A Determinantal Expansion for a Class of Definite Integral

Part 5. Recurrence Relations

By L. R. Shenton

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1. We develop here the recurrence relations for the generalised C.F.'s introduced in Part 3 (Shenton[1] 1956). In the main the discussion will be limited to second order C.F.'s, but results for higher orders will be given when these are not complicated.

We shall give three forms of recurrence relation, one involving *recurrent* determinants, and another corresponding to the *even* and *odd* parts of a Stieltjes C.F. In addition we shall show how to write down directly the recurrence relations for a second order C.F. being given the first order C.F. Several numerical examples are given in illustration.

2.0. We consider the C.F. "corresponding"[2] to a determined Stieltjes moment problem, and write

$$F(z) = \int_0^\infty \frac{d\psi(x)}{x+z} = \frac{b_1}{z\,+}\,\frac{b_2}{1\,+}\,\frac{b_3}{z\,+}\,\frac{b_4}{1\,+}\cdots,\; z > 0, \tag{1}$$

and for the "contracted" form

$$F(z) = \frac{a_0}{z+c_1\,-}\,\frac{a_1}{z+c_2\,-}\,\frac{a_2}{z+c_3\,-}\cdots \tag{2}$$

where $\quad a_s = b_{2s}\,b_{2s+1},\ s > 0; \qquad a_0 = b_1,$

$$c_s = b^*_{2s-1} + b_{2s},\ s > 1; \qquad b^*_s = b_s,\ s > 1, \qquad b^*_1 = 0.$$

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[1] We shall refer to the previous four papers on this subject as **S1, S2, S3,** and **S4,** respectively.

[2] Stieltjes preferred to write the "corresponding" C.F. in the form

$$F(z) = \frac{1}{a_1 z\,+}\,\frac{1}{a_2\,+}\,\frac{1}{a_3 z\,+}\,\frac{1}{a_4\,+}\cdots,$$

in which case the Stieltjes moment problem is determined if $\sum_1^\infty a_s$ diverges, the a's being positive.

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The s^{th} convergent of (1) will be written $\chi_s(z)/w_s(z)$, and that of the *even* part $\chi_{2s}(z)/w_{2s}(z)$, where $\chi_0(z) = 0$, $\chi_1(z) = b_1$, $w_0(z) = 1$, $w_1(z) = z$. In the notation of §3 the expression (2) becomes

$$F(z) = \underset{s \to \infty}{\text{l. i. s.}} \frac{a_0 K_{s-1}(\beta_1, \alpha_1)}{K_s(\beta_0, \alpha_0)} \tag{3}$$

where $\beta_s = z + c_{s+1}$, $\alpha_s = \sqrt{a_{s+1}}$ and $K_s(\beta_0, \alpha_0)$ is a continuant determinant of order s with elements $\beta_0, \beta_1, \ldots$ along the diagonal through (1, 1) and elements $\alpha_0, \alpha_1, \ldots$ along the diagonals through (2, 1) and (1, 2).

The second order C.F. can be written [1]

$$F(z_1, z_2) = \int_0^\infty \frac{d\psi(x)}{(x+z_1)(x+z_2)} = \underset{s \to \infty}{\text{l. i. s.}}\ a_0 \frac{K_{s-1}(\gamma_1, \beta_1, \alpha_1)}{K_s(\gamma_0, \beta_0, \alpha_0)} \tag{4a}$$

where
$$\begin{aligned}
\alpha_s &= \sqrt{(a_{s+1}\, a_{s+2})}, && (4b)\\
\beta_s &= (p + c_{s+1} + c_{s+2})\sqrt{a_{s+1}},\\
\gamma_s &= q + pc_{s+1} + c^2_{s+1} + a^*_s + a_{s+1},\\
a^*_s &= a_s,\ s > 0, \qquad a^*_0 = 0,\\
(x+z_1)&(x+z_2) \equiv x^2 + px + q > 0,\ x \geqslant 0.
\end{aligned}$$

Similarly the third order C.F. may be expressed as

$$F(z_1, z_2, z_3) = \int_0^\infty \frac{d\psi(x)}{(x+z_1)(x+z_2)(x+z_3)} = \underset{s \to \infty}{\text{l. i. s.}}\ a_0 \frac{K_{s-1}(\delta_1, \gamma_1, \beta_1, \alpha_1)}{K_s(\delta_0, \gamma_0, \beta_0, \alpha_0)} \tag{5a}$$

where

$$\begin{aligned}
\alpha_s &= \sqrt{(a_{s+1}\, a_{s+2}\, a_{s+3})},\\
\beta_s &= (p + c_{s+1} + c_{s+2} + c_{s+3})\sqrt{(a_{s+1}\, a_{s+2})}, && (5b)\\
\gamma_s &= \{q + p(c_{s+1} + c_{s+2}) + c^2_{s+2} + c_{s+2}c_{s+1} + c^2_{s+1} + a_{s+2} + a_{s+1} + a^*_s\}\sqrt{a_{s+1}},\\
\delta_s &= r + qc_{s+1} + pc^2_{s+1} + c^3_{s+1} + p(a_{s+1} + a^*_s) + a_{s+1}c_{s+2} + 2a_{s+1}c_{s+1}\\
&\qquad + 2a^*_s\, c_{s+1} + a^*_s\, c,\\
a^*_s &= a_s,\ s > 0, \qquad a^*_0 = 0,\\
(x+z_1)&(x+z_2)(x+z_3) \equiv x^3 + px^2 + qx + r > 0,\ x \geqslant 0.
\end{aligned}$$

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[1] $K_s(\gamma_0, \beta_0, \alpha_0)$ is a determinant of order s with elements $\gamma_0, \gamma_1, \ldots$. along the diagonal through (1, 1), $\beta_0, \beta_1 \ldots$. along the diagonals through (2, 1) and (1, 2), and $\alpha_0, \alpha_1, \ldots$ along the diagonals through (3, 1) and (1, 3). The determinant $K_s(\gamma_0, \beta_0, \alpha_0)$ is symmetric with elements in five diagonals only, and may be regarded as a form of generalised continuant. The extension of the notation is obvious.

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2.1 To develop the numerators and denominators of (4a) and (5a) in terms of recurrent determinants we require the following lemma.

LEMMA. *If* $K_s\left(h_1, \begin{smallmatrix} g_1, f_1 \\ g_1^1, f_1^1 \end{smallmatrix}\right)$ *is a determinant of order s with elements* $f_1, f_2, \ldots$ *along the diagonal through* (1, 3), $g_1, g_2, \ldots$ *along the diagonal through* (1, 2), $h_1, h_2, \ldots$ *along the diagonal through* (1, 1) *and so on, then*[1]

$$K_{s-2}\left(g_1, \begin{smallmatrix} f_1 \\ h_2, g_2^1, f_2^1 \end{smallmatrix}\right) K_s\left(g_0, \begin{smallmatrix} f_0 \\ h_1, g_1^1, f_1^1 \end{smallmatrix}\right) = \begin{vmatrix} K_{s-1}\left(g_1, \begin{smallmatrix} f_1 \\ h_2, g_2^1, f_2^1 \end{smallmatrix}\right) & K_{s-1}\left(h_1, \begin{smallmatrix} g_1, f_1 \\ g_1^1, f_1^1 \end{smallmatrix}\right) \\ \prod\limits_{\lambda=0}^{s-2} f_\lambda & K_{s-1}\left(g_0, \begin{smallmatrix} f_0 \\ h_1, g_1^1, f_1^1 \end{smallmatrix}\right) \end{vmatrix}$$

