

Oscillating simply connected wandering domains

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Abstract. Although detailed descriptions of the possible types of behaviour inside periodic Fatou components have been known for over 100 years, a classification of wandering domains has only recently been given. Recently, simply connected wandering domains were classified into nine possible types and examples of escaping wandering domains of each of these types were constructed. Here we consider the case of oscillating wandering domains, for which only six of these types are possible. We use a new technique based on approximation theory to construct examples of all six types of oscillating simply connected wandering domains. This requires delicate arguments since oscillating wandering domains return infinitely often to a bounded part of the plane. Our technique is inspired by that used by Eremenko and Lyubich to construct the first example of an oscillating wandering domain, but with considerable refinements which enable us to show that the wandering domains are bounded, to specify the degree of the mappings between wandering domains and to give precise descriptions of the dynamical behaviour of these mappings.

Key words: oscillating wandering domains, internal dynamics, Approximation Theory

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1. Introduction

Let f be a transcendental entire function. We consider the iterates of f , which we denote by f^n , $n \geq 1$. The complex plane is divided into two sets: the Fatou set, $F(f)$, where the iterates (f^n) form a normal family in a neighbourhood of every point, and its complement, the Julia set $J(f)$. An introduction to the theory of iteration of transcendental entire and meromorphic functions can be found in [Ber93].

The Fatou set is open and consists of connected components, which are called *Fatou components*. Fatou components can be periodic, preperiodic or wandering domains. A Fatou component U is called a *wandering domain* if $f^n(U) \cap f^m(U) = \emptyset$, for all

$n \neq m$. Although Sullivan showed in [Sul85] that rational maps have no wandering domains, transcendental entire functions can have wandering domains. The first example of such a function was given by Baker [Bak76] who proved that a certain entire function given by an infinite product has a multiply connected wandering domain. Several examples of simply connected wandering domains have been constructed since then (see, for example, [Her84, p. 104], [Sul85, p. 414], [Bak84, pp. 564 and 567], [Dev90, p. 222], [EL87, Examples 1 and 2], [FH06]).

In [BRS13] the authors gave a complete description of the dynamical behaviour in multiply connected wandering domains. Recently, in [BEF+21] the authors gave a detailed classification of simply connected wandering domains in terms of the hyperbolic distance between orbits of points and in terms of convergence to the boundary. More specifically, they classified simply connected wandering domains into *contracting*, *semi-contracting* and *eventually isometric* depending on whether, for almost all pairs of points in the wandering domain, the hyperbolic distances between the orbits of these points, tend to 0, decrease but do not tend to 0, or are eventually constant, respectively (see Theorem 5.1). In terms of convergence to the boundary, the orbits of all points *stay away* from the boundary, come arbitrarily close to the boundary but do not converge to it (*bungee*), or *converge* to the boundary (see Theorem 5.2). These two classifications give nine possible types of simply connected wandering domains. Using a new technique, based on approximation theory, they show that all of these nine possible types are indeed realizable.

All the examples constructed in [BEF+21] were *escaping* wandering domains. Hence it is natural to ask whether there exist *oscillating* wandering domains of all nine types. (It remains a major open question as to whether it is possible to have wandering domains of bounded orbit.) A wandering domain U is called oscillating if there exist $(n_k), (m_k)$ such that $f^{n_k}(z) \rightarrow \infty$ and $(f^{m_k}(z))$ stays bounded for all $z \in U$. Since in oscillating wandering domains the iterates of f have finite limit points in $J(f)$, as well as ∞ , it is impossible for the orbit of a point in such a wandering domain to stay away from the boundary. Thus three of the nine possible types are not realizable. In this paper we show that the remaining six possible types of oscillating wandering domains are all realizable.

The first transcendental entire function with oscillating wandering domains was given by Eremenko and Lyubich in [EL87]; this was also the first application of approximation theory in complex dynamics. The authors used sequences of discs and half-annuli and a model function which was constant on the half-annuli and a translation on the discs. This model function was approximated on the closure of every disc and half-annulus by a transcendental entire function using an extended version of Runge's approximation theorem. Their technique did not show though whether their wandering domains are bounded or not, and did not give information on the degree of the entire function on each of the wandering domains.

Motivated by the construction in [EL87], we adapt the new techniques from [BEF+21] to construct bounded oscillating wandering domains, which, moreover, have the property that the degree of f on each of the wandering domains is equal to that of our model map. We state and prove our main construction theorem in §3.

It is worth pointing out that, in order for the wandering domains to be oscillating, the set-up needs to be much more complicated than that used for escaping wandering domains

in [BEF+21]. Although some of the building blocks of our proof are similar to those used in the analogous result for escaping wandering domains, the proof here requires several additional techniques. In particular, great care has to be taken over the accumulating errors in the approximation, as each of the discs on which the approximation takes place contains infinitely many domains in the orbit of the wandering domain.

In §5 we use the main construction theorem to construct all six types of oscillating wandering domains, proving the following result. This requires several preliminary results concerning Blaschke products which we prove in §4.

THEOREM 1.1. *For each of the six possible types of simply connected oscillating wandering domains, there exists a transcendental entire function with a bounded, simply connected oscillating wandering domain of that type.*

Oscillating wandering domains for functions in the Eremenko–Lyubich class \mathcal{B} have been constructed, first by Bishop in [Bis15], using the novel technique of quasiconformal folding, and more recently in [Boc21, FJL19, MPS20]. It would be interesting to see whether their methods can be adapted to classify the resulting wandering domains as one of the six possible types described above.

2. Preliminary results

In this section we give some existing results which are used in the proof of the main construction theorem. The following theorem, which is [BEF+21, Theorem D], plays a key role in the proof. We say that a curve σ surrounds a curve σ' if σ' is contained in a bounded complementary component of σ . Also, for two compact sets S, S' we say that S lies exterior to S' if S is contained in the unbounded complementary component of S' .

THEOREM 2.1. *Let f be a transcendental entire function and suppose that there exist Jordan curves γ_n and Γ_n , $n \geq 0$, compact sets L_k , $k \geq 0$, and a bounded domain D such that:*

- (a) Γ_n surrounds γ_n , for $n \geq 0$;
- (b) the sets Γ_n , $n \geq 0$, L_k , $k \geq 0$, and \overline{D} all lie exterior to each other;
- (c) γ_{n+1} surrounds $f(\gamma_n)$, for $n \geq 0$;
- (d) $f(\Gamma_n)$ surrounds Γ_{n+1} , for $n \geq 0$;
- (e) $f(\overline{D} \cup \bigcup_{k \geq 0} L_k) \subset D$;
- (f) there exists $n_k \rightarrow \infty$ such that

$$\max\{\text{dist}(z, L_k) : z \in \Gamma_{n_k}\} = o(\text{dist}(\gamma_{n_k}, \Gamma_{n_k})) \quad \text{as } k \rightarrow \infty.$$

Then there exists an orbit of simply connected wandering domains U_n such that $\overline{\text{int } \gamma_n} \subset U_n \subset \text{int } \Gamma_n$, for $n \geq 0$.

Moreover, if there exists $z_n \in \text{int } \gamma_n$ such that both $f(\gamma_n)$ and $f(\Gamma_n)$ wind d_n times around $f(z_n)$, then $f : U_n \rightarrow U_{n+1}$ has degree d_n , for $n \geq 0$.

In order to obtain the transcendental entire function with the required properties, we consider an analytic function which is our model function and then apply the following

result which is an extension of the well-known Runge approximation theorem and was the Main Lemma in [EL87].

LEMMA 2.2. *Let (E_n) be a sequence of compact subsets of \mathbb{C} with the following properties:*

- (i) $\mathbb{C} \setminus E_n$ is connected, for $n \geq 0$;
- (ii) $E_n \cap E_m = \emptyset$, for $n \neq m$;
- (iii) $\min\{|z| : z \in E_n\} \rightarrow \infty$ as $n \rightarrow \infty$.

Suppose ψ is holomorphic on $E = \bigcup_{n=0}^{\infty} E_n$. For $n \geq 0$, let $\varepsilon_n > 0$ and let $z_n \in E_n$. Then there exists an entire function f satisfying, for $n \geq 0$,

$$|f(z) - \psi(z)| < \varepsilon_n \quad \text{for } z \in E_n; \tag{2.1}$$

$$f(z_n) = \psi(z_n), \quad f'(z_n) = \psi'(z_n). \tag{2.2}$$

Remark 2.3. We note that if the sets E_n are each real-symmetric (that is, $\overline{E_n} = E_n$), the function ψ is real-symmetric in E (that is, $\overline{\psi(\bar{z})} = \psi(z)$, for $z \in E$), and the points z_n , $n \geq 0$, are all real, then we can take the entire function f to be real-symmetric on \mathbb{C} . Indeed, if f satisfies the conclusions of Lemma 2.2, then $g(z) = \frac{1}{2}(f(z) + \overline{f(\bar{z})})$ is real-symmetric and entire, and satisfies the conclusions of Lemma 2.2.

We also need the following result, which is a version of [EL87, Lemma 2].

LEMMA 2.4. *Let g be an analytic function in the disc $\{z : |z| < R\}$ such that $g(0) = g'(0) = 0$ and $|g(z)| < \epsilon R$ for $|z| < R$ and some $\epsilon < 1/4$. Then*

$$|g(z)| \leq \frac{\epsilon}{R} |z|^2 \quad \text{for } |z| < R.$$

Finally, we need the following lemma about hyperbolic distances in discs, which is [BEF+21, Lemma 5.2]. Here we define $D_r = D(0, r)$ and $\mathbb{D} = D_1$.

LEMMA 2.5. *Suppose that $0 < s < r < 1 < R$ and set*

$$c(s, R) = \frac{1 - s^2}{R - s^2/R}, \quad D_r = D(0, r) \quad \text{and} \quad D_R = D(0, R).$$

If $|z|, |w| \leq s$, then

$$\text{dist}_{D_R}(z, w) = \text{dist}_{\mathbb{D}}(z/R, w/R) \geq c(s, R) \text{dist}_{\mathbb{D}}(z, w) \tag{2.3}$$

and

$$\text{dist}_{D_r}(z, w) = \text{dist}_{\mathbb{D}}(z/r, w/r) \leq \frac{1}{c(s/r, 1/r)} \text{dist}_{\mathbb{D}}(z, w). \tag{2.4}$$

Also $0 < c(s, R) < 1$, and if the variables s, r and R satisfy in addition

$$1 - r = o(1 - s) \quad \text{as } s \rightarrow 1 \quad \text{and} \quad R - 1 = O(1 - r) \quad \text{as } r \rightarrow 1, \tag{2.5}$$

then

$$c(s, R) \rightarrow 1 \quad \text{as } s \rightarrow 1, \tag{2.6}$$

and

$$c(s/r, 1/r) \rightarrow 1 \quad \text{as } s \rightarrow 1. \tag{2.7}$$

3. Main construction

In this section we state and prove our construction result. Throughout, $D(z, r)$ denotes the open disc with centre z and radius r .

