsilon_i| < \delta_1$, $i = 1, \dots, k-1, \epsilon_k = (-1)^k\delta$; and we choose $\delta > 0$, $\delta < \delta_1$, small enough so as to have $s_k > r_{k-1}^+$.

Let $A_{n,\epsilon_0,\dots,\epsilon_k}(\theta)$ and $R_{n,\epsilon_0,\dots,\epsilon_k}(\theta)$ the corresponding maps be introduced in 2 for the field $\pi(X_{n,\epsilon_0,\dots,\epsilon_k})$; then

$$A_{n,\epsilon_0,\dots,\epsilon_k}(\theta) = (-1)^k[\cos^{2k+2}(\theta) + \sin^{2k+2}(\theta)] \quad \text{and}$$
$$R_{n,\epsilon_0,\dots,\epsilon_k}(\theta) = (-1)^k \sin(\theta)\cos(\theta)[\sin^{2k}(\theta) - \cos^{2k}(\theta)] + \epsilon^k$$

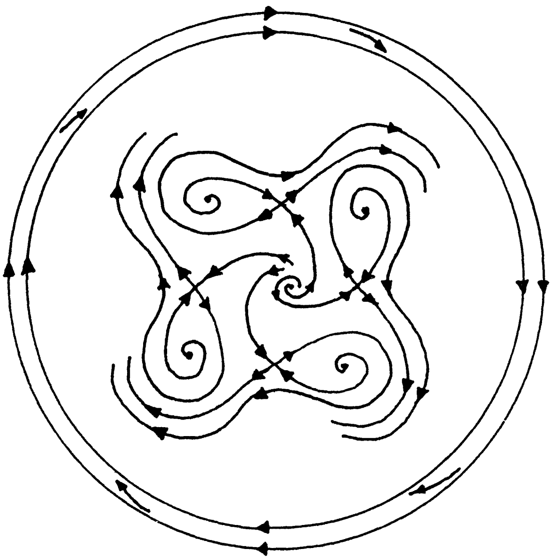

FIGURE 2.

Therefore the equator S^1 is a periodic orbit for $\pi(X_{n,\epsilon_0,\ldots,\epsilon_k})$ and since

$$I_{n,\epsilon_0,\ldots,\epsilon_k} = \int_0^{2\pi} \frac{A_{n,\epsilon_0,\ldots,\epsilon_k}(\theta)}{R_{n,\epsilon_0,\ldots,\epsilon_k}(\theta)}\, d\theta = (-1)^k \epsilon_k \int_0^{2\pi} \frac{d\theta}{\cos^{k+2}(\theta) + \sin^{k+2}(\theta)}$$

this orbit is an attracting (resp. repelling) limit cycle when k is even (resp. odd).

On the other hand, when $(x, y) \in \gamma_i, i = 0, \ldots, k-1$, the inner product

$$\langle X_{n,\epsilon_0,\ldots,\epsilon_k}(x,y),\; X_n^{\perp}(x,y)\rangle = g((x^2+y^2)^{1/2}) \left(x^2 \prod_{j=1}^{k} (j^2 - x^2) + y^2 \prod_{j=1}^{k} (j^2 - y^2) \right)$$

is always positive (resp. negative) if $i + k$ is even (resp. odd), where $X_n^{\perp} = (Q, -P)$. Then, for $i = 0, \ldots, k-2$, the field $X_{n,\epsilon_0,\ldots,\epsilon_k}$ has at least one limit cycle in the annular region bounded by γ_i and γ_{i+1}.

Finally, when k is even (resp. odd) the paths of the field $X_{n,\epsilon_0,\ldots,\epsilon_k}$ cross γ_{k-1} and enter γ_{k-2} (resp. leave γ_{k-2}) and our field $X_{n,\epsilon_0,\ldots,\epsilon_k}$ has at least one limit cycle outside the compact region bounded by γ_{k-1}. So the proof is complete. □

In figure 2 we show the phase portrait of $X_{3,\epsilon_0,\epsilon_1}$.

With the notations introduced in 1, we have the following.

COROLLARY. *When n is odd, $f(n) \geqq (n^2 + 1)/2$ (therefore $f(3) \geqq 5$) and $S(n) = n^2$.*

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