The proof is straightforward. For consider $K_s\left(g_0, \begin{smallmatrix} f_0 \\ h_1, g_1^1, f_1^1 \end{smallmatrix}\right)$. Delete the first and last rows and columns, and use the remaining array as a pivot.[2] The result then follows. We now deduce that

$$\prod_{\lambda=0}^{s-2} f_\lambda K_{s-1}\left(h_1, \begin{smallmatrix} g_1, f_1 \\ g_1^1, f_1^1 \end{smallmatrix}\right) = \begin{vmatrix} K_{s-1}\left(g_1, \begin{smallmatrix} f_1 \\ h_2, g_2^1, f_2^1 \end{smallmatrix}\right) & K_{s-2}\left(g_1, \begin{smallmatrix} f_1 \\ h_2, g_2^1, f_2^1 \end{smallmatrix}\right) \\ K_s\left(g_0, \begin{smallmatrix} f_0 \\ h_1, g_1^1, f_1^1 \end{smallmatrix}\right) & K_{s-1}\left(g_0, \begin{smallmatrix} f_0 \\ h_1, g_1^1, f_1^1 \end{smallmatrix}\right) \end{vmatrix}. \tag{6}$$

Applying (6) to the numerator and denominator of (4a) we find

$$F(z_1, z_2) = \underset{s \to \infty}{\text{l. i. s.}}\ a_0 \frac{| V_s, W_{s+1} |}{| U_s, V_{s+1} |}, \tag{7}$$

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[1] A determinant of the form $K_s\left(g_0, \begin{smallmatrix} f_0 \\ h_1, g_1^1, f_1^1 \end{smallmatrix}\right)$ of order s, with elements $f_0, f_1, \ldots$ along the first superdiagonal, $g_0, g_1, \ldots$ along the principal diagonal, $h_1, h_2, \ldots$ along the first subdiagonal, and so on, will be referred to as a *recurrent determinant*, or simply a *recurrent*. By expanding a *recurrent* of this form by its last row, it will be seen that it follows a fourth order recurrence relation. Similarly a *recurrent* with n subdiagonals may be shown to follow a *recurrence* relation of order $n + 1$.

[2] See for example A. C. Aitken, *Determinants and Matrices* (fourth edition, Edinburgh, 1946,), pp. 48-49.

where $$U_s = K_s \left(\beta^*_{-1}, \begin{matrix} a^*_{-1} \\ \gamma_0, \end{matrix} \beta_0, a_0\right)$$

$$V_s = K_{s-1}\left(\beta_0, \begin{matrix} a_0 \\ \gamma_1, \end{matrix} \beta_1, a_1\right)$$

$$W_s = K_{s-2}\left(\beta_1, \begin{matrix} a_1 \\ \gamma_2, \end{matrix} \beta_2, a_2\right),$$

$\beta^*_s = \beta_s,\ a^*_s = a_s,\ s \geqslant 0;\qquad \beta^*_{-1} = 0,\qquad a^*_{-1} = 1,$

and the *recurrents* U_s, V_s, W_s follow the relation

$$y_s = \beta_{s-2} y_{s-1} - \gamma_{s-2} a^*_{s-3} y_{s-2} + \beta_{s-3} a_{s-3} a_{s-4} y_{s-3} - a_{s-3} a^2_{s-4} a_{s-5} y_{s-4}, \tag{8}$$

with
$$\begin{array}{llll} U_0 = 1, & U_1 = 0, & U_s = 0, & s < 0, \\ V_1 = 1, & V_2 = \beta_0, & V_s = 0, & s < 1, \\ W_2 = 1, & W_3 = \beta_1, & W_s = 0, & s < 2, \end{array}$$

the values of a_s, β_s, γ_s being given in (4b).

The s^{th} approximant to $F(z_1, z_2)$ depends upon the six terms U_s, U_{s+1}, V_s, V_{s+1}, W_s, W_{s+1}, each of which follows a recurrence relation of order four. Hence to advance the approximation process one stage, it is necessary to evaluate a value of each of U_s, V_s, W_s, and this will involve twelve calculations. We shall show in a later section that $| V_s, W_{s+1} |$ and $| U_s, V_{s+1} |$ (or equivalent expressions) follow recurrence relations of order five, so that there is perhaps an economy to be gained by this method.

A decreasing sequence of *upper bounds* may be derived from the expression.

$$F(z_1, z_2) = (q - \tfrac{1}{4}p^2)^{-1} a_0 - (q - \tfrac{1}{4}p^2)^{-1} \int_0^\infty \frac{(x + \frac{1}{2}p)^2\, d\psi(x)}{x^2 + px + q}, \tag{9}$$

and it is not difficult to show that the difference between the s^{th} approximations that arise from (9) and (4a) is

$$\frac{(q - \frac{1}{4}p^2)^{-1} \prod_{\lambda=0}^{s} a_\lambda}{K_s(\gamma_0, \beta_0, a_0)}, \tag{10}$$

it being assumed that $q - \frac{1}{4}p^2 > 0$.

2.2 For third order C.F's we require the following extension of the lemma:

$$Q_s^{(0)} K_s\left(i_1, \begin{matrix} h_1, g_1, f_1 \\ h_1^1, g_1^1, f_1^1 \end{matrix}\right) = \begin{vmatrix} T_s^{(1)} & T_{s+1}^{(0)} \\ T_{s-1}^{(1)} & T_s^{(0)} \end{vmatrix}, \tag{11}$$

where $Q_s^{(\lambda)} \equiv K_s\left(g_\lambda, \begin{matrix} f_\lambda \\ h_{\lambda+1}, i_{\lambda+2}, h_{\lambda+2}^1, g_{\lambda+2}^1; f_{\lambda+2}^1 \end{matrix}\right)$,

$$T_s^{(\lambda)} \equiv K_s\left(h_\lambda, \begin{matrix} g_\lambda, f_\lambda \\ i_{\lambda+1}, h_{\lambda+1}^1, g_{\lambda+1}^1; f_{\lambda+1}^1 \end{matrix}\right).$$

The proof depends on pivotal condensation methods and closely follows that of the lemma. Each determinant in the compound determinant in (11) is replaced by a compound determinant, using identities similar to (6). For example

$$\prod_{\lambda=0}^{s-1} f_\lambda . \; T_s^{(1)} = \begin{vmatrix} Q_s^{(0)} & Q_{s-1}^{(1)} \\ Q_{s+1}^{(0)} & Q_s^{(1)} \end{vmatrix}.$$

We find in this way the relation

$$\prod_{\lambda=-1}^{s-2} f_\lambda . \prod_{\lambda=0}^{s-1} f_\lambda \, K_s\left(i_1, \begin{matrix} h_1, g_1, f_1 \\ h_1^1, g_1^1, f_1^1 \end{matrix}\right)$$

$$= \begin{vmatrix} Q_{s-2}^{(1)} & Q_{s-1}^{(0)} & Q_s^{(-1)} \\ Q_{s-1}^{(1)} & Q_s^{(0)} & Q_{s+1}^{(-1)} \\ Q_s^{(1} & Q_{s+1}^{(0)} & Q_{s+2}^{(-1)} \end{vmatrix}, \tag{12}$$

it being assumed that $f_\lambda \neq 0$, $\lambda = -1, 0, 1, \ldots, s-1$.