THEOREM 3.1. (Main construction) *Let $(b_n)_{n \geq 0}$ be a sequence of Blaschke products of corresponding degree $d_n \geq 1$, and let $(\alpha_n)_{n \geq 0}$ be a sequence of real numbers with $\alpha_0 = 1$ and $\alpha_{n+1}/\alpha_n \leq 1/6$. For $n \geq 0$, let*

$$D_n = D(9n, \alpha_n),$$

$$\Delta_n = D(a_n, \alpha_n) \quad \text{and} \quad \Delta'_n = D(a_n, 2\alpha_n) \quad \text{where } a_n = 9n + 4\alpha_n,$$

and

$$G_n = D(\kappa_n, 1) \quad \text{and} \quad G'_n = D(\kappa_n, 5/4) \quad \text{where } \kappa_n = a_n + 3.$$

We consider the function

$$\varphi(z) = \begin{cases} z + 9 & \text{if } z \in \overline{D_n}, n \geq 0, \\ \frac{z - a_n}{\alpha_n} + \kappa_n & \text{if } z \in \overline{\Delta'_n}, n \geq 0, \\ \alpha_{n+1}b_n(z - \kappa_n) + 4\alpha_{n+1} & \text{if } z \in \overline{G'_n}, n \geq 0, \end{cases}$$

and define V_m, ζ_m and ρ_m by

$$\begin{aligned} V_m &= D(\zeta_m, \rho_m) \\ &= \varphi^m(\Delta_0) = \begin{cases} \Delta_n & \text{if } m = \ell_n - 1, n \geq 0, \\ G_n & \text{if } m = \ell_n, n \geq 0, \\ D(9k + 4\alpha_{n+1}, \alpha_{n+1}) \subset D_k & \text{if } m = \ell_n + k + 1, 0 \leq k \leq n, \end{cases} \end{aligned}$$

where (ℓ_n) is defined by $\ell_0 = 1$ and $\ell_{n+1} = \ell_n + n + 3, n \geq 0$.

For a suitable choice of (α_n) , there exists a transcendental entire function f having an orbit of bounded, simply connected, oscillating wandering domains U_m such that, for $m, n \geq 0$:

- (i) $\overline{D(\zeta_m, r_m)} \subset U_m \subset D(\zeta_m, R_m)$, where $0 < r_m < \rho_m < R_m$, and $r_m \sim \rho_m$ and $R_m \sim \rho_m$ as $m \rightarrow \infty$;
- (ii) $|f(z) - \varphi(z)| \leq \varepsilon_m$ on $\overline{D(\zeta_m, R_m)}$, where $\varepsilon_0 \leq 1/24$ and $\varepsilon_{\ell_n+k} = \alpha_{n+1}^2/2^{k+1}$, for $0 \leq k \leq n + 2$;
- (iii) $f(9n) = \varphi(9n) = 9(n + 1)$ and $f'(9n) = \varphi'(9n) = 1$;
- (iv) $f : U_m \rightarrow U_{m+1}$ has degree q_m , where $q_{\ell_n} = d_n$, and $q_m = 1$ otherwise.

Finally, if $z, z' \in U_0$ and there exists $N \in \mathbb{N}$ such that $f^{\ell_N}(z), f^{\ell_N}(z') \in \overline{D(\kappa_N, r_{\ell_N})}$, then, for $n \geq N$, we have

$$k_n \operatorname{dist}_{G_n}(f^{\ell_n}(z), f^{\ell_n}(z')) \leq \operatorname{dist}_{U_{\ell_n}}(f^{\ell_n}(z), f^{\ell_n}(z')) \leq K_n \operatorname{dist}_{G_n}(f^{\ell_n}(z), f^{\ell_n}(z')), \tag{3.1}$$

where $0 < k_n < 1 < K_n$ with $k_n, K_n \rightarrow 1$ as $n \rightarrow \infty$.

Remark 3.2. If the Blaschke products b_n are real-symmetric for each $n \geq 0$, then f can be taken to be real-symmetric; see Remark 2.3.

Proof. We consider the sets $V_m = \varphi^m(\Delta_0)$, where $\Delta_0 = D(4, 1)$, as defined in the statement of Theorem 3.1, and construct a function f which is sufficiently close to φ in parts of the plane in order to ensure that f has a bounded wandering domain U with $f^m(U)$ close to V_m , for $m \geq 0$, in the sense that the Hausdorff distance between U_m and V_m tends to 0 as $m \rightarrow \infty$. □

3.1. *The sets V_m .* We begin by noting that it follows from the definition of φ and the fact that $\alpha_{m+1}/\alpha_m \leq 1/6$, for $m \geq 0$, that, for each $n \geq 0$,

$$\begin{aligned} \varphi(\Delta_n) &= G_n, \\ \varphi^2(\Delta_n) &= D(4\alpha_{n+1}, \alpha_{n+1}) \subset D(0, \alpha_0) = D_0, \end{aligned}$$

so, for $0 \leq k \leq n$,

$$\varphi^{k+2}(\Delta_n) = D(9k + 4\alpha_{n+1}, \alpha_{n+1}) \subset D(9k, \alpha_k) = D_k$$

and

$$\varphi^{n+3}(\Delta_n) = D(9(n + 1) + 4\alpha_{n+1}, \alpha_{n+1}) = \Delta_{n+1}.$$

This gives the following properties of V_m , stated in Theorem 3.1:

$$\begin{aligned} V_m &= D(\zeta_m, \rho_m) \\ &= \varphi^m(\Delta_0) = \begin{cases} \Delta_n & \text{if } m = \ell_n - 1, n \geq 0, \\ G_n & \text{if } m = \ell_n, n \geq 0, \\ D(9k + 4\alpha_{n+1}, \alpha_{n+1}) \subset D_k & \text{if } m = \ell_n + k + 1, 0 \leq k \leq n, \end{cases} \end{aligned} \tag{3.2}$$

where (ℓ_n) is defined by $\ell_0 = 1$ and $\ell_{n+1} = \ell_n + n + 3$, for $n \geq 0$.

In words, if $V_m \subset D_0$, then φ repeatedly translates V_m to the right by 9 until the translated image lands on Δ_n , for some $n \in \mathbb{N}$, at which point φ maps the disc Δ_n onto G_n and then maps G_n into D_0 (see Figure 1).

3.2. *Construction of the circles γ_n and Γ_n .* We now give an inductive definition of the values r_m and R_m described in Theorem 3.1(i) and define α_n inductively at the same time. We will choose these values in such a way that, if we define

$$\gamma_m = \{z : |z - \zeta_m| = r_m\} \quad \text{and} \quad \Gamma_m = \{z : |z - \zeta_m| = R_m\},$$

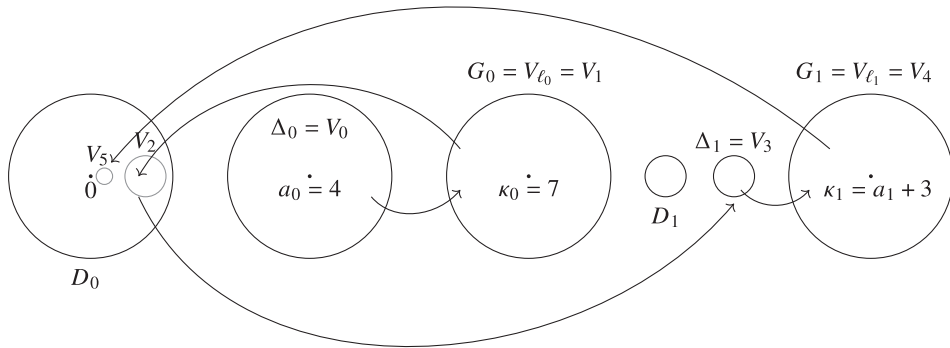


FIGURE 1. The action of the model function φ .

then, for $m \geq 0$,

$$\gamma_{m+1} \text{ surrounds } \varphi(\gamma_m) \tag{3.3}$$

and

$$\varphi(\Gamma_m) \text{ surrounds } \Gamma_{m+1}. \tag{3.4}$$

Further, we choose these values in such a way that we are able to use Lemma 2.2 and Lemma 2.4 to approximate the map φ by an entire function f such that φ can be replaced by f in (3.3) and (3.4). This in turn allows us to apply Theorem 2.1 to deduce that f has wandering domains with the required properties.

Our construction uses the Blaschke products b_n which, for $n \geq 0$, we write as

$$b_n(z) = e^{i\theta_n} \prod_{j=1}^{d_n} \frac{z + p_{n,j}}{1 + \overline{p_{n,j}}z},$$

where $p_{n,j} \in \mathbb{D} = \{z : |z| < 1\}$ are not necessarily different from each other, and $\theta_n \in [0, 2\pi)$. We also use the maps defined by

$$B_n(z) = b_n(z - \kappa_n) \quad \text{for } n \geq 0, \tag{3.5}$$

noting that B_n has degree d_n and maps G_n to $D_0 = \mathbb{D}$.

First take

$$r_0 \in (5/6, 1) \quad \text{and} \quad R_0 \in (1, 7/6), \tag{3.6}$$

and recall that $\alpha_0 = 1$. We then choose r_1 such that

$$0 < 1 - r_1 \leq \min \left\{ \frac{1 - r_0}{2}, \text{dist}(\varphi(\gamma_0), \partial G_0)^2 \right\} \tag{3.7}$$

and

$$B_0(\gamma_1) \text{ winds exactly } d_0 \text{ times round } D(0, 1/2),$$

so

$\varphi(\gamma_1)$ winds exactly d_0 times round $D(\zeta_2, \rho_2/2)$,

and choose R_1 such that

$$0 < R_1 - 1 \leq \min \left\{ \frac{R_0 - 1}{2}, \text{dist}(\varphi(\Gamma_0), \partial G_0), \frac{1}{\max_j |p_0, j| - 1} \right\}. \tag{3.8}$$

Now assume that, for some $n \geq 0$, α_k has been chosen for $0 \leq k \leq n$, and r_m and R_m have been chosen for $0 \leq m \leq \ell_n$. (Note that $\ell_0 = 1$ and we have already specified α_0, r_0, R_0, r_1 and R_1 .) We shall give a rule for choosing α_{n+1} and also for choosing r_m and R_m for $\ell_n + 1 \leq m \leq \ell_{n+1}$. There are three different cases, depending on the value of m .

Case 1. First we consider the case when $m = \ell_n + 1$ (and so $V_m \subset D_0$). We also specify α_{n+1} as part of this case.

We begin by choosing c_{n+1}, C_{n+1} to be circles centred at 0, lying in the interior and exterior of D_0 respectively, such that

$$\text{dist}(c_{n+1}, \partial D_0) \leq \min \left\{ \frac{\rho_{\ell_n} - r_{\ell_n}}{6}, \frac{1}{2} \text{dist}(B_n(\gamma_{\ell_n}), \partial D_0) \right\} \tag{3.9}$$

and

$$\text{dist}(C_{n+1}, \partial D_0) \leq \min \left\{ \frac{R_{\ell_n} - \rho_{\ell_n}}{6}, \text{dist}(c_{n+1}, \partial D_0), \frac{1}{2} \text{dist}(B_n(\Gamma_{\ell_n}), \partial D_0) \right\}. \tag{3.10}$$

We set

$$\alpha_{n+1} = \text{dist}(C_{n+1}, \partial D_0) \tag{3.11}$$

and note, using the fact that $\varphi(z) = \alpha_{n+1} B_n(z) + 4\alpha_{n+1}$, for $z \in G'_n$, that $V_{\ell_{n+1}} = \varphi(V_{\ell_n}) = D(4\alpha_{n+1}, \alpha_{n+1})$, so $\rho_{\ell_{n+1}} = \alpha_{n+1}$. We then set

$$r_{\ell_{n+1}} = \rho_{\ell_{n+1}} - \alpha_{n+1}^2 \tag{3.12}$$

and

$$R_{\ell_{n+1}} = \rho_{\ell_{n+1}} + \alpha_{n+1}^2. \tag{3.13}$$

Note that, together with (3.9), (3.11) and (3.10), these definitions imply that

$$\rho_{\ell_{n+1}} - r_{\ell_{n+1}} = \alpha_{n+1}^2 \leq \alpha_{n+1} \text{dist}(c_{n+1}, \partial D_0) \leq \frac{1}{2} \text{dist}(\varphi(\gamma_{\ell_n}), \partial V_{\ell_{n+1}}), \tag{3.14}$$

$$R_{\ell_{n+1}} - \rho_{\ell_{n+1}} = \alpha_{n+1}^2 = \alpha_{n+1} \text{dist}(C_{n+1}, \partial D_0) \leq \frac{1}{2} \text{dist}(\varphi(\Gamma_{\ell_n}), \partial V_{\ell_{n+1}}), \tag{3.15}$$

and hence

$$R_{\ell_{n+1}} - r_{\ell_{n+1}} = 2\alpha_{n+1}^2 \leq \min\{\text{dist}(\varphi(\gamma_{\ell_n}), \partial V_{\ell_{n+1}}), \text{dist}(\varphi(\Gamma_{\ell_n}), \partial V_{\ell_{n+1}})\}. \tag{3.16}$$