Returning now to (5a) we derive from (12) an expansion for a third order C.F. in terms of *recurrent determinants*, namely

$$F(z_1, z_2, z_3) = \int_0^\infty \frac{d\psi(x)}{(x+z_1)(x+z_2)(x+z_3)}$$

$$= \underset{s \to \infty}{\text{l. i. s.}} \; a_0 \frac{| U_{s-1}, V_s, W_{s+1} |}{| X_{s-1}, V_s, W_{s+1} |} \tag{13}$$

where (i) $U_s = K_{s-2}\left(\beta_1, \begin{matrix} a_1 \\ \gamma_2, \delta_3, \gamma_3, \beta_3, a_3 \end{matrix}\right)$,

$$V_s = K_{s-1}\left(\beta_0, \begin{matrix} a_0 \\ \gamma_1, \delta_2, \gamma_2, \beta_2, a_2 \end{matrix}\right),$$

$$W_s = K_s \left(\beta_{-1}^*, \begin{matrix} a_{-1}^* \\ \gamma_0, \delta_1, \gamma_1, \beta_1, a_1 \end{matrix}\right),$$

$$X_s = K_{s+1}\left(\beta_{-2}^*, \begin{matrix} a_{-2}^* \\ \gamma_{-1}^*, \delta_0, \gamma_0, \beta_0, a_0 \end{matrix}\right);$$

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(ii) $\alpha_s^* = \alpha_s,\ \beta_s^* = \beta_s,\ \gamma_s^* = \gamma_s,\ s \geqslant 0;$

$\alpha_{-1}^* = \alpha_{-2}^* = 1,\ \beta_{-1}^* = \beta_{-2}^* = 0,\ \gamma_{-1}^* = 0,$

$\alpha_s, \beta_s, \gamma_s, \delta_s$ being given in (5b);

(iii) the *recurrents* U_s, V_s, W_s, X_s follow

$$y_s = \beta_{s-2}^* y_{s-1} - \gamma_{s-2}^* \alpha_{s-3}^* y_{s-2} + \delta_{s-2} \alpha_{s-3} \alpha_{s-4} y_{s-3} - \gamma_{s-3} \alpha_{s-3} \alpha_{s-4} \alpha_{s-5} y_{s-4}$$
$$+ \beta_{s-4} \alpha_{s-3} \alpha_{s-4} \alpha_{s-5} \alpha_{s-6} y_{s-5} - \alpha_{s-3} \alpha_{s-4} \alpha_{s-5}^2 \alpha_{s-6} \alpha_{s-7} y_{s-6},$$

with
$$\begin{array}{lll} U_2 = 1, & U_3 = \beta_1, & U_s = 0,\ s < 2, \\ V_1 = 1, & V_2 = \beta_0, & V_s = 0,\ s < 1, \\ W_0 = 1, & W_1 = 0, & W_s = 0,\ s < 0, \\ X_{-1} = 1, & X_0 = 0, & X_s = 0,\ s < -1. \end{array}$$

It will be seen that each of the elements U_s, V_s, W_s, X_s, occurring in the s^{th} convergent of a third order C.F. follows a sixth order recurrence relation, so that in setting up approximations to $F(z_1, z_2, z_3)$ we have to perform in general twenty-four calculations to obtain each new approximation. Similarly for a C.F. of order n associated with the function $F(z_1, z_2, \ldots, z_n)$, each approximation consists of the ratio of two n^{th} order determinants, an element of either determinant consisting of a *recurrent* which satisfies a recurrence relation of order $2n$. In general then each new approximation to $F(z_1, z_2, \ldots, z_n)$ will involve $2n(n+1)$ calculations, followed by the evaluation of two n^{th} order determinants.[1] We shall consider these more general C.F.'s and the associated recurrence relations in a forthcoming paper.

3. *A Fifth Order Recurrence Relation.*

3.1 We now establish a recurrence relation for the symmetric determinant $K_s(h_1, g_1, f_1)$. Expand K_s by its last row and column.

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[1] A referee has indicated to me that the recurrence relation followed by these two n^{th} order determinants will be of order $\binom{2n}{n}$ in general, or a little less owing to the symmetry involved. Thus for a third order C.F. the numerator and denominator of the s^{th} convergent will very likely satisfy a recurrence relation of order nineteen. Even if this recurrence could be found it might well be too complicated to be of much value, and the method of compound *recurrent* determinants seems to have a distinct advantage for C.F.'s of order three or more.

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Then

$$K_s = h_s K_{s-1} - g_{s-1}^2 K_{s-2} + 2g_{s-1} f_{s-2} K_{s-2}^* - f_{s-2}^2 h_{s-1} K_{s-3} + f_{s-2}^2 f_{s-3}^2 K_{s-4}, \quad (14)$$

$$s = 4, 5, \ldots,$$

where

$$K_s^* = \left| \begin{array}{ccc:cc} & \mathbf{K}_{s-1} & & & \\ & & & f_{s-2} & \\ & & & g_{s-1} & \\ \hdashline & & & f_{s-1} & g_s \end{array} \right|,$$

and $\mathbf{K}_{s-1}$ is the matrix consisting of the elements of $K_{s-1}(h_1, g_1, f_1)$. For example,

$$K_2^* = \begin{vmatrix} h_1 & g_1 \\ f_1 & g_2 \end{vmatrix}, \qquad K_3^* = \begin{vmatrix} h_1 & g_1 & f_1 \\ g_1 & h_2 & g_2 \\ & f_2 & g_3 \end{vmatrix}.$$

But expanding K_s^* by its last row, we have

$$K_s^* = g_s K_{s-1} - f_{s-1} K_{s-1}^*, \qquad s = 3, 4, \ldots. \quad (15)$$

Eliminating K^* from (14) and (15), we find

$$g_{s-2} K_s = \left(h_s g_{s-2} - f_{s-2} g_{s-1}\right) K_{s-1}$$
$$-\left(g_{s-1} g_{s-2} - h_{s-1} f_{s-2}\right)\left(g_{s-1} K_{s-2} - g_{s-2} f_{s-2} K_{s-3}\right)$$
$$-f_{s-3}^2 f_{s-2}\left(h_{s-2} g_{s-1} - f_{s-2} g_{s-2}\right) K_{s-4} + f_{s-2} f_{s-3}^2 f_{s-4}^2 g_{s-1} K_{s-5}, \quad (16)$$
$$s = 3, 4, \ldots, \qquad K_{-2} = K_{-1} = 0,\ K_0 = 1.$$

The recurrence relation (16) satisfied by $K_s(h_1, g_1, f_1)$ is of order five. By a slight modification of the method employed here it may be shown that the recurrence relation for the asymmetric determinant $K_s\left(h_1, \begin{matrix} g_1, & f_1 \\ g_1^1, & f_1^1 \end{matrix}\right)$ is of order six, but we do not require it in the present context.

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There are three interesting special cases:

(a) $f_j = 0$, when (16) reduces to

$$K_s = h_s K_{s-1} - g^2_{s-1} K_{s-2},$$

as we should expect since $K_s(h_1, g_1, 0)$ is now a "continuant" type of determinant.

(b) $g_j = 0$, when (16) becomes

$$K_s = h_s K_{s-1} - h_{s-1} f^2_{s-2} K_{s-3} + f^2_{s-2} f^2_{s-3} K_{s-4},$$

which is the recurrence relation for the product of two "continuants," and indeed [1]

$$K_{2s}(h_1, 0, f_1) = K_s(h^*_1, f^*_1) K_s(h^{**}_2, f^{**}_2),$$

$$K_{2s+1}(h_1, 0, f_1) = K_{s+1}(h^*_1, f^*_1) K_s(h^{**}_2, f^{**}_2),$$

where

$$h^*_s = h_{2s-1}, \qquad f^*_s = f_{2s-1},$$

$$h^{**}_s = h_{2s}, \qquad f^{**}_s = f_{2s}.$$

We shall treat the third example of reducibility in § 4, for it turns out to have several applications.

3.2 Now applying (16) to (4a) we may write the second order C.F. as

$$F(z_1, z_2) = \underset{s\to\infty}{\text{l.i.s.}}\ a_0 \frac{u_s}{v_s}, \tag{17}$$

where $\quad K_s(\gamma_0, \beta_0, a_0) = v_s \prod_{\lambda=1}^{s} a_\lambda, \qquad K_{s-1}(\gamma_1, \beta_1, a_1) = u_s \prod_{\lambda=1}^{s} a_\lambda,$

and u_s, v_s follow the recurrence

$$a_s\beta^1_{s-3} y_s = (\gamma_{s-1}\beta^1_{s-3} - a_{s-1}\beta^1_{s-2}) y_{s-1} - (\beta^1_{s-2}\beta^1_{s-3} - \gamma_{s-2})(\beta^1_{s-2} y_{s-2} - \beta^1_{s-3} y_{s-3})$$
$$- (\gamma_{s-3}\beta^1_{s-2} - a_{s-2}\beta^1_{s-3}) y_{s-4} + a_{s-3}\beta^1_{s-2} y_{s-5}, \tag{18}$$

where $\beta^1_s \surd a_{s+1} = \beta_s$, and a_s, β_s, γ_s, a_s are given in (4b), the initial values being

$$u_0 = 0, \qquad a_1 u_1 = 1, \qquad a_1 a_2 u_2 = \gamma_1, \qquad u_s = 0,\ s < 0;$$

$$v_0 = 1, \qquad a_1 v_1 = \gamma_0, \qquad a_1 a_2 v_2 = \gamma_0\gamma_1 - \beta^2_0, \qquad v_s = 0,\ s < 0.$$

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[1] The result has been noted by T. Muir, *Proc. Edinburgh Math. Soc.* ii (1884), 16-18.

3.3 As an illustration consider the "J" fraction expansion, convergent for $z > 0$.

$$\int_0^1 \frac{dx}{x+z} = \frac{1}{z+\frac{1}{2}-}\,\frac{a_1}{z+\frac{1}{2}-}\,\frac{a_2}{z+\frac{1}{2}-}\,\frac{a_3}{z+\frac{1}{2}-}\cdots, \tag{19}$$

where $a_s = s^2/(16s^2-4)$, from which we deduce the second order C.F. expansion

$$\int_0^1 \frac{dx}{x^2+z^2} = z^{-1} \text{ arc tan } z^{-1} = \underset{s\to\infty}{\text{l.i.s.}} \frac{u_s}{v_s},\ z \neq 0, \tag{20}$$

where u_s and v_s follow

$$\begin{aligned} a_s y_s = (z^2+a_s+\tfrac{1}{4})\, y_{s-1} &+ (z^2+a_{s-1}+a_{s-2}-\tfrac{3}{4})\,(y_{s-2}-y_{s-3}) \\ &- (z^2+a_{s-3}+\tfrac{1}{4})\, y_{s-4} + a_{s-3}\, y_{s-5}, \qquad s = 3, 4, \ldots, \end{aligned} \tag{21}$$

and the initial values are

$$\begin{aligned} &u_s=0,\ s<0, \qquad && u_0=0,\ u_1=12,\ u_2=3\,(60z^2+24); \\ &v_s=0,\ s<0, \qquad && v_0=1,\ v_1=4\,(3z^2+1),\ v_2=3\,(60z^2+44z+3). \end{aligned} \tag{21a}$$

For example, using (21) it will be found that

$$\begin{cases} 3u_3=16\,(525z^4+410z^2+45), \\ 3v_3=16\,(525z^6+585z^4+135z^2+3), \end{cases}$$

$$\begin{cases} 3u_4=132{,}300z^6+153{,}300z^4+41{,}300z^2+1{,}800, \\ 3v_4=132{,}300z^8+197{,}400z^6+80{,}640z^4+8{,}100z^2+75. \end{cases}$$

The expansion indicated in (20)-(21) is not the same (apart from the approximations for $s = 0, 1$) as the even part of the *hypergeometric* C.F.

$$z^{-1} \text{ arctan } z^{-1} = \frac{1}{z^2+}\,\frac{b_1}{1+}\,\frac{b_2}{z^2+}\,\frac{b_3}{1+}\,\frac{b_4}{z^2+}\cdots,$$

where $b_s = s^2/(4s^2-1)$.

4. *A Reducible Case of the Fifth Order Recurrence Relation.*

4.1 The recurrence relation (16) reduces to a fourth order one when the C.F. in (2) takes on the special form

$$F(z) = \frac{a_0}{z-}\,\frac{a_1}{z-}\,\frac{a_2}{z-}\,\frac{a_3}{z-}\cdots. \tag{22}$$

Proceeding formally at first and writing $K_{s-1}(\gamma_1, \beta_1, a_1) = T_s^*$, $K_s(\gamma_0, \beta_0, a_0) = T_s$, we find that the recurrence for T_s^* and T_s becomes

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$$y_s = (q + a_s)\, y_{s-1} + a_{s-1}\,(q - p^2 + a_{s-1} + a_{s-2})\, y_{s-2}$$
$$- a_{s-1}\, a_{s-2}\,(q - p^2 + a_{s-1} + a_{s-2})\, y_{s-3} - a_{s-1}\, a_{s-2}\, a_{s-3}\,(q + a_{s-3})\, y_{s-4}$$
$$+ a_{s-1}\, a_{s-2}\, a^2_{s-3}\, a_{s-4}\, y_{s-5}, \qquad s = 3, 4, \ldots, \tag{23}$$

where $q = z_1 z_2$, $p = z_1 + z_2$, $\qquad T_s = 0,\ s < 0, \qquad T^*_s = 0,\ s \leqslant 0.$

Now (23) may be written

$$\Phi_s(y) - a_{s-1}\, \Phi_{s-1}(y) = 0, \tag{24}$$

where $\quad \Phi_s(y) \equiv y_s - (q + a_s - a_{s-1})\, y_{s-1} - a_{s-1}(2a_{s-1} - p^2 + 2q)\, y_{s-2}$

$$- a_{s-1}\, a_{s-2}\,(q + a_{s-2} - a_{s-1})\, y_{s-3} + a_{s-1}\, a^2_{s-2}\, a_{s-3}\, y_{s-4}. \tag{25}$$

But it is easily verified that $\Phi_2(T) = 0$. Hence from (25)

$$T_s = (q + a_s - a_{s-1})\, T_{s-1} + a_{s-1}\,(2a_{s-1} - p^2 + 2q)\, T_{s-2}$$
$$+ a_{s-1}\, a_{s-2}\,(q + a_{s-2} - a_{s-1})\, T_{s-3} - a_{s-1}\, a^2_{s-2}\, a_{s-3}\, T_{s-4}, \tag{26}$$
$$s = 2, 3, \ldots,$$

with $\qquad T_0 = 1, \qquad T_1 = q + a_1, \qquad T_s = 0,\ s < 0.$

Similarly it will be found that

$$\Psi_s(T^*) - a_{s-1}\, \Psi_{s-1}(T^*) = 0, \tag{27}$$

where $$\Psi_s(T^*) \equiv \Phi_s(T^*) - 2 \prod_{\lambda=0}^{s-1} a_\lambda.$$

But since $\Psi_2(T^*) = 0$, it follows that the recurrence for T^*_s is

$$T^*_s = (q + a_s - a_{s-1}) T^*_{s-1} + a_{s-1}\,(2a_{s-1} - p^2 + 2q)\, T^*_{s-2}$$
$$+ a_{s-1} a_{s-2}\,(q + a_{s-2} - a_{s-1})\, T^*_{s-3} - a_{s-1} a^2_{s-2} a_{s-3}\, T^*_{s-4} + 2 \prod_{\lambda=0}^{s-1} a_\lambda,$$
$$s = 2, 3, \ldots, \tag{28}$$

with $\qquad T^*_1 = a_0, \qquad T^*_s = 0,\ s \leqslant 0.$

4.2 Returning to §2 we observe that in (1) $b_s > 0$, $s = 1, 2, \ldots$. Hence c_s cannot be zero for a Stieltjes C.F. We may ask the question then as to how the value of a_s in (22) must be restricted so that for certain values of z_1 and z_2 it will be true to assert that

$$\frac{F(z_1) - F(z_2)}{z_2 - z_1} = \lim_{s \to \infty} \frac{T^*_s}{T_s}.$$

We can give at this stage only a partial answer to this question. First we refer to the theory of the *Hamburger moment problem.* Let the expansion in descending powers of z of $F(z)$ be

$$F(z) \sim \frac{\mu_0}{z} + \frac{\mu_2}{z^3} + \frac{\mu_4}{z^5} + \dots \tag{29}$$

and assume that

$$\begin{cases} a_s \geqslant 0, & s = 0, 1, 2, \dots, \\ \sum\limits_{n=1}^{\infty} \mu_{2n}^{-1/2n} = \infty . \end{cases} \tag{30}$$

Then there exists a unique bounded non-decreasing function $\psi(x)$ in the interval $(-\infty, \infty)$ such that[1] $\int_{-\infty}^{\infty} x^{2n} d\psi(x) = \mu_{2n}, \int_{-\infty}^{\infty} x^{2n+1} d\psi(x) = 0.$