Case 2. We now consider the cases when $m = \ell_n + k + 1$, for $1 \leq k \leq n + 1$. Then

$$V_{\ell_n+k+1} = \varphi^k(V_{\ell_n+1}) \subset D_k \quad \text{for } 1 \leq k \leq n,$$

and

$$V_{\ell_n+n+2} = V_{\ell_{n+1}-1} = D(9(n + 1) + 4\alpha_{n+1}, \alpha_{n+1}) = \Delta_{n+1}.$$

In all these cases, we simply choose r_m and R_m to satisfy

$$\rho_m - r_m = \frac{\rho_{m-1} - r_{m-1}}{2} \tag{3.17}$$

and

$$R_m - \rho_m = \frac{R_{m-1} - \rho_{m-1}}{2}. \tag{3.18}$$

Case 3. Finally, we consider the case when $m = \ell_{n+1} = \ell_n + n + 3$, so $V_m = G_{n+1}$. In this case, we choose $r_{\ell_{n+1}}$ and $R_{\ell_{n+1}}$ so that

$$0 < \rho_{\ell_{n+1}} - r_{\ell_{n+1}} \leq \min \left\{ \frac{\rho_{\ell_{n+1}-1} - r_{\ell_{n+1}-1}}{2}, \text{dist}(\varphi(\gamma_{\ell_{n+1}-1}), \partial G_{n+1})^2 \right\}, \tag{3.19}$$

$$B_{n+1}(\gamma_{\ell_{n+1}}) \text{ winds exactly } d_{n+1} \text{ times round } D(0, 1/2), \tag{3.20}$$

$$\begin{aligned} &0 < R_{\ell_{n+1}} - \rho_{\ell_{n+1}} \\ &\leq \min \left\{ \frac{R_{\ell_{n+1}-1} - \rho_{\ell_{n+1}-1}}{2}, \text{dist}(\varphi(\Gamma_{\ell_{n+1}-1}), \partial G_{n+1}), \frac{1}{\max_j \{ |p_{n+1,j}| \}} - 1 \right\}. \end{aligned} \tag{3.21}$$

This inductive process defines the values r_m and R_m , and hence the circles γ_m and Γ_m , for $m \geq 0$. Note that it follows from (3.10), (3.11), (3.18), (3.13) and (3.21) that

$$\alpha_{n+1} \leq \frac{R_{\ell_n} - \rho_{\ell_n}}{6} < \frac{R_{\ell_{n-1}+1} - \rho_{\ell_{n-1}+1}}{6} = \frac{\alpha_n^2}{6} < \frac{\alpha_n}{6} \text{ for } n \geq 1.$$

Moreover, it follows from the definition of φ together with (3.5) and (3.2) that we have

$$\varphi(z) = \alpha_{n+1} B_n(z) + 4\alpha_{n+1} = \rho_{\ell_{n+1}} B_n(z) + \zeta_{\ell_{n+1}} \text{ for } z \in G_n, n \geq 0. \tag{3.22}$$

So (3.20) implies that, for $n \geq 0$,

$$\varphi(\gamma_{\ell_{n+1}}) \text{ winds exactly } d_{n+1} \text{ times round } D(\zeta_{\ell_{n+1}+1}, \rho_{\ell_{n+1}+1}/2). \tag{3.23}$$

We also note that it follows from (3.15), (3.18) and (3.2) that, for $m = \ell_n + k + 1$, where $0 \leq k \leq n + 1$, we have $R_m - \rho_m \leq R_{\ell_{n+1}} - \rho_{\ell_{n+1}} < \alpha_{n+1} = \rho_m$. So, for $m = \ell_n + k + 1$, where $0 \leq k \leq n$, we have

$$V'_m = D(\zeta_m, R_m) \subset D(\zeta_m, 2\rho_m) = D(9k + 4\alpha_{n+1}, 2\alpha_{n+1}) \subset D(9k, 6\alpha_{n+1}) \subset D_k \tag{3.24}$$

and

$$V'_{\ell_n+n+2} = V'_{\ell_{n+1}-1} = D(\zeta_{\ell_{n+1}-1}, R_{\ell_{n+1}-1}) \subset \Delta'_{n+1} \tag{3.25}$$

by the definitions of $D(\zeta_m, \rho_m)$ and Δ'_n in the statement of Theorem 3.1.

It then follows from (3.16), and the fact that $\alpha_n < 1/2$ for $n > 0$, that

$$V'_{\ell_{n+1}} = D(\zeta_{\ell_{n+1}}, R_{\ell_{n+1}}) \subset G'_{n+1}. \tag{3.26}$$

Also, φ is analytic in $V'_{\ell_{n+1}}$ by (3.22) together with the last condition in (3.21).

It follows from (3.14) that, for $n \geq 0$,

$$\gamma_{\ell_{n+1}} \text{ surrounds } \varphi(\gamma_{\ell_n}), \tag{3.27}$$

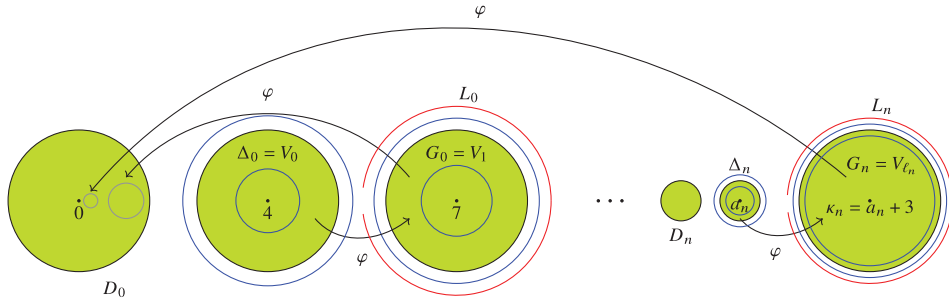


FIGURE 2. Sketch of the set-up of Theorem 3.1, showing the location of the circles γ_n and Γ_n (in blue), and the arcs L_n (in red). (Colour available online.)

and from (3.15) that, for $n \geq 0$,

$$\varphi(\Gamma_{\ell_n}) \text{ surrounds } \Gamma_{\ell_n+1}. \tag{3.28}$$

Thus (3.3) and (3.4) hold when $m = \ell_n$, where $n \geq 0$.

Also, if $m = \ell_n + k + 1$, where $n \geq 0, 0 \leq k \leq n + 1$, then φ is a translation on γ_m and Γ_m , by (3.24) and the definition of φ . Since, by (3.17), we have

$$\rho_{m+1} - r_{m+1} \leq \frac{\rho_m - r_m}{2}$$

and, by (3.18),

$$R_{m+1} - \rho_{m+1} \leq \frac{R_m - \rho_m}{2},$$

it follows that (3.3) and (3.4) hold for these values of m too.

Finally, it follows from (3.25) that, on $\gamma_{\ell_{n+1}-1}$ and $\Gamma_{\ell_{n+1}-1}$, $n \geq 0$, the function φ is a scaling by a factor of $1/\alpha_{n+1} > 1$ followed by a translation, and so it follows from (3.19) and (3.21) that (3.3) and (3.4) hold in this case too.

We note that the sets $\overline{V'_m}$ are disjoint since, if $V'_{\ell_n+k+1}, V'_{\ell_{n+1}+k+1} \subset D_k$, for some $n \geq 0, 0 \leq k \leq n$, then $V'_{\ell_n+k+1} \subset D(9k + 4\alpha_{n+1}, 2\alpha_{n+1})$ and $V'_{\ell_{n+1}+k+1} \subset D(9k + 4\alpha_{n+2}, 2\alpha_{n+2})$, and

$$D(9k + 4\alpha_{n+1}, 2\alpha_{n+1}) \cap D(9k + 4\alpha_{n+2}, 2\alpha_{n+2}) = \emptyset,$$

since

$$4\alpha_{n+2} + 2\alpha_{n+2} = 6\alpha_{n+2} \leq \alpha_{n+1} < 4\alpha_{n+1} - 2\alpha_{n+1}.$$

3.3. Construction of the function f . Our aim now is to use Lemma 2.2 and Lemma 2.4 to approximate the map φ by a single entire function f such that, for $m \geq 0$, γ_{m+1} surrounds $f(\gamma_m)$ and $f(\Gamma_m)$ surrounds Γ_{m+1} . We also require f to map certain curves L_n near $\overline{G_n}$ in such a way that we can apply Theorem 2.1.

We define L_n , for $n \geq 0$, to be the circular arc

$$L_n := \{z : |z - a_n| = R_{\ell_n} + \delta_{\ell_n}^2/2, |\arg(z - a_n)| \leq \pi - \delta_{\ell_n}^2\}, \tag{3.29}$$

where $\delta_m = R_m - r_m \rightarrow 0$ as $m \rightarrow \infty$; see Figure 2.

We also define approximation error quantities ε_m , for $m \geq 0$, by

$$\varepsilon_m = \min\{\frac{1}{4} \text{dist}(\varphi(\gamma_m), \partial V_{m+1}), \frac{1}{4} \text{dist}(\varphi(\Gamma_m), \partial V_{m+1}), \frac{1}{4} \delta_{m+1}\} > 0. \tag{3.30}$$

We now show that these errors have the upper bounds stated in part (ii) of our theorem. First, it follows from (3.6), (3.7) and (3.8) that

$$\varepsilon_0 \leq (R_1 - r_1)/4 \leq (R_0 - r_0)/8 < 1/24.$$

Next we note that it follows from (3.16) that, for $n \geq 0$,

$$\varepsilon_{\ell_n} = \delta_{\ell_{n+1}}/4 = (R_{\ell_{n+1}} - r_{\ell_{n+1}})/4 = 2\alpha_{n+1}^2/4 = \alpha_{n+1}^2/2.$$

It then follows from (3.17), (3.18), (3.19) and (3.21) that, for $0 \leq k \leq n + 1$, $n \geq 0$,

$$\varepsilon_{\ell_{n+k+1}} = (R_{\ell_{n+k+2}} - r_{\ell_{n+k+2}})/4 = (R_{\ell_{n+1}} - r_{\ell_{n+1}})/2^{k+3} = \alpha_{n+1}^2/2^{k+2}. \tag{3.31}$$

Thus

$$\varepsilon_{\ell_{n+k}} = \alpha_{n+1}^2/2^{k+1} \quad \text{for } 0 \leq k \leq n + 2, \quad n \geq 0, \tag{3.32}$$

as required for part (ii).