All we have to do now is to justify the Parseval expansion for $\int_{-\infty}^{\infty} \{f(x)\}^2 d\bar{\psi}(x)$ where $f(x) = \{(x+z_1)\ (x+z_2)\}^{-1}$,

$$\bar{\psi}(x) = \int_{-\infty}^{x} (z_1+t)\ (z_2+t)\ d\psi(t),$$

the argument being similar to that used in §2 of **S4**. It turns out then, using the theorem of M. Riesz (Shohat and Tamarkin, *loc. cit.*, p. 62) regarding the solution of a *determined* Hamburger moment problem, that under the conditions in (30) we have

$$F(z_1, z_2) = \int_{-\infty}^{\infty} \frac{d\psi(x)}{(x+z_1)\ (x+z_2)} = \underset{s\to\infty}{\text{l. i. s.}} \frac{T_s^*}{T_s}, \begin{cases} z_1 = x_1 + iy_1, \\ z_2 = x_1 - iy_1, \end{cases} \tag{31}$$

provided in addition $(x+z_1)\,(x+z_2) > 0$ for all real x. Again using (34) and (35) of **S4** we can set up a decreasing sequence of upper bounds, the difference between corresponding s^{th} approximants being $\prod_{\lambda=0}^{s} a_\lambda/(y_1^2 T_s)$, where $y_1 = \text{Im}z$.

Secondly it may be possible to justify (31) if we are given a definite integral for the C.F. in (22) of the form $\int_a^b \frac{d\psi(x)}{x+z}$ and can justify the application of Parseval's theorem as in §B of **S2**. It would

[1] See for example Shohat, J. A. and Tamarkin, J.D., *The Problem of Moments*, p. 5 and p. 19 (New York : American Mathematical Society, 1943).

be of some interest to know what are the weakest restrictions on the a's in (22) to justify a statement similar to (31).

4.2. We now give several examples in illustration.

Example 1. Let
$$F(z) = \tan(\tfrac{1}{2}z^{-1}),$$
so that
$$\mu_{2n} = 2B_{n+1}(2^{2n+2}-1)/(2n+2)!,$$
where B_n is a *Bernoulli* number and $B_1 = 1/6$, $B_2 = 1/30$ etc. Using Lambert's C.F.[1] for $F(z)$ we have $a_0 = \frac{1}{2}$, $a_s = (16s^2-4)^{-1}$, $s = 1, 2, \ldots$ Moreover it is easily verified that $\mu_{2n}^{1/2n} \sim 1/\pi$, so that (30) is satisfied. Hence with the appropriate value of a_s in (26) and (28) we have
$$\frac{\sinh y_1^1}{y_1(\cosh y_1^1 + \cos x_1^1)} = \underset{s\to\infty}{\text{l.i.s.}} \frac{T_s^*}{T_s}, \begin{cases} y_1 \neq 0, \\ z_1 = x_1 + iy_1 \\ z_2 = x_1 - iy_1. \end{cases} \tag{32}$$
where $x_1^1(x_1^2 + y_1^2) = x_1$, $y_1^1(x_1^2 + y_1^2) = y_1$. In particular if $x_1 = 3$, $y_1 = 4$ then $\lambda T_4^* = 398, 499, 385, 800$, $\lambda T_4 = 19, 896, 118, 681, 110$ where $\lambda^{-1} = a_4 a_3 a_2 a_1 a_0$, giving the lower bound $0{\cdot}020, 029, 001, 243, 260$, and using the corresponding upper bound it turns out that the error in this cannot exceed $1{\cdot}6 \times 10^{-15}$.

In a similar way, taking $F(z)$ to be $2z - \cot(\frac{1}{2}z^{-1})$, so that $a_0 = 1/6$, $a_s = [(4s+2)(4s+6)]^{-1}$, $s = 1, 2, \ldots.$, it may be shown that
$$\frac{\sinh y_1^1}{y_1(\cosh y_1^1 - \cos x_1^1)} - 2 = \text{l.i.s.} \frac{T_s^*}{T_s}, \; y_1 \neq 0. \tag{33}$$
In each case the recurrence relation for T_s is (26) and that for T_s^* is (28) with the appropriate value of a_s, $s = 0, 1, 2, \ldots.$

Example 2. It has been indicated by Stieltjes[2] that
$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\operatorname{sech}(\frac{1}{2}\pi x)dx}{x+z} = \frac{1}{z-}\,\frac{1^2}{z-}\,\frac{2^2}{z-}\,\frac{3^2}{z-}\ldots, \; \text{Im}(z) \neq 0, \tag{34a}$$
$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{x\operatorname{cosech}(\frac{1}{2}\pi x)\,dx}{x+z} = \frac{1}{z-}\,\frac{1.2}{z-}\,\frac{2.3}{z-}\,\frac{3.4}{z-}\ldots, \; \text{Im}(z) \neq 0. \tag{34b}$$
It may be shown that the Hamburger moment problem is determined in each case[3], and it follows that for $z_1 = x_1 + iy_1$, $z_2 = x_1 - iy$, $y_1 \neq 0$

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[1] See for example Perron, O., *Die Lehre von den Kettenbrüchen*, p. 354, (Berlin, 1913).

[2] *Correspondance d'Hermite et de Stieltjes*, p. 360 (Paris, 1905).

[3] Compare also Wall, H. S., *Continued Fractions*, p. 366, Example 2 (New York, 1948). We may also recall that the Hamburger moment problem
$$\mu_n = \int_{-\infty}^{\infty} x^n e^{-by}\,dx,\; y = |x|^a,\; a > 1,\; b > 0,\; n = 0, 1, 2, \ldots \text{ is determined.}$$

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the second order C.F.'s indicated in (31) converge to the corresponding value of $F(z_1, z_2)$.

Example 3. Let $F(z) = \int_{-\infty}^{\infty} \frac{g(x)dx}{x+z}$,

where $g(x) = e^{-\frac{1}{2}x^2}/\sqrt{(2\pi)}$, with Im $(z) \neq 0$, so that

$$F(z) = \frac{1}{z-}\,\frac{1}{z-}\,\frac{2}{z-}\,\frac{3}{z-}\cdots. \tag{35}$$

From P. 3, §B of **S2** we can conclude that the second order C.F. converges for $y_1 \neq 0$, and indeed

$$\int_{-\infty}^{\infty} \frac{g(x)dx}{(x+x_1)^2+y_1^2} = \underset{s\to\infty}{\text{l. i. s.}} \frac{T_s^*}{T_s},$$

where

$$T_s^* = (x_1^2+y_1^2+1)\,T_{s-1}^* + 2\,(s-1)\,(s-1+y_1^2-x_1^2)\,T_{s-2}^*$$
$$+\,(s-1)\,(s-2)\,(x_1^2+y_1^2-1)\,T_{s-3}^* - (s-1)\,(s-2)^2\,(s-3)\,T_{s-4}^*$$
$$+\,2(s-1)!, \qquad s = 2, 3, \ldots, \tag{36}$$

and the recurrence for T_s is exactly the same except that the factorial term is omitted. The initial values are

$$T_s^* = 0,\ s<1, \qquad T_1^* = 1$$
$$T_s = 0,\ s<0, \qquad T_0 = 1, \qquad T_1 = x_1^2+y_1^2+1.$$

A numerical example will be found in §C 4 of **S2**.