Since φ is analytic in each set $\overline{V'_{\ell_{n+1}}}$, for $n \geq 0$, it follows from Lemma 2.2 that there exists an entire function f such that, for $n \geq 0$,

$$|f(z) - \varphi(z)| < \varepsilon_{\ell_{n+n+1}} \quad \text{for } z \in \overline{D_n}, \tag{3.33}$$

$$|f(z) - \varphi(z)| < \varepsilon_{\ell_{n-1}} \quad \text{for } z \in \overline{\Delta_n}, \tag{3.34}$$

$$|f(z) - \varphi(z)| < \varepsilon_{\ell_n} \quad \text{for } z \in \overline{G_n}, \tag{3.35}$$

$$f(9n) = 9(n + 1), \tag{3.36}$$

$$f'(9n) = 1, \tag{3.37}$$

and such that

$$|f(z) + 4| \leq 1/2 \quad \text{for } z \in \overline{D(-4, 1)} \cup \bigcup_{n \geq 0} L_n. \tag{3.38}$$

It follows from (3.33), (3.36) and (3.37) that for each $k \geq 0$ we can apply Lemma 2.4 in the disc $D_k = D(9k, \alpha_k)$, with $g(z) = f(z) - \varphi(z)$, $R = \alpha_k$ and associated constant $\epsilon = \varepsilon_{\ell_{k+k+1}}/\alpha_k$. Note that the conditions of Lemma 2.4 are satisfied since it follows from (3.32) that

$$\epsilon = \varepsilon_{\ell_{k+k+1}}/\alpha_k = \frac{\alpha_{k+1}^2}{\alpha_k 2^{k+2}} < \frac{\alpha_{k+1}}{6} < 1/4 \quad \text{for } k \geq 0.$$

So, by Lemma 2.4, for all $z \in D_k$, $k \geq 0$, we have

$$|f(z) - \varphi(z)| \leq \frac{\varepsilon_{\ell_{k+k+1}}}{\alpha_k^2} |z - 9k|^2. \tag{3.39}$$

We will now show that this implies that, for each $m \geq 0$,

$$|f(z) - \varphi(z)| < \varepsilon_m \quad \text{for } z \in \overline{V'_m}. \tag{3.40}$$

First we note that (3.40) follows from (3.25) and (3.26) together with (3.34) and (3.35) when $m = \ell_n$ or $m = \ell_n - 1$, for some $n \geq 0$. Other values of m are of the form $m = \ell_n + k + 1$, for some $n \geq 0, 0 \leq k \leq n$, and it follows from (3.24) that, in this case,

$$V'_m \subset D(9k, 6\alpha_{n+1}) \subset D_k.$$

Therefore, by (3.39), (3.32) and using the fact that $\alpha_{k+1} \leq \alpha_k/6$, we have, for $z \in V_{\ell_n+k+1}, n \geq 0$ and $0 \leq k \leq n$,

$$\begin{aligned} |f(z) - \varphi(z)| &\leq \frac{\varepsilon_{\ell_n+k+1}}{\alpha_k^2} (6\alpha_{n+1})^2 \\ &= \frac{\alpha_{k+1}^2}{2^{k+2}\alpha_k^2} 36\alpha_{n+1}^2 \\ &\leq \frac{\alpha_{n+1}^2}{2^{k+2}} = \varepsilon_{\ell_n+k+1}. \end{aligned}$$

Thus (3.40) holds for all $m \geq 0$.

It now follows from (3.3), (3.4), (3.30) and (3.40) that, for $m \geq 0$,

$$\gamma_{m+1} \text{ surrounds } f(\gamma_m), \tag{3.41}$$

$$f(\Gamma_m) \text{ surrounds } \Gamma_{m+1}. \tag{3.42}$$

We now apply Theorem 2.1 to the Jordan curves $\gamma_m, \Gamma_m, m \geq 0$, the compact curves $L_n, n \geq 0$, and the bounded domain $D = D(-4, 1)$, noting that these sets satisfy the required hypotheses by construction and by (3.41), (3.42), (3.29) and (3.38). Part (i) of Theorem 3.1 now follows from Theorem 2.1, part (ii) follows from (3.40) together with the upper bounds for the errors that we obtained earlier, and part (iii) follows from (3.36) and (3.37).

Next we outline the proof of part (iv). The fact that $f : U_{\ell_{n+1}} \rightarrow U_{\ell_{n+1}+1}$ has degree d_{n+1} follows from the final statement of Theorem 2.1, since (3.23), (3.30) and (3.35) together imply that $f(\gamma_{\ell_{n+1}})$ and $f(\Gamma_{\ell_{n+1}})$ both wind exactly d_{n+1} times round the disc $D(\zeta_{\ell_{n+1}+1}, \rho_{\ell_{n+1}+1}/2)$; for the details of this argument see the proof of [BEF+21, Theorem 5.3]. Since φ is univalent in all other cases, the same argument applies to show that $f : U_m \rightarrow U_{m+1}$ is univalent in all other cases.

To complete the proof of Theorem 3.1, we note that the double inequality that compares the hyperbolic distances in U_{ℓ_n} between points of two orbits under f with the corresponding hyperbolic distances in the discs G_n follows by applying Lemma 2.5 with

$$s = 1 - \frac{3}{4} \text{dist}(\varphi(\gamma_{\ell_n-1}), \partial G_n), \quad r = r_{\ell_n} \quad \text{and} \quad R = R_{\ell_n},$$

and noting that $f^{\ell_{n+1}-\ell_n}(\overline{D(\kappa_n, r_{\ell_n})}) \subset D(\kappa_{n+1}, r_{\ell_{n+1}})$; we omit the details, which are similar to those given in the proof of the final statement of [BEF+21, Theorem 5.3]. \square

4. Preliminary results for Theorem 1.1

In this section we prove some results which we use in order to construct our examples. In particular, we obtain estimates on the orbits of points in a wandering domain U of a transcendental entire function f obtained by applying Theorem 3.1 with specific Blaschke

products b_n . Our first result is used repeatedly in our constructions and gives estimates of the distances between orbits under the function f and under the model function φ .

LEMMA 4.1. *Let f be a transcendental entire function with a wandering domain U arising from applying Theorem 3.1 with the Blaschke products b_n . Then, using the notation of Theorem 3.1,*

(a) *if $z, z' \in U_{\ell_n}$, for some $n \geq 0$, we have*

$$|f^{n+3}(z) - \varphi^{n+3}(z')| \leq \alpha_{n+1} + |b_n(z - \kappa_n) - b_n(z' - \kappa_n)|;$$

(b) *and hence, if $z, z' \in \overline{D(\zeta_0, r_0)} \subset U_0$, we have*

$$|f^{\ell_{n+1}}(z) - \varphi^{\ell_{n+1}}(z')| \leq \alpha_{n+1} + |b_n(f^{\ell_n}(z) - \kappa_n) - b_n(\varphi^{\ell_n}(z') - \kappa_n)|.$$

Proof. To prove part (a), we begin by considering the case that $z = z'$. We first use induction to show that, if $z \in U_{\ell_n}$, for some $n \geq 0$, then

$$|f^m(z) - \varphi^m(z)| \leq \sum_{k=0}^{m-1} \varepsilon_{\ell_n+k} \quad \text{for } 1 \leq m \leq n+2. \tag{4.1}$$

We note that, for $m = 1$, this holds by Theorem 3.1(ii). Now assume that (4.1) holds for some m , $1 \leq m < n+2$. We have

$$|f^{m+1}(z) - \varphi^{m+1}(z)| \leq |f(f^m(z)) - \varphi(f^m(z))| + |\varphi(f^m(z)) - \varphi^{m+1}(z)|. \tag{4.2}$$

Since $z \in U_{\ell_n}$, we have $f^m(z) \in U_{\ell_n+m} \subset D(\zeta_{\ell_n+m}, R_{\ell_n+m})$ and $\varphi^m(z) \in D(\zeta_{\ell_n+m}, R_{\ell_n+m})$. Also, φ is a translation on $D(\zeta_{\ell_n+m}, R_{\ell_n+m})$ and so, together with Theorem 3.1(ii), we can deduce from (4.2) that

$$|f^{m+1}(z) - \varphi^{m+1}(z)| \leq \varepsilon_{\ell_n+m} + |f^m(z) - \varphi^m(z)| \leq \sum_{k=0}^m \varepsilon_{\ell_n+k}.$$

Thus (4.1) holds as claimed.

Next, we note that for $z \in U_{\ell_n}$ we have $f^{n+2}(z), \varphi^{n+2}(z) \in \Delta'_n$, on which φ is a scaling by a factor of $1/\alpha_{n+1}$ followed by a translation and so, by (4.1) and Theorem 3.1(ii),

$$\begin{aligned} |f^{n+3}(z) - \varphi^{n+3}(z)| &\leq |f(f^{n+2}(z)) - \varphi(f^{n+2}(z))| + |\varphi(f^{n+2}(z)) - \varphi^{n+3}(z)| \\ &\leq \varepsilon_{\ell_n+n+2} + \frac{1}{\alpha_{n+1}} \sum_{k=0}^{n+1} \varepsilon_{\ell_n+k} \\ &\leq \frac{\alpha_{n+1}}{2^{n+3}} + \alpha_{n+1} \sum_{k=0}^{n+1} \frac{1}{2^{k+1}} \\ &< \alpha_{n+1}. \end{aligned}$$

This shows that

$$|f^{n+3}(z) - \varphi^{n+3}(z)| \leq \alpha_{n+1}, \tag{4.3}$$

which is the result of part (a) in the case that $z = z'$.

We now use this fact to prove part (a) in general. If $z, z' \in U_{\ell_n}$, for some $n \geq 0$, then it follows from (4.3) and the definition of φ that

$$\begin{aligned} |f^{n+3}(z) - \varphi^{n+3}(z')| &\leq |f^{n+3}(z) - \varphi^{n+3}(z)| + |\varphi^{n+3}(z) - \varphi^{n+3}(z')| \\ &\leq \alpha_{n+1} + |b_n(z - \kappa_n) - b_n(z' - \kappa_n)|. \end{aligned}$$

This completes the proof of part (a).

Now we suppose that $z, z' \in \overline{D(\zeta_0, r_0)}$. It follows from Theorem 3.1(i) that $z \in U_0$ and hence $f^{\ell_n}(z) \in U_{\ell_n}$, for $n \geq 0$. It also follows from (3.3) in the proof of Theorem 3.1 that $\varphi^{\ell_n}(z') \in D(\zeta_{\ell_n}, r_{\ell_n})$ and hence, by Theorem 3.1(i), that $\varphi^{\ell_n}(z') \in U_{\ell_n}$, for $n \geq 0$. So part (b) follows from part (a) by replacing z and z' by $f^{\ell_n}(z)$ and $\varphi^{\ell_n}(z')$, respectively. \square

Our next result gives a precise estimate for a Blaschke product that is used in one of our examples. We use this result in the proof of Lemma 4.3(a).

LEMMA 4.2. *Let $b(z) = ((z + 1/3)/(1 + z/3))^2$ and suppose that $0 < r < 1$. Then $0 < r < b(r) < 1$ and*

$$\left| \frac{b(x) - b(r)}{x - r} \right| < \frac{b^2(r) - b(r)}{b(r) - r} \quad \text{for } 0 < x < b(r). \tag{4.4}$$

Proof. Our proof is based on a useful relationship between the cross-ratio of four points $a < b < c < d$ on \mathbb{R} , defined as

$$(a, b, c, d) = \frac{(b - a)(d - c)}{(d - a)(c - b)},$$

and the Schwarzian derivative of a real function f , defined as

$$Sf = \frac{f'''}{f''} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2.$$

It is well known that if f is monotonic on an interval I and $Sf < 0$ on I , then

$$(f(a), f(b), f(c), f(d)) < (a, b, c, d) \quad \text{whenever } a, b, c, d \in I, a < b < c < d. \tag{4.5}$$

See, for example, de Melo and van Strien [dMvS89, §1] for a good account of the relationship between functions with negative Schwarzian and the cross-ratio, including a proof of the above fact. Other key properties (also mentioned in [dMvS89]) are that Möbius maps have zero Schwarzian and the composition rule for Schwarzians is

$$S(g \circ f)(x) = Sg(f(x))(f'(x))^2 + Sf(x).$$

Since the Schwarzian derivative of a Möbius map is zero on its domain in \mathbb{R} , it follows immediately from this composition rule that the function b has negative Schwarzian on the interval $(-3, \infty)$.

It is straightforward to check that 1 is a fixed point of the function b and that $r < b(r) < 1$, for $r \in (0, 1)$. Note also that b is increasing on $(-1/3, \infty)$ and convex on $(-3, 1)$. We first prove (4.4) in the case when $r < x < b(r)$, by considering the four points $r, x, b(r), 1$.

Since b is increasing on $(-1/3, \infty)$ and has negative Schwarzian there, we deduce that

$$\frac{(b(x) - b(r))(1 - b^2(r))}{(1 - b(r))(b^2(r) - b(x))} < \frac{(x - r)(1 - b(r))}{(1 - r)(b(r) - x)}.$$

Since b is convex on $(0, 1)$ we deduce that

$$\frac{1 - b(r)}{1 - r} < \frac{1 - b^2(r)}{1 - b(r)}.$$

We deduce from the previous two inequalities that

$$\frac{b(x) - b(r)}{b^2(r) - b(x)} < \frac{(1 - b(r))^2}{(1 - r)(1 - b^2(r))} \frac{x - r}{b(r) - x} < \frac{x - r}{b(r) - x},$$

and hence (by taking reciprocals and adding 1 to both sides) that

$$\frac{b(x) - b(r)}{b^2(r) - b(r)} < \frac{x - r}{b(r) - r}.$$

This proves (4.4) in the case when $r < x < b(r)$.

For the case when $0 < x < r$, similar reasoning can be used with the points $x, r, b(r), 1$, to deduce that

$$\frac{b(r) - b(x)}{b^2(r) - b(r)} < \frac{r - x}{b(r) - r}.$$

This completes the proof of Lemma 4.2. □

The following lemma describes dynamical properties of the transcendental entire functions arising from Theorem 3.1 when using specific Blaschke products of a certain form. Two of our examples will be constructed using these Blaschke products. The proof of this result takes several pages.

LEMMA 4.3. *Let b be a Blaschke product of degree 2, and let f be an entire function arising by applying Theorem 3.1 with $b_n = b$, for $n \geq 0$.*

- (a) *If $b(z) = ((z + 1/3)/(1 + z/3))^2$ then there exist $x, y \in U_0 \cap \mathbb{R}$, $N \in \mathbb{N}$ and $c > 0$, with $f^n(x) \neq f^n(y)$ for $n \geq 0$, such that*

$$f^{\ell_N}(x) = \kappa_N \quad \text{and} \quad \kappa_n + 1 - f^{\ell_n}(x) \sim \frac{c}{n^{1/2}} \quad \text{as } n \rightarrow \infty,$$

and

$$f^{\ell_N}(y) = \kappa_N + 1/9 \quad \text{and} \quad \kappa_n + 1 - f^{\ell_n}(y) \sim \frac{c}{(n + 1)^{1/2}} \quad \text{as } n \rightarrow \infty,$$

and moreover,

$$f^{\ell_n}(y) - f^{\ell_n}(x) = \frac{O(1)}{n^{3/2}} \quad \text{as } n \rightarrow \infty.$$

- (b) *If $b(z) = ((z + 1/2)/(1 + z/2))^2$ then there exist $x, y \in U_0 \cap \mathbb{R}$ and $N \in \mathbb{N}$, with $f^n(x) \neq f^n(y)$ for $n \geq 0$, such that*

$$f^{\ell_N}(x) = \kappa_n \quad \text{and} \quad \kappa_n + 1 - f^{\ell_n}(x) = c\lambda^n(1 + \eta_n) \quad \text{for } n \geq N,$$

and

$$f^{\ell_N}(y) = \kappa_n + 1/4 \quad \text{and} \quad \kappa_n + 1 - f^{\ell_n}(y) = c\lambda^{n+1}(1 + \xi_n) \quad \text{for } n \geq N,$$

where $c > 0$, $\lambda = 2/3$ and $\max\{|\eta_n|, |\xi_n|\} \leq 1/10$, for $n \geq N$.

Proof. First we observe that in both parts it is sufficient to prove the stated results about the behaviours of $f^{\ell_n - \ell_N}(x_N)$ and $f^{\ell_n - \ell_N}(y_N)$ when $x_N, y_N \in U_{\ell_N}$ for some particular positive integer N .

(a) Recall that, by the analysis of the behaviour in \mathbb{D} of the iterates of b near its parabolic fixed point 1 (see [BEF+21, Lemma 6.2(c)], for example), there are positive constants c and d such that

$$1 - b^n(0) \sim \frac{c}{n^{1/2}} \quad \text{as } n \rightarrow \infty, \tag{4.6}$$

and

$$b^{n+1}(0) - b^n(0) \sim \frac{d}{n^{3/2}} \quad \text{as } n \rightarrow \infty. \tag{4.7}$$

Therefore, we can choose N so large that

$$\frac{d}{2n^{3/2}} < b^{n+1}(0) - b^n(0) < \frac{2d}{n^{3/2}} \quad \text{for } n \geq N, \tag{4.8}$$

and also such that

$$\frac{4}{d6^n} \leq \frac{1}{10} \quad \text{for } n \geq N. \tag{4.9}$$

We then take $r_n = b^{n-N}(0)$, for $n \geq N$, and define

$$x_n = \kappa_N \quad \text{and} \quad x_{n+1} = f^{n+3}(x_n) \quad \text{for } n \geq N, \tag{4.10}$$

and

$$x'_n := \kappa_n + r_n \in G_n \cap \mathbb{R} \quad \text{for } n \geq N.$$

It follows from the definition of φ that

$$x'_{n+1} = \varphi^{n+3}(x'_n) = b(x'_n - \kappa_n) \quad \text{for } n \geq N.$$

We use Lemma 4.2 to show that the orbit of $x_N = \kappa_N$ under f closely follows that of x_N under φ . More precisely, we shall show that

$$|x_n - x'_n| < \frac{1}{10}(r_{n+1} - r_n) \quad \text{for } n \geq N + 1. \tag{4.11}$$

Note that it follows from (4.10) that $x'_n \in U_{\ell_n} \cap \mathbb{R}$ since $x'_N = x_N \in U_{\ell_N} \cap \mathbb{R}$ and f is a real entire function.

We shall prove (4.11) by using induction to show that

$$|x_n - x'_n| \leq (r_{n+1} - r_n) \sum_{k=N+1}^n \frac{1}{6^k(r_{k+1} - r_k)} \quad \text{for } n \geq N + 1. \tag{4.12}$$

Before proving (4.12), we show that it implies (4.11). Using (4.8) and (4.9), it follows that, for $n \geq N + 1$,

$$\begin{aligned} \frac{|x_n - x'_n|}{r_{n+1} - r_n} &\leq \sum_{k=N+1}^n \frac{1}{6^k(r_{k+1} - r_k)} \\ &\leq \sum_{k=N+1}^n \frac{2(k - N)^{3/2}}{d6^k} \\ &= \frac{2}{d6^{N+1}} \sum_{j=0}^{n-N-1} \frac{(j + 1)^{3/2}}{6^j} \\ &\leq \frac{4}{d6^{N+1}} < \frac{1}{10}, \end{aligned}$$

since the sum in the penultimate expression is dominated by the geometric series $1 + 1/2 + 1/4 + \dots$. Thus (4.11) holds.

To start the proof of (4.12), we have

$$|x_{N+1} - x'_{N+1}| = |f^{N+3}(x_N) - \varphi^{N+3}(x_N)| \leq \frac{1}{6^{N+1}},$$

by Theorem 3.1(iv), since $x_N \in U_{\ell_N}$. Now we assume that (4.12) holds for some $n \geq N + 1$ and deduce that it holds for $n + 1$. Note that, whenever (4.11) holds (and so whenever (4.12) holds), we have $x'_n \in [\kappa_n, \kappa_n + 1)$, by the definition of x_n .

By the definition of φ , Lemma 4.1(a), Lemma 4.2 and Theorem 3.1(iv), we have

$$\begin{aligned} |x_{n+1} - x'_{n+1}| &\leq \alpha_{n+1} + |b(x_n - \kappa_n) - b(x'_n - \kappa_n)| \\ &\leq \frac{1}{6^{n+1}} + |x_n - x'_n| \left(\frac{b^2(x'_n - \kappa_n) - b(x'_n - \kappa_n)}{b(x'_n - \kappa_n) - (x'_n - \kappa_n)} \right) \\ &= \frac{1}{6^{n+1}} + |x_n - x'_n| \left(\frac{r_{n+2} - r_{n+1}}{r_{n+1} - r_n} \right) \\ &\leq \frac{1}{6^{n+1}} + (r_{n+2} - r_{n+1}) \sum_{k=N+1}^n \frac{1}{6^k(r_{k+1} - r_k)} \\ &= (r_{n+2} - r_{n+1}) \sum_{k=N+1}^{n+1} \frac{1}{6^k(r_{k+1} - r_k)}. \end{aligned}$$

This proves (4.12), so (4.11) holds.

Next, we define

$$y_N = \kappa_N + b(0) = \kappa_N + 1/9 \quad \text{and} \quad y_{n+1} = f^{n+3}(y_n) \text{ for } n \geq N,$$

and

$$y'_n := \kappa_n + r_{n+1} \in G_n \cap \mathbb{R} \quad \text{for } n \geq N,$$

with the same value of N as used earlier. Then

$$y'_n - x'_n = r_{n+1} - r_n = b^{n+1-N}(0) - b^{n-N}(0) \quad \text{for } n \geq N, \tag{4.13}$$

and

$$y'_{n+1} = \varphi^{n+3}(y'_n) = b(y'_n - \kappa_n) \quad \text{for } n \geq N.$$

Reasoning as above, we obtain

$$|y_n - y'_n| \leq \frac{1}{10}(r_{n+2} - r_{n+1}) \quad \text{for } n \geq N + 1. \tag{4.14}$$

Combining (4.6) and (4.7) with (4.11) and (4.14), we obtain

$$\kappa_n + 1 - x_n \sim \frac{c}{n^{1/2}} \quad \text{as } n \rightarrow \infty$$

and

$$\begin{aligned} |y_n - x_n| &\leq |y_n - y'_n| + |y'_n - x'_n| + |x'_n - x_n| \\ &\leq \frac{1}{10}(r_{n+2} - r_{n+1}) + (r_{n+1} - r_n) + \frac{1}{10}(r_{n+1} - r_n) \\ &= \frac{O(1)}{n^{3/2}} \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which gives the required result by taking $x, y \in U_0$ such that $f^{\ell_N}(x) = x_N = \kappa_N$ and $f^{\ell_N}(y) = y_N = \kappa_N + b(0) = \kappa_N + 1/9$. Note that $y_n \neq x_n$ for $n \geq N$, by (4.11), (4.13) and (4.14), so we deduce that $f^n(x) \neq f^n(y)$ for $n \geq 0$.

(b) The proof of part (b) is similar to that of part (a), and we outline the argument briefly.

As in part (a), we take $r_n = b^{n-N}(0)$, for $n \geq N$, for some sufficiently large $N \in \mathbb{N}$ to be specified later in the proof, and put

$$x_N = \kappa_N \quad \text{and} \quad x_{n+1} = f^{n+3}(x_n) \quad \text{for } n \geq N, \tag{4.15}$$

and

$$x'_n := \kappa_n + r_n \in G_n \cap \mathbb{R} \quad \text{for } n \geq N,$$

so once again

$$x'_{n+1} = \varphi^{n+3}(x'_n) = b(x'_n - \kappa_n) \quad \text{for } n \geq N.$$

Now note that the function b has fixed point 1 with multiplier $\lambda = 2/3$. It follows that

$$1 - r_{n+1} = \lambda(1 - r_n)(1 + O(1 - r_n)) \quad \text{as } n \rightarrow \infty, \tag{4.16}$$

so, for some constant $c > 0$,

$$\kappa_n + 1 - x_n = 1 - r_n \sim c\lambda^n \quad \text{as } n \rightarrow \infty. \tag{4.17}$$

Also, since b is univalent in the disc $\{z : |z - 1| < 1\}$ (or by a direct calculation), we have

$$|b'(z)| \leq \lambda(1 + C|z - 1|) \quad \text{for } |z - 1| < 1/2,$$

where C is a positive constant, and so

$$|b(w) - b(z)| \leq \lambda(1 + C|z - 1|)|w - z| \quad \text{for } |w - z| < 1/4, |z - 1| < 1/4. \quad (4.18)$$

As in part (a), we show that the orbit of x_N under f closely follows that of x_N under φ . To be precise, we claim that for N sufficiently large we have

$$|x_n - x'_n| \leq \frac{C}{10}\lambda^n \quad \text{for } n \geq N. \quad (4.19)$$

Indeed, for $n \geq N$, we have

$$\begin{aligned} |x_{n+1} - x'_{n+1}| &\leq \alpha_{n+1} + |b(x_n - \kappa_n) - b(x_n - \kappa_n)| \\ &\leq \frac{1}{6^{n+1}} + \lambda(1 + C(1 - r_n))|x_n - x'_n|, \end{aligned}$$

by Lemma 4.1(a), Theorem 3.1(iv), (4.17) and (4.18), provided that N is sufficiently large. Since $x_N = x'_N = \kappa_N$, it follows easily by induction that, for $n \geq N + 1$, we have

$$\delta_n \leq \left(\prod_{k=N+1}^{\infty} (1 + C(1 - r_k)) \right) \left(\sum_{k=N+1}^n \frac{1}{(6\lambda)^k} \right) \quad \text{where } \delta_n = \frac{|x_n - x'_n|}{\lambda^n},$$

and (4.19) easily follows by (4.17) and by taking N sufficiently large.