5. *A Recurrence Relation with* Even *and* Odd *Parts.*

5.1 For the C.F. given in (1) there are recurrence relations corresponding to the *even* and *odd* parts, namely

$$\begin{aligned} w_{2s}(z) &= w_{2s-1}(z) + b_{2s}\,w_{2s-2}(z), \\ w_{2s+1}(z) &= zw_{2s}(z) + b_{2s+1}\,w_{2s-1}(z). \end{aligned} \tag{37}$$

The question naturally arises as to whether there is a similar structure for higher order C.F.'s. We give here the result for a second order C.F. Starting with the form of the denominator of (4a) given in (30) of **S4**, we have, assuming for the moment $z_1 \neq z_2$,

$$\begin{aligned} K_s(\gamma_0, \beta_0, \alpha_0) &= \mid w_{2s}(z_1),\ w_{2s+2}(z_2) \mid \div (z_2 - z_1), \\ &= k_{2s}, \end{aligned} \tag{38}$$

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say. Using (37) we readily find that

$$k_{2s} = y_{2s} + b_{2s+1}\, b_{2s}\, k_{2s-2}, \tag{39}$$

where $$y_s = w_s(z_1)w_s(z_2).$$

Similarly if

$$k_{2s-1} = \mid w_{2s-1}(z_1),\ w_{2s+1}(z_2) \mid \div (z_2 - z_1), \tag{40}$$

then $$k_{2s-1} = y_{2s-1} + b_{2s}\, b_{2s-1} k_{2s-3}. \tag{41}$$

But $$y_{2s} = y_{2s-1} + b_{2s}^2\, y_{2s-2} + b_{2s}\, \Theta_{2s-1}, \tag{42}$$

where $$\Theta_{2s-1} = w_{2s-1}\,(z_1)\; w_{2s-2}\;(z_2) + w_{2s-1}\;(z_2)\; w_{2s-2}\;(z_1)$$
$$= (z_1 + z_2)\; y_{2s-2} + b_{2s-1}\;[w_{2s-2}(z_1)w_{2s-3}(z_2) + w_{2s-2}(z_2)w_{2s-3}(z_1)].$$

Hence $$\Theta_{2s-1} = (z_1 + z_2)y_{2s-2} + 2b_{2s-1}y_{2s-3} + b_{2s-1}b_{2s-2}\Theta_{2s-3},$$

and so from (42) we have

$$y_{2s} - y_{2s-1} - b_{2s}^2 y_{2s-2} = b_{2s}(z_1+z_2)y_{2s-2} + 2b_{2s}b_{2s-1}y_{2s-3}$$
$$+ b_{2s}b_{2s-1}(y_{2s-2} - y_{2s-3} - b_{2s-2}^2 y_{2s-4}),$$

from which, using (39) and (41), we deduce

$$k_{2s} = k_{2s-1} + b_{2s}(z_1 + z_2 + b_{2s+1} + b_{2s} + b_{2s-1})k_{2s-2}$$
$$- b_{2s}b_{2s-1}b_{2s-2}(z_1 + z_2 + b_{2s} + b_{2s-1} + b_{2s-2})k_{2s-4}$$
$$- b_{2s}b_{2s-1}b^2{}_{s-2}\, b_{2s-3}k_{2s-5} + b_{2s}b_{2s-1}b_{2s-2}^2 b_{2s-3}b_{2s-4}k_{2s-6}. \tag{43}$$

5.2 Similarly from (37) it follows that

$$y_{2s+1} = z_1 z_2 y_{2s} + b_{2s+1}^2 y_{2s-1} + b_{2s+1}\Phi_{2s}, \tag{44}$$

where $$\Phi_{2s} = z_1 w_{2s}(z_1)w_{2s-1}(z_2) + z_2 w_{2s}(z_2)\, w_{2s-1}(z_1)$$
$$= (z_1 + z_2)y_{2s-1} + b_{2s}[z_1 w_{2s-2}(z_1)w_{2s-1}(z_2) + z_2 w_{2s-2}(z_2)w_{2s-1}(z_1)],$$

or $$\Phi_{2s} = (z_1 + z_2)y_{2s-1} + 2z_1 z_2 b_{2s}y_{2s-2} + b_{2s}b_{2s-1}\Phi_{2s-2}. \tag{45}$$

Thus

$$y_{2s+1} = z_1 z_2 y_{2s} + b_{2s+1}^2 y_{2s-1} + b_{2s+1}(z_1 + z_2)y_{2s-1} + 2z_1 z_2 b_{2s+1}b_{2s}y_{2s-2}$$
$$+ b_{2s+1}b_{2s}(y_{2s-1} - z_1 z_2 y_{2s-2} - b_{2s-1}^2 y_{2s-3}), \tag{46}$$

and so by (39) and (41) it appears that

$$k_{2s+1} = z_1 z_2 k_{2s} + b_{2s+1}(z_1 + z_2 + b_{2s+2} + b_{2s+1} + b_{2s})k_{2s-1}$$
$$- b_{2s+1}b_{2s}b_{2s-1}(z_1 + z_2 + b_{2s+1} + b_{2s} + b_{2s-1})k_{2s-3} \tag{47}$$
$$- b_{2s+1}b_{2s}b_{2s-1}b_{2s-2}z_2 z_1 k_{2s-4} + b_{2s+1}b_{2s}b_{2s-1}^2 b_{2s-2}b_{2s-3}k_{2s-5}.$$

In the derivation of the recurrence formulæ (43) and (47) it has been assumed that s is large enough to avoid initial value idiosyncrasies. Allowance being made for these we finally have the theorem :

If the Stieltjes moment problem is determined and

$$F(z) = \int_0^\infty \frac{d\psi(x)}{x+z} = \frac{b_1}{z} + \frac{b_2}{1} + \frac{b_3}{z} + \frac{b_4}{1} + \frac{b_5}{z} + \frac{b_6}{1} + \ldots,$$

then

$$\underset{s\to\infty}{\text{l. i. s.}} \frac{k_{2s}^*}{k_{2s}} = \lim_{s\to\infty} \frac{k_s^*}{k_s} = \frac{F(z_2)-F(z_1)}{z_1-z_2}, \tag{48}$$

where

(i) k_s^* *and* k_s *follow, for* $s = 2, 3, \ldots,$

$$w_{2s-1} = z_1 z_2 w_{2s-2} + \alpha_{2s-1} w_{2s-3} - \beta_{2s-1} w_{2s-5} - z_1 z_2 \gamma_{2s-1} w_{2s-6} + \delta_{2s-1} w_{2s-7}$$

$$w_{2s} = w_{2s-1} + \alpha_{2s} w_{2s-2} - \beta_{2s} w_{2s-4} - \gamma_{2s} w_{2s-5} + \delta_{2s} w_{2s-6},$$

(ii) $k_0^* = 0,\ k_1^* = k_2^* = b_1, \qquad k_s^* = 0,\ s<0,$

$k_0 = 1,\ k_1 = z_1 z_2,\ k_2 = z_1 z_2 + b_2(z_1 + z_2 + b_3 + b_2),\ k_s = 0,\ s<0,$

(iii) $\alpha_s = b_s(z_1 + z_2 + b_{s+1} + b_s + b_{s-1}),\ \beta_s = b_s b_{s-2} \alpha_{s-1},$

$\gamma_s = b_s b_{s-1} b_{s-2} b_{s-3}, \qquad \delta_s = b_s b_{s-1} b_{s-2}^2 b_{s-3} b_{s-4},$

(iv) $(x + z_1)(x + z_2) > 0$ for[1] $x \geqslant 0$.

It may be established that the 'odd' part k_{2s-1}^*/k_{2s-1} arises from the second order C.F. associated with the integral in the expression

$$z_1 z_2 \left(\frac{F(z_2) - Fz_1)}{z_1 - z_2} \right) = b_1 - \int_0^\infty \frac{x(x + z_1 + z_2) d\psi(x)}{(x + z_1)(x + z_2)}$$

where $x d\psi(x)$ is taken as the weight function. The 'odd' part of the sequence, unlike that of a Stieltjes C.F., does not in general provide a set of decreasing upper bounds, but there is a remarkable property which we now consider.