We obtain the first estimate in part (b) by taking $x \in U_0$ such that $f^{\ell_N}(x) = x_N = \kappa_N$. The second estimate follows by a similar argument, but this time we use an orbit under f whose subsequence passing through U_{ℓ_n} , $n \geq N$, closely follows the sequence $y'_n := \kappa_n + r_{n+1}$, $n \geq N$, by taking $y \in U_0$ such that $f^{\ell_N}(y) = y_N = \kappa_N + b(0) = \kappa_N + 1/4$. The proof that $f^n(x) \neq f^n(y)$ for $n \geq 0$ uses (4.19) and is similar to that in part (a). \square

Finally in this section, we give several estimates for a Blaschke product used in another of our examples.

LEMMA 4.4. For $n \geq 0$, let $b_n(z) = \tilde{\mu}_n(\mu_n(z)^2)$, where

$$\mu_n(z) = \frac{z + s_n}{1 + s_n z} \quad \text{and} \quad \tilde{\mu}_n(z) = \frac{z - s_n^2}{1 - s_n^2 z},$$

and let

$$\lambda_n = \frac{2s_n}{1 + s_n^2},$$

where $s_n \in (0, 1)$. Then, for $n \geq 0$,

$$\lambda_n x \leq \left(\frac{x + \lambda_n}{1 + \lambda_n x} \right) x = b_n(x) \leq x \quad \text{for } 0 < x < 1, \quad (4.20)$$

and

$$\lambda_n(y - x) \leq b_n(y) - b_n(x) \leq \frac{2}{1 + \lambda_n}(y - x) \quad \text{for } 0 \leq x < y < 1. \quad (4.21)$$

Proof. For $x \in (0, 1)$ and $n \geq 0$, we have

$$\begin{aligned} b_n(x) &= \frac{((x + s_n)/(1 + s_n x))^2 - s_n^2}{1 - s_n^2((x + s_n)/(1 + s_n x))^2} \\ &= \frac{(x + s_n)^2 - s_n^2(1 + s_n x)^2}{(1 + s_n x)^2 - s_n^2(x + s_n)^2} \\ &= \frac{x^2 + 2s_n x - 2s_n^3 x - s_n^4 x^2}{1 + 2s_n x - 2s_n^3 x - s_n^4} \\ &= \left(\frac{(1 - s_n^4)x + (1 - s_n^2)2s_n}{1 - s_n^4 + (1 - s_n^2)2s_n x} \right) x \\ &= \left(\frac{(1 + s_n^2)x + 2s_n}{1 + s_n^2 + 2s_n x} \right) x \\ &= \left(\frac{x + \lambda_n}{1 + \lambda_n x} \right) x. \end{aligned}$$

Since

$$a \leq \frac{x + a}{1 + ax} \leq 1,$$

for $x, a \in [0, 1]$, part (a) follows.

For part (b), we deduce from the expression for b_n given in part (a) that, for $0 \leq x < y < 1$ and $n \geq 0$,

$$\begin{aligned} b_n(y) - b_n(x) &= y \left(\frac{y + \lambda_n}{1 + \lambda_n y} \right) - x \left(\frac{x + \lambda_n}{1 + \lambda_n x} \right) \\ &= (y - x) \frac{y + x + \lambda_n(1 + xy)}{(1 + \lambda_n y)(1 + \lambda_n x)}, \end{aligned}$$

and the conclusion then easily follows from the facts that $0 < \lambda_n < 1$, for $n \geq 0$, and $0 \leq x < y < 1$. □

5. Proof of Theorem 1.1.

In this section we construct six examples of bounded oscillating wandering domains, based on the two simply connected wandering domain classifications given in [BEF+21]. First, in terms of hyperbolic distances between orbits of points, simply connected wandering domains are classified as follows [BEF+21, Theorem A].

THEOREM 5.1. (First classification theorem) *Let U be a simply connected wandering domain of a transcendental entire function f and let U_n be the Fatou component containing $f^n(U)$, for $n \in \mathbb{N}$. Define the countable set of pairs*

$$E = \{(z, z') \in U \times U : f^k(z) = f^k(z') \text{ for some } k \in \mathbb{N}\}.$$

Then exactly one of the following statements holds.

- (1) $\text{dist}_{U_n}(f^n(z), f^n(z')) \xrightarrow{n \rightarrow \infty} c(z, z') = 0$ for all $z, z' \in U$, and we say that U is (hyperbolically) contracting.

- (2) $\text{dist}_{U_n}(f^n(z), f^n(z')) \xrightarrow[n \rightarrow \infty]{} c(z, z') > 0$ and $\text{dist}_{U_n}(f^n(z), f^n(z')) \neq c(z, z')$ for all $(z, z') \in (U \times U) \setminus E$, $n \in \mathbb{N}$, and we say that U is (hyperbolically) semi-contracting.
- (3) There exists $N > 0$ such that for all $n \geq N$, $\text{dist}_{U_n}(f^n(z), f^n(z')) = c(z, z') > 0$ for all $(z, z') \in (U \times U) \setminus E$, and we say that U is (hyperbolically) eventually isometric.

Next, in terms of convergence of orbits to the boundary there are again three types of simply connected wandering domains (see [BEF+21, Theorem C]), though only the latter two types (b) and (c) are realizable for oscillating wandering domains as explained in the introduction.

THEOREM 5.2. (Second classification theorem) *Let U be a simply connected wandering domain of a transcendental entire function f and let U_n be the Fatou component containing $f^n(U)$, for $n \in \mathbb{N}$. Then exactly one of the following statements holds.*

- (a) $\liminf_{n \rightarrow \infty} \text{dist}(f^n(z), \partial U_n) > 0$ for all $z \in U$, that is, all orbits stay away from the boundary.
- (b) There exists a subsequence $n_k \rightarrow \infty$ for which $\text{dist}(f^{n_k}(z), \partial U_{n_k}) \rightarrow 0$ for all $z \in U$, while for a different subsequence $m_k \rightarrow \infty$ we have that

$$\liminf_{k \rightarrow \infty} \text{dist}(f^{m_k}(z), \partial U_{m_k}) > 0 \quad \text{for } z \in U.$$

- (c) $\text{dist}(f^n(z), \partial U_n) \rightarrow 0$ for all $z \in U$, that is, all orbits converge to the boundary.

Each of the examples in this section is constructed by applying Theorem 3.1 with an appropriate choice of the Blaschke products b_n . We make repeated use of the following two results.

LEMMA 5.3. *Let f be a transcendental entire function with an orbit of wandering domains (U_n) arising from applying Theorem 3.1 with the Blaschke products $(b_n)_{n \geq 0}$ and suppose that there exist $s, t \in U_0$, $N \in \mathbb{N}$ with*

$$f^{\ell_N}(s), f^{\ell_N}(t) \in \overline{D(\kappa_N, r_{\ell_N})},$$

where the sequences (ℓ_n) , (κ_n) and (r_n) are as defined in Theorem 3.1.

- (a) If $\text{dist}_{G_n}(f^{\ell_n}(s), f^{\ell_n}(t)) \xrightarrow[n \rightarrow \infty]{} 0$ and $f^{\ell_n}(s) \neq f^{\ell_n}(t)$, for $n \geq 0$, then U_0 is contracting.
- (b) If $\liminf_{n \rightarrow \infty} \text{dist}_{G_n}(f^{\ell_n}(s), f^{\ell_n}(t)) > 0$ and $f : U_n \rightarrow U_{n+1}$ has degree greater than 1 for infinitely many $n \in \mathbb{N}$, then U_0 is semi-contracting.

Proof. (a) In this case it follows from the last part of Theorem 3.1 that

$$\text{dist}_{U_n}(f^{\ell_n}(s), f^{\ell_n}(t)) \xrightarrow[n \rightarrow \infty]{} 0.$$

It now follows from Theorem 5.1 that the only possibility is for U_0 to be contracting.

- (b) In this case it follows from the last part of Theorem 3.1 that

$$\liminf_{n \rightarrow \infty} \text{dist}_{U_n}(f^{\ell_n}(s), f^{\ell_n}(t)) > 0$$

and so U_0 is not contracting. Since $f : U_n \rightarrow U_{n+1}$ has degree greater than 1 for infinitely many $n \in \mathbb{N}$, we know that U_0 is not eventually isometric, and so it follows from Theorem 5.1 that U_0 is semi-contracting. \square

LEMMA 5.4. *Let f be a transcendental entire function with an orbit of wandering domains (U_n) arising from applying Theorem 3.1 with the Blaschke products $(b_n)_{n \geq 0}$ and let $s \in U_0$ with*

$$f^{\ell_n}(s) \in G_n \quad \text{for } n \geq 0,$$

where the sets G_n and the sequence (ℓ_n) , $n \geq 0$ are as defined in Theorem 3.1.

- (a) *If $\liminf_{n \rightarrow \infty} \text{dist}(f^{\ell_n}(s), \partial G_n) > 0$, then orbits of points in U_0 behave as described in Theorem 5.2(b).*
- (b) *If $\text{dist}(f^{\ell_n}(s), \partial G_n) \rightarrow 0$ as $n \rightarrow \infty$, then orbits of points in U_0 behave as described in Theorem 5.2(c).*

Proof. We begin by noting that it follows from Theorem 3.1 that

$$\begin{aligned} \varphi^m(\Delta_0) &= D(\zeta_m, \rho_m) \\ &= \begin{cases} \Delta_n = D(a_n, \alpha_n) & \text{if } m = \ell_n - 1 \text{ where } n \geq 0, \\ G_n = D(\kappa_n, 1) & \text{if } m = \ell_n, \text{ where } n \geq 0, \\ D(9k + 4\alpha_{n+1}, \alpha_{n+1}) \subset D_k & \text{if } m = \ell_n + k + 1, \text{ where } 0 \leq k \leq n. \end{cases} \end{aligned} \tag{5.1}$$

Since we know from Theorem 3.1(i) that the wandering domains U_m are approximated increasingly well by the sets $\varphi_m(\Delta_0)$ as $m \rightarrow \infty$, it follows that $\text{diam}U_m \rightarrow 0$ as $m \rightarrow \infty$ for $m \neq \ell_n, n \geq 0$. So, if $s \in U_0$, then

$$\text{dist}(f^m(s), \partial U_m) \rightarrow 0 \quad \text{as } m \rightarrow \infty, \quad m \neq \ell_n, \quad n \geq 0. \tag{5.2}$$

- (a) In this case it follows from (5.1) together with Theorem 3.1(i) that

$$\liminf_{n \rightarrow \infty} \text{dist}(f^{\ell_n}(s), \partial U_{\ell_n}) > 0.$$

Together with (5.2), this implies that orbits of points in U_0 behave as described in Theorem 5.2(b).

- (b) In this case it follows from (5.1) together with Theorem 3.1(i) that

$$\text{dist}(f^{\ell_n}(s), \partial U_{\ell_n}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Together with (5.2), this implies that orbits of points in U_0 behave as described in Theorem 5.2(c). \square

In some of the examples we make use of the following estimate for the hyperbolic distance in the unit disc.

Observation 1. For two points $r, s \in (0, 1)$ with $r < s$ we have that

$$\text{dist}_{\mathbb{D}}(r, s) = \int_r^s \frac{2dt}{1-t^2},$$

and so

$$\log \frac{1-r}{1-s} = \int_r^s \frac{dt}{1-t} \leq \text{dist}_{\mathbb{D}}(r, s) = \int_r^s \frac{2dt}{1-t} = 2 \log \frac{1-r}{1-s}. \tag{5.3}$$

We now give the examples that together prove Theorem 1.1. Examples 1, 2 and 3, which follow, correspond to the three cases of Theorem 5.1. Within each example we give two functions, corresponding to the two realizable cases of Theorem 5.2.

Example 1. (Two contracting wandering domains) For each of the cases (b) and (c) of Theorem 5.2, there exists a transcendental entire function f having a sequence of bounded, simply connected, oscillating contracting wandering domains (U_n) with the stated behaviour.

Proof. First example. We construct an oscillating contracting wandering domain U_0 with the behaviour described in Theorem 5.2(b) by applying Theorem 3.1 with $b_n(z) = z^2$, for $n \geq 0$.

We begin by considering the orbits of points in the disc $D(4, 1/12) \subset D(4, r_0)$ under iteration by φ . We note that, if $z \in D(\kappa_n, R_{\ell_n})$, for some $n \geq 0$, then

$$|\varphi^{n+3}(z) - \kappa_{n+1}| = |b_n(z - \kappa_n)| = |z - \kappa_n|^2.$$

So, if $z \in D(4, 1/12)$, then, for $n \geq 0$, we have

$$|\varphi^{\ell_n}(z) - \kappa_n| = |\varphi(z) - \kappa_0|^{2^n} = |z - 7|^{2^n} \leq (1/12)^{2^n} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{5.4}$$

Next we claim that, if $z \in D(4, 1/12)$, then

$$|f^{\ell_n}(z) - \varphi^{\ell_n}(z)| \leq \sum_{i=1}^n \frac{\alpha_i}{3^{n-i}} + \frac{|f(z) - \varphi(z)|}{3^n} \text{ for } n \geq 0, \tag{5.5}$$

and

$$|f^{\ell_n}(z) - \kappa_n| \leq \frac{1}{4} \text{ for } n \geq 0. \tag{5.6}$$

We prove (5.6) and (5.5) together using induction. First, we note that they are true when $n = 0$, since if $z \in D(4, 1/12)$, then $|f^{\ell_0}(z) - \varphi^{\ell_0}(z)| = |f(z) - \varphi(z)| \leq \varepsilon_0 \leq 1/24$, by Theorem 3.1(ii), and $|\varphi^{\ell_0}(z) - \kappa_0| = |\varphi(z) - \kappa_0| \leq 1/12$ by (5.4).

Next, we suppose that (5.5) and (5.6) hold for $n = m \geq 0$. It follows from these two estimates and Lemma 4.1(b), together with (5.4), that if $z \in D(4, 1/12)$, then

$$\begin{aligned} |f^{\ell_{m+1}}(z) - \varphi^{\ell_{m+1}}(z)| &\leq \alpha_{m+1} + |b_m(f^{\ell_m}(z) - \kappa_m) - b_m(\varphi^{\ell_m}(z) - \kappa_m)| \\ &= \alpha_{m+1} + |(f^{\ell_m}(z) - \kappa_m)^2 - (\varphi^{\ell_m}(z) - \kappa_m)^2| \\ &\leq \alpha_{m+1} + |f^{\ell_m}(z) - \varphi^{\ell_m}(z)| (|f^{\ell_m}(z) - \kappa_m| + |\varphi^{\ell_m}(z) - \kappa_m|) \\ &\leq \alpha_{m+1} + \frac{1}{3} |f^{\ell_m}(z) - \varphi^{\ell_m}(z)| \end{aligned}$$

$$\begin{aligned} &\leq \alpha_{m+1} + \frac{1}{3} \left(\sum_{i=1}^m \frac{\alpha_i}{3^{m-i}} + \frac{|f(z) - \varphi(z)|}{3^m} \right) \\ &= \sum_{i=1}^{m+1} \frac{\alpha_i}{3^{m+1-i}} + \frac{|f(z) - \varphi(z)|}{3^{m+1}}, \end{aligned}$$

which gives (5.5) with $n = m + 1$, and also

$$\begin{aligned} |f^{\ell_{m+1}}(z) - \kappa_{m+1}| &\leq |f^{\ell_{m+1}}(z) - \varphi^{\ell_{m+1}}(z)| + |\varphi^{\ell_{m+1}}(z) - \kappa_{m+1}| \\ &\leq \sum_{i=1}^{m+1} \frac{\alpha_i}{3^{m+1-i}} + \frac{|f(z) - \varphi(z)|}{3^{m+1}} + \left(\frac{1}{12}\right)^{2^{m+1}} \\ &\leq \sum_{i=1}^n \frac{1}{3^{m+1-i} 6^i} + \frac{1}{12} \frac{1}{3^{m+1}} + \left(\frac{1}{12}\right)^{2^{m+1}} \\ &\leq \frac{1}{6} + \frac{1}{12} \frac{1}{3^{m+1}} + \left(\frac{1}{12}\right)^{2^{m+1}} < \frac{1}{4}, \end{aligned}$$

which gives (5.6) with $n = m + 1$.

Since $\alpha_n \leq 1/6^n$ for $n \geq 0$, it follows from (5.5) that if $z \in D(4, 1/12)$, then

$$|f^{\ell_n}(z) - \varphi^{\ell_n}(z)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Together with (5.4), this implies that, if $z \in D(4, 1/12)$, then

$$|f^{\ell_n}(z) - \kappa_n| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (5.7)$$

We now use (5.7) together with Lemma 5.3 and Lemma 5.4 to show that the wandering domain U_0 has the required properties. First we take $s, t \in D(4, 1/12)$ such that $f(s), f(t) \in D(\kappa_0, r_1)$ with $f^{\ell_n}(s) \neq f^{\ell_n}(t)$, for $n \geq 0$. Since $G_n = D(\kappa_n, 1)$, for $n \geq 0$, it follows from (5.7) that

$$\text{dist}_{G_n}(f^{\ell_n}(s), f^{\ell_n}(t)) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and hence, by Lemma 5.3(a), U_0 is contracting.

Also, it follows from (5.7) that

$$\lim_{n \rightarrow \infty} \text{dist}(f^{\ell_n}(s), \partial G_n) = 1$$

and hence, by Lemma 5.4(a), orbits of points in U_0 behave as described in Theorem 5.2(b).

Second example. We construct an oscillating contracting wandering domain U_0 with the behaviour described in Theorem 5.2(c) by applying Theorem 3.1 with $b_n(z) = ((z + 1/3)/(1 + z/3))^2$, for $n \geq 0$. Let $x, y \in U_0$ be as in Lemma 4.3(a). Since $G_n = D(\kappa_n, 1)$, for $n \geq 0$, it follows from Lemma 4.3(a) and the hyperbolic metric

estimate (5.3) that

$$\begin{aligned} \text{dist}_{G_n}(f^{\ell_n}(x), f^{\ell_n}(y)) &\leq 2 \log \frac{\kappa_n + 1 - f^{\ell_n}(x)}{\kappa_n + 1 - f^{\ell_n}(y)} \\ &= 2 \log \left(1 + \frac{f^{\ell_n}(t) - f^{\ell_n}(x)}{\kappa_n + 1 - f^{\ell_n}(y)} \right) \\ &\sim 2 \log \left(1 + \frac{O(1)/n^{3/2}}{c(n+1)^{1/2}} \right) \\ &= \frac{O(1)}{n} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

It now follows from Lemma 5.3(a) that U_0 is contracting. We also know from Lemma 4.3(a) that

$$\text{dist}(f^{\ell_n}(x), \partial G_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and so it follows from Lemma 5.4(b) that orbits of points in U_0 behave as described in Theorem 5.2(c). □

Example 2. (Two semi-contracting wandering domains) For each of the cases (b) and (c) of Theorem 5.2, there exists a transcendental entire function f having a sequence of bounded, simply connected, oscillating semi-contracting wandering domains (U_n) with the stated behaviour.

Proof. First example. We construct an oscillating semi-contracting wandering domain with the behaviour described in Theorem 5.2(b) by applying Theorem 3.1 with $b_n(z) = \tilde{\mu}_n(\mu_n(z)^2)$, for $n \geq 0$, where

$$\mu_n(z) = \frac{z + s_n}{1 + s_n z} \quad \text{and} \quad \tilde{\mu}_n(z) = \frac{z - s_n^2}{1 - s_n^2 z}.$$

We shall use the estimates for b_n obtained in Lemma 4.4, and once again put

$$\lambda_n = \frac{2s_n}{1 + s_n^2} \quad \text{for } n \geq 0.$$

We now choose $s_n \in (0, 1)$ with $s_n \rightarrow 1$ as $n \rightarrow \infty$ so quickly that

$$\prod_{j=0}^{\infty} \lambda_j \geq 8/9 \quad \text{and} \quad \prod_{j=0}^{\infty} \frac{2}{1 + \lambda_j} \leq 4/3. \tag{5.8}$$

We first consider the orbit of the point 4 under iteration by f , noting that

$$\varphi^{\ell_n}(4) = \kappa_n \quad \text{for } n \geq 0. \tag{5.9}$$

It follows from (5.9), Lemma 4.1(b) and (4.20) in Lemma 4.4 that, for $n \geq 0$,

$$\begin{aligned} |f^{\ell_{n+1}}(4) - \kappa_{n+1}| &= |f^{\ell_{n+1}}(4) - \varphi^{\ell_{n+1}}(4)| \\ &\leq \alpha_{n+1} + |b_n(f^{\ell_n}(4) - \kappa_n) - b_n(\varphi^{\ell_n}(4) - \kappa_n)| \end{aligned}$$

$$\begin{aligned}
 &= \alpha_{n+1} + |b_n(f^{\ell_n}(4) - \kappa_n)| \\
 &\leq \alpha_{n+1} + |f^{\ell_n}(4) - \kappa_n|.
 \end{aligned}$$

Together with Theorem 3.1(ii) and (5.9), this implies that, for $n \geq 0$,

$$|f^{\ell_n}(4) - \kappa_n| \leq |f(4) - \kappa_0| + \sum_{i=1}^{n-1} \alpha_i = |f(4) - \varphi(4)| + \sum_{i=1}^{n-1} \alpha_i \leq \frac{1}{24} + \sum_{i=1}^{n-1} \frac{1}{6^i} < \frac{1}{4}. \tag{5.10}$$

Now we consider the orbit of $19/4$ under f . Once again, we begin by considering the orbit under φ . We claim that

$$\varphi^{\ell_n}(19/4) - \kappa_n \geq \frac{3}{4} \prod_{i=0}^{n-1} \lambda_i. \tag{5.11}$$

We prove (5.11) by induction, first noting that it holds for $n = 0$, since $\varphi^{\ell_0}(19/4) = \varphi(19/4) = \kappa_0 + 3/4$. Next, suppose that (5.11) holds for $n = m$. Then, by (4.20),

$$\begin{aligned}
 \varphi^{\ell_{m+1}}(19/4) - \kappa_{m+1} &= b_n(\varphi^{\ell_m}(19/4) - \kappa_m) \\
 &\geq \lambda_m(\varphi^{\ell_m}(19/4) - \kappa_m) \\
 &\geq \frac{3}{4} \lambda_m \prod_{i=0}^{m-1} \lambda_i = \frac{3}{4} \prod_{i=0}^m \lambda_i.
 \end{aligned}$$