5.2 We shall prove the following identities :

$$t_{2s+1} - t_{2s} = \frac{B_{2s+1} U_{2s+1}^{(1)} U_{2s+2}^{(1)}}{k_{2s}\, k_{2s+1}},\ s = 0, 1, 2, \ldots \tag{49a}$$

$$t_{2s} - t_{2s-1} = -\frac{B_{2s} U_{2s}^{(1)} U_{2s+1}^{(1)}}{k_{2s-1}\, k_{2s}},\ s = 1, 2, \ldots \tag{49b}$$

where

$$U_s^{(1)} = \frac{w_s(z_2) - w_s(z_1)}{z_2 - z_1}, \qquad B_s = \prod_{\lambda=1}^{s} b_\lambda, \qquad t_s = k_s^* / k_s.$$

[1] It has been assumed throughout that $z_1 \neq z_2$, but it is easily shown that the theorem still holds if $z_1 = z_2$ and $(x+z_1)^2 > 0$ for $x \geqslant 0$.

Introducing the expressions appearing in (30) and (47) of §4 for t_s, we have ($z_1 \neq z_2$)

$$b_{2s+2}(z_1 - z_2)^3 k_{2s} k_{2s-1}(t_{2s+1} - t_{2s}) \tag{50}$$

$$= | w_{2s+1}(z_1), w_{2s+2}(z_2) | X_s - \{z_1 w_{2s+1}(z_2) w_{2s+2}(z_1) - z_2 w_{2s+1}(z_1) w_{2s+2}(z_2)\} Y_s,$$

where

$$X_s = z_1 \xi_{2s+1}(z_1, z_2) + z_2 \xi_{2s+1}(z_2, z_1) - (z_1 + z_2) B_{2s+2},$$

$$Y_s = 2B_{2s+2} - \eta_{2s+1}(z_1, z_2) - \eta_{2s+1}(z_2, z_1),$$

$$\xi_{2s+1}(z_1, z_2) = \chi_{2s+1}(z_2) w_{2s+2}(z_1) - \chi_{2s+2}(z_1) w_{2s+1}(z_2),$$

$$\eta_{2s+1}(z_1, z_2) = \chi_{2s+1}(z_1) w_{2s+2}(z_2) - \chi_{2s+2}(z_1) w_{2s+1}(z_2).$$

Now since $\eta_{2s+1}(z, z) = B_{2s+2}$, the right member of (50) becomes

$$(z_2 - z_1)\{w_{2s+1}(z_1) w_{2s+2}(z_2) \mu_{2s+1}(z_1, z_2) + w_{2s+1}(z_2) w_{2s+2}(z_1) \mu_{2s+1}(z_2, z_1)\},$$

where $\quad \mu_{2s+1}(z_1, z_2) = \chi_{2s+2}(z_1) w_{2s+1}(z_2) - \chi_{2s+1}(z_2) w_{2s+2}(z_1) + B_{2s+2},$

and from which (49a) follows after simplification.[1] The proof of (49b) is similar. We now deduce the expansion, valid under the conditions of the theorem in (48),

$$F(z_1, z_2) = \sum_{s=1}^{\infty} \frac{(-1)^{s+1} B_s U_s^{(1)} U_{s+1}^{(1)}}{k_{s-1} k_s}. \tag{51}$$

This is the third type of expansion for $F(z_1, z_2)$, the other two, given in earlier parts, being

$$F(z_1, z_2) = \sum_{s=0}^{\infty} \frac{B_{2s+1}\{U_{2s+2}^{(1)}\}^2}{k_{2s} k_{2s+2}}, \tag{52}$$

$$= \frac{b_1}{z_1 z_2} - \sum_{s=1}^{\infty} \frac{B_{2s}\, U_{2s+1}^{(0)}\, U_{2s+1}^{(1)}}{k_{2s-1}\, k_{2s+1}}, \tag{53}$$

where $\quad U_s^{(0)} = \begin{vmatrix} z_1 & z_2 \\ w_s(z_1) & w_s(z_2) \end{vmatrix} \div (z_2 - z_1).$

The expansions in (51)-(53) bear a striking resemblance to those for a first order C.F., namely

$$F(z) = \sum_{s=1}^{\infty} \frac{(-1)^{s+1} B_s}{w_{s-1} w_s} \tag{54}$$

$$= \sum_{s=0}^{\infty} \frac{B_{2s+1}}{w_{2s}\, w_{2s-2}} \tag{55}$$

$$= \frac{b_1}{z} - z \sum_{s=1}^{\infty} \frac{B_{2s}}{w_{2s-1}\, w_{2s-1}} \tag{56}$$

where $w_s \equiv w_s(z)$.

[1] The results in (49) still hold if $z_1 = z_2$, and we merely introduce the *confluent* forms of k_s and $U_s^{(1)}$.

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Under the conditions set down in §2, (55) gives an increasing sequence (56) a decreasing sequence and (54) an enveloping sequence[1], provided z is real and positive. Correspondingly (52) gives an increasing sequence and (53) a converging sequence when z_1 and z_2 are complex conjugates with $\operatorname{Im} z_1 \neq 0$. We can go a little further than this. Suppose the interval $(0, \infty)$ is "reducible," i.e., it may reduce to a sub-interval (a, b) if $\psi(x)=\psi(a)$ for $x<a$, $a\geqslant 0$, and $\psi(x)=\psi(b)$ for $x>b$, $b>a$. Now using the fact that the zeros of $w_s(x)$ are distinct and lie entirely within[2] $(-b, -a)$ and recalling that the degrees of the highest terms in $w_{2s}(x)$ and $w_{2s+1}(x)$ are s and $s+1$ respectively, we easily see that[3]

$$\begin{cases} U^{(1)}_{4s+1} > 0, \; U^{(1)}_{4s+2} > 0, \text{ for } z_1 \leqslant z_2 < -b, \\ U^{(1)}_{4s} < 0, \quad U^{(1)}_{4s+3} < 0, \end{cases} \tag{57a}$$

$$U^{(1)}_{4s} > 0, \text{ for } z_2 \geqslant z_1 > -a. \tag{57b}$$

Hence for $z_1 \leqslant z_2 < -b$ it follows from (49) that $\{t_s\}$ is an increasing sequence, whereas this is not necessarily so for $z_2 \geqslant z_1 > -a$. Again suppose $z_2 > 0$, $z_1 < -b$ but $z_2 \geqslant |z_1|$. Then $U^{(1)}_s$ is clearly positive. But it is known (see **S1**, (18)) that

$$\int_a^b (x+z_1)(x+z_2)\, q_s^2(x)\, d\psi(x) = B_{2s+1}(z_2 - z_1)^2 k_{2s} k_{2s+2}, \tag{58}$$

where $\{q_s(x)\}$is an orthogonal system with respect to the weight function $(x+z_1)(x+z_2)\,d\psi(x)$. We deduce from (58), since $k_0 = 1$, the inequalities for the even denominators

$$\begin{cases} k_{4s} > 0, & k_{4s+2} < 0, \\ k_{4s+1} < 0, & k_{4s+3} > 0. \end{cases} \qquad \begin{cases} z_2 > 0 \\ z_1 < -b. \end{cases} \tag{59}$$

The result given for the odd denominators may be proved in a similar way. Hence $\{t_s\}$ is a decreasing sequence for $z_2 > 0$, $z_1 < -b$, $z_2 \geqslant |z_1|$. Lastly suppose z_1 and $z_2 > 0$. Then $U^{(1)} > 0$, $U^{(0)}_s > 0$, and so from (52) and (53) $\{t_s\}$ is an enveloping sequence. A summary of the various possibilities appears in Table 1.

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[1] i.e. $s_{2r} < F < s_{2r+1}$ where s_r is the sum of the first r terms of the series.

[2] Exceptionally, $w_{2s+1}(x)$ always has a zero $x = 0$.

[3] It is assumed now that z_1, z_2 are entirely real.

TABLE 1.

NATURE OF SERIES FOR SECOND ORDER C.F.

Arguments	Series (51)	(52)	(53)
$z_1 = \bar{z}_2$, $\mathrm{Im}(z_1) \neq 0$.	C	I	C
$z_1 < -b$, $z_2 < -b$.	I	I	I
$z_2 > 0$, $z_1 < -b$, $z_2 \geqslant \mid z_1 \mid$.	D	D	D
$z_1 > 0$, $z_2 > 0$.	E	I	D

C = Converges, D = Decreases, I = Increases, E = Envelopes.