Thus (5.11) holds for $n = m + 1$ and hence, by induction, for all $n \geq 0$. Together with (5.8), this implies that

$$\varphi^{\ell_n}(19/4) - \kappa_n \geq \frac{2}{3} \quad \text{for } n \geq 0. \tag{5.12}$$

Next we claim that, for $n \geq 0$,

$$|f^{\ell_n}(19/4) - \varphi^{\ell_n}(19/4)| \leq \sum_{i=1}^n \alpha_i \prod_{j=i}^{n-1} \frac{2}{1 + \lambda_j} + \prod_{i=0}^{n-1} \frac{2}{1 + \lambda_i} |f(19/4) - \varphi(19/4)|. \tag{5.13}$$

We prove (5.13) by induction, noting that it holds for $n = 0$. Next, suppose that (5.13) holds for $n = m \geq 0$. Then it follows from Lemma 4.1(b) and (4.21) that

$$\begin{aligned}
 &|f^{\ell_{m+1}}(19/4) - \varphi^{\ell_{m+1}}(19/4)| \\
 &\leq \alpha_{m+1} + |b_m(f^{\ell_m}(19/4) - \kappa_m) - b_m(\varphi^{\ell_m}(19/4) - \kappa_m)| \\
 &\leq \alpha_{m+1} + \frac{2}{1 + \lambda_m} |f^{\ell_m}(19/4) - \varphi^{\ell_m}(19/4)| \\
 &\leq \alpha_{m+1} + \frac{2}{1 + \lambda_m} \left(\sum_{i=1}^m \alpha_i \prod_{j=i}^{m-1} \frac{2}{1 + \lambda_j} + \prod_{i=0}^{m-1} \frac{2}{1 + \lambda_i} |f(19/4) - \varphi(19/4)| \right) \\
 &= \sum_{i=1}^{m+1} \alpha_i \prod_{j=i}^m \frac{2}{1 + \lambda_j} + \prod_{i=0}^m \frac{2}{1 + \lambda_i} |f(19/4) - \varphi(19/4)|.
 \end{aligned}$$

Thus (5.13) holds for $n = m + 1$ and hence, by induction, for all $n \geq 0$. It now follows from Theorem 3.1(ii) and (5.8) that

$$|f^{\ell_n}(19/4) - \varphi^{\ell_n}(19/4)| \leq \prod_{j=0}^{\infty} \frac{2}{1 + \lambda_j} \left(\sum_{i=1}^{\infty} \frac{1}{6^i} + \frac{1}{24} \right) \leq \frac{1}{4} \prod_{j=0}^{\infty} \frac{2}{1 + \lambda_j} \leq \frac{1}{3}. \tag{5.14}$$

It follows from (5.12), (5.14) and (5.10) that, for $n \geq 0$,

$$\begin{aligned} |f^{\ell_n}(19/4) - f^{\ell_n}(4)| &= |\varphi^{\ell_n}(19/4) - \kappa_n + f^{\ell_n}(19/4) - \varphi^{\ell_n}(19/4) + \kappa_n - f^{\ell_n}(4)| \\ &\geq |\varphi^{\ell_n}(19/4) - \kappa_n| - |f^{\ell_n}(19/4) - \varphi^{\ell_n}(19/4)| - |f^{\ell_n}(4) - \kappa_n| \\ &\geq \frac{2}{3} - \frac{1}{3} - \frac{1}{4} = \frac{1}{12}. \end{aligned}$$

Since $G_n = D(\kappa_n, 1)$, for $n \geq 0$, together with (5.10) this implies that

$$\liminf_{n \rightarrow \infty} \text{dist}_{G_n}(f^{\ell_n}(19/4), f^{\ell_n}(4)) > 0.$$

Also, it follows from Theorem 3.1(iv) that $f : U_{\ell_n} \rightarrow U_{\ell_{n+1}}$ has degree greater than 1, for $n \geq 0$. So, by Lemma 5.3(b), U_0 is semi-contracting.

Finally, it follows from (5.10) that

$$\liminf_{n \rightarrow \infty} \text{dist}(f^{\ell_n}(4), \partial G_n) > 0$$

and so, by Lemma 5.4(a), orbits of points in U_0 behave as described in Theorem 5.2(b).

Second example. We construct an oscillating semi-contracting wandering domain U_0 with the behaviour described in Theorem 5.2(c) by applying Theorem 3.1 with $b_n(z) = ((z + 1/2)/(1 + z/2))^2$, for $n \geq 0$. Let $x, y \in U$ be as in Lemma 4.3(b). Since $G_n = D(\kappa_n, 1)$, for $n \geq 0$, it follows from Lemma 4.3(b) and the hyperbolic metric estimate (5.3) that for n sufficiently large we have

$$\begin{aligned} \text{dist}_{G_n}(f^{\ell_n}(x), f^{\ell_n}(y)) &\geq \log \frac{\kappa_n + 1 - f^{\ell_n}(x)}{\kappa_n + 1 - f^{\ell_n}(y)} \\ &\geq \log \frac{c\lambda^n(1 - 1/10)}{c\lambda^{n+1}(1 + 1/10)} \\ &= \log \frac{9}{11\lambda} = \log \frac{27}{22} > 0, \end{aligned}$$

recalling that $\lambda = 2/3$.

Also, it follows from Theorem 3.1 that $f : U_{\ell_n} \rightarrow U_{\ell_{n+1}}$ has degree greater than 1, for $n \geq 0$, so U_0 is semi-contracting by Lemma 5.3(b). Finally, we know from Lemma 4.3(b) that

$$\text{dist}(f^{\ell_n}(x), \partial G_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and so, by Lemma 5.4(b), orbits of points in U_0 behave as described in Theorem 5.2(c). \square

Example 3. (Two eventually isometric wandering domains) For each of the cases (b) and (c) of Theorem 5.2, there exists a transcendental entire function f having a sequence of

bounded, simply connected, oscillating eventually isometric wandering domains (U_n) with the stated behaviour.

Proof. First example. We construct an oscillating eventually isometric wandering domain U_0 with the behaviour described in Theorem 5.2(b) by applying Theorem 3.1 with $b_n(z) = z$, for $n \geq 0$.

Since b_n is univalent, for $n \geq 0$, it follows from Theorem 3.1(iv) that $f : U_m \rightarrow U_{m+1}$ is also univalent, for $m \geq 0$. Thus U is eventually isometric.

We now consider the orbit of 4 under iteration by f . We claim that, for $n \geq 0$,

$$|f^{\ell_n}(4) - \varphi^{\ell_n}(4)| \leq \sum_{i=1}^n \alpha_i + |f(4) - \varphi(4)|. \tag{5.15}$$

We prove (5.15) by induction, noting that it is true for $n = 0$. Next, suppose that (5.15) holds for $n = m \geq 0$. Then it follows from Lemma 4.1(b) that

$$\begin{aligned} |f^{\ell_{m+1}}(4) - \varphi^{\ell_{m+1}}(4)| &= \alpha_{m+1} + |b_m(f^{\ell_m}(4) - \kappa_m) - b_m(\varphi^{\ell_m}(4) - \kappa_m)| \\ &= \alpha_{m+1} + |f^{\ell_m}(4) - \varphi^{\ell_m}(4)| \\ &= \alpha_{m+1} + \sum_{i=1}^m \alpha_i + |f(4) - \varphi(4)| \\ &= \sum_{i=1}^{m+1} \alpha_i + |f(4) - \varphi(4)|. \end{aligned}$$

Thus (5.15) holds for $n = m + 1$ and hence, by induction, for all $n \geq 0$.

Since $\alpha_n \leq 1/6^n$ for $n \geq 0$, and $|f(4) - \varphi(4)| \leq 1/24$, by Theorem 3.1(ii), it follows from (5.15) that

$$|f^{\ell_n}(4) - \varphi^{\ell_n}(4)| \leq 1/2 \quad \text{for } n \geq 0.$$

Since $\varphi^{\ell_n}(4) = \kappa_n$ and $G_n = D(\kappa_n, 1)$, for $n \geq 0$, it follows from Lemma 5.4(a) that orbits of points in U_0 behave as described in Theorem 5.2(b).

Second example. We construct an oscillating eventually isometric wandering domain U_0 with the behaviour described in Theorem 5.2(c) by applying Theorem 3.1 with $b_n(z) = b(z) = (z + 5/6)/(1 + 5z/6)$, for $n \geq 0$.

Since b_n is univalent, for $n \geq 0$, it follows from Theorem 3.1(iv) that $f : U_m \rightarrow U_{m+1}$ is also univalent, for $m \geq 0$. Thus U_0 is eventually isometric.

We now consider the orbit of 4 under iteration by φ , noting that

$$\varphi^{\ell_n}(4) = \kappa_n + b^n(0) \quad \text{for } n \geq 0.$$

The Blaschke product b has an attracting fixed point at 1 and we have

$$b^n(0) \rightarrow 1 \text{ as } n \rightarrow \infty \quad \text{and} \quad b^n(0) \geq 5/6 \quad \text{for } n \in \mathbb{N}, \tag{5.16}$$

and so

$$\text{dist}(\varphi^{\ell_n}(4), \partial G_n) \rightarrow 0 \text{ as } n \rightarrow \infty, \quad \text{and} \quad \varphi^{\ell_n}(4) - \kappa_n \geq 5/6 \quad \text{for } n \in \mathbb{N}. \tag{5.17}$$

We also note that if $0 \leq z_1, z_2 < 1$, then

$$|b(z_1) - b(z_2)| = \left| \frac{11(z_1 - z_2)}{(6 + 5z_1)(6 + 5z_2)} \right| \leq \frac{11|z_1 - z_2|}{36}. \tag{5.18}$$

Next, we take a point $x \in D(4, r_0)$ such that $f(x) = \kappa_0$, which is possible by Theorem 3.1(i), and consider the orbit of x under iteration by f . We claim that, for $n \geq 0$,

$$|f^{\ell_n}(x) - \varphi^{\ell_n}(4)| \leq \sum_{i=1}^n \alpha_i \left(\frac{11}{36}\right)^{n-i} \leq \frac{1}{2^n}. \tag{5.19}$$

We prove (5.19) by induction. First, we note that it is true if $n = 0$, since $f(x) - \varphi(4) = 0$. Next, suppose that (5.19) holds for $n = m \geq 0$. Then it follows from (5.17) and (5.19) that $f^{\ell_m}(x) > 0$, and so it follows from Lemma 4.1(b) together with (5.18) that, for $m \geq 0$,

$$\begin{aligned} |f^{\ell_{m+1}}(x) - \varphi^{\ell_{m+1}}(4)| &\leq \alpha_{m+1} + |b_m(f^{\ell_m}(x) - \kappa_m) - b_m(\varphi^{\ell_m}(4) - \kappa_m)| \\ &\leq \alpha_{m+1} + \frac{11|f^{\ell_m}(x) - \varphi^{\ell_m}(4)|}{36} \\ &\leq \alpha_{m+1} + \sum_{i=1}^m \alpha_i \left(\frac{11}{36}\right)^{m+1-i} \\ &= \sum_{i=1}^{m+1} \alpha_i \left(\frac{11}{36}\right)^{m+1-i} \\ &\leq \sum_{i=1}^{m+1} \frac{1}{6^i} \left(\frac{11}{36}\right)^{m+1-i} \leq \frac{m+1}{3^{m+1}} \leq \frac{1}{2^{m+1}}. \end{aligned}$$

Thus (5.19) holds for $n = m + 1$ and hence, by induction, for all $n \geq 0$.

It follows from (5.19) together with (5.17) that

$$\text{dist}(f^{\ell_n}(s), \partial G_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and so, by Lemma 5.4(b), orbits of points in U_0 behave as described in Theorem 5.2(c). \square

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