6. *Numerical Illustrations.*

Example 1. $F(z) = \frac{1}{z+}\frac{1}{1+}\frac{1}{z+}\frac{1}{1+}\cdots = \frac{1}{2\pi}\int_0^4 \frac{\sqrt{(4x^{-1}-1)}\,dx}{x+z}$,

for $z > 0$.

The second order C.F. for $F(z, z) = \{z\sqrt{(z^2+4z)}\}^{-1}$, $z > 0$, is given by $\lim_{s\to\infty} t_s$ where

$$k_1^* = 1,\ k_2^* = 1,\ k_1 = z^2,\ k_2 = z^2 + 2z + 2,$$

and k_s^*, k_s follow

$$w_{2s-1} = z^2 w_{2s-2} + (2z+3)w_{2s-3} - (2z+3)w_{2s-5} - z^2 w_{2s-6} + w_{2s-7},$$

$$w_{2s} = w_{2s-1} + (2z+3)w_{2s-2} - (2z+3)w_{2s-4} - w_{2s-5} + w_{2s-6}, \qquad s = 2, 3, \ldots,$$

$$k_s^* = 0,\ s \leqslant 0, \qquad k_s = 0,\ s < 0.$$

From Table 1 the sequence $\{t_s\}$ is enveloping. In particular with $z=1$ the limit of the convergents is $1/\sqrt{5}$ and the first twenty are shown in Table 2

TABLE 2.

s	k_s^*	k_s	t_s	s	k_s^*	k_s	t_s
1	1	1	1·0	11	10866	24276	0·4476
2	1	5	0·2	12	28416	63565	0·4470
3	6	10	0·6	13	74431	166405	0·447288
4	11	30	0·37	14	194821	435665	0·447181
5	36	74	0·49	15	510096	1140574	0·447227
6	85	199	0·43	16	1335395	2986074	0·447208
7	235	515	0·456	17	3496170	7817630	0·447216
8	600	1355	0·443	18	9153025	20466835	0·4472125
9	1590	3540	0·449	19	23963005	53582855	0·4472140
10	4140	9276	0·4463	20	62735880	140281751	0·4472134
				∞	———	———	0·4472136

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Example 2. $$F(z) = \frac{1}{z} + \frac{\lambda}{1} + \frac{\lambda}{z} + \frac{\lambda}{1} + \ldots,$$
$$= \frac{1}{2\pi\lambda}\int_0^{4\lambda} \frac{\sqrt{\{x(4\lambda - x)\}}dx}{x(x+z)}, \quad \begin{cases}\lambda > 0\\ z > 0\cdot\end{cases}$$

With $\lambda = \frac{1}{4}$ we consider $F(\sqrt{2}, -\sqrt{2}) = \lim_{s\to\infty} t_s$,

where $k_s^* = X_s/4^s$, $k_s = Y_s/4^s$, and the recurrence for X_s and Y_s is

$$w_{2s-1} = -8w_{2s-2} + 3w_{2s-3} - 3w_{2s-5} + 8w_{2s-6} + w_{2s-7},$$
$$w_{2s} = 4w_{2s-1} + 3w_{2s-2} - 3w_{2s-4} - 4w_{2s-5} + w_{2s-6}, \qquad s = 2.\ 3, \ldots,$$

with

$$X_1 = 4,\ X_2 = 16,\ Y_0 = 1,\ Y_1 = -8,\ Y_2 = -30,$$
$$X_s = 0,\ s \leqslant 0, \qquad Y_s = 0,\ s < 0.$$

The sequence $\{-t_s\}$ steadily increases to $\frac{1}{2}\{\sqrt{(2+\sqrt{2})} - \sqrt{(2-\sqrt{2})}\}$, and a few values are stated in Table 3.

TABLE 3.

s	$-t_s$	s	$-t_s$
1	0·5	8	0·541186
2	0·53	9	0·541193
3	0·537	10	0·541195
4	0·5396	11	0·5411958
5	0·5408	12	0·54119602
6	0·5411	13	0·54119608
7	0·54116	∞	0·54119610

Example 3. From

$$F(z) = \frac{1}{z} + \frac{a}{1} + \frac{1}{z} + \frac{a+1}{1} + \frac{2}{z} + \frac{a+2}{1} + \frac{3}{z} + \frac{a+3}{1} + \ldots$$
$$= \frac{1}{\Gamma(a)}\int_0^\infty \frac{e^{-x}x^{a-1}dx}{x+z}, \qquad a > 0,\ z > 0,$$

we derive the C.F. for $F(iz, -iz)$ which we write

$$\Phi(a, z) = \frac{1}{\Gamma(a)}\int_0^\infty \frac{e^{-x}x^{a-1}\,dx}{x^2+z^2} = \underset{s\to\infty}{\text{l. i. s.}}\ (t_{2s}), \qquad z \neq 0,$$

where k_s^* and k_s follow

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$$\begin{aligned} w_{2s-1} = z^2 w_{2s-2} + (s-1)(2a+3s-4) w_{2s-3} \\ - (s-1)(s-2)(a+s-2)(a+3s-5) w_{2s-5} \\ - (s-1)(s-2)(a+s-2)(a+s-3) z^2 w_{2s-6} \\ + (s-1)(s-2)^2(s-3)(a+s-2)(a+s-3) w_{2s-7}, \end{aligned}$$

$$\begin{aligned} w_{2s} = w_{2s-1} + (a+s-1)(a+3s-2) w_{2s-2} \\ - (s-1)(a+s-1)(a+s-2)(2a+3s-4) w_{2s-4} \\ - (s-1)(s-2)(a+s-1)(a+s-2) w_{2s-5} \\ + (s-1)(s-2)(a+s-1)(a+s-2)^2(a+s-3) w_{2s-6}, \\ s = 2, 3, \ldots, \end{aligned}$$

with

$$k_0^* = 0,\ k_1^* = k_2^* = 1,\ k_0 = 1,\ k_1 = z^2,\ k_2 = z^2 + a(a+1);$$

$$k_s^* = 0,\ k_s = 0 \text{ for } s < 0.$$

The sequence $\{t_{2s}\}$ is increasing and $\{t_{2s+1}\}$ is convergent. In particular the coefficients in the recurrence relations for $\Phi(1, 1)$ are set out in Table 4, and they are to be read off from the penultimate row upwards. Thus, suppose we have found the values of k_s for $s = 1, 2, 3$ and 4; then from the column for $s = 5$ we see that

$k_5 = 1.k_4 + 14k_3 + 0.k_2 - 20.k_1 - 4k_0 + 0.k_{-1}$, and similarly for k_5^*.

TABLE 4.

RECURRENCE COEFFICIENTS FOR $\Phi(1, 1)$.

				0	24	72	432	864	2880	4800	12000	18000
			0	−4	−12	−36	−72	−144	−240	−400	−600	−900
	0	0	−8	−20	−84	−144	−360	−528	−1040	−1400	−2400	−3060
	0	0	0	0	0	0	0	0	0	0	0	0
	2	4	10	14	24	30	44	52	70	80	102	114
	1	1	1	1	1	1	1	1	1	1	1	1
s	2	3	4	5	6	7	8	9	10	11	12	13

A sequence of decreasing upper bounds is made available from

$$z^2 \Phi(1, z) = 1 - 2\Phi(3, z)$$

using the appropriate second order C.F. arising from

$$\Phi(3, z) = \frac{1}{z+}\,\frac{3}{1+}\,\frac{1}{z+}\,\frac{4}{1+}\,\frac{2}{z+}\,\frac{5}{1+}\,\frac{3}{z+}\,\frac{6}{1+}\,\frac{4}{z+}\,\frac{7}{1+}\cdots.$$

The corresponding multipliers in the recurrence formulae are, when $z_1 = i$, $z_2 = -i$, those in Table 5.

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TABLE 5.

RECURRENCE COEFFICIENTS FOR $\Phi(3, 1)$.

					480	240	360	2160	15120	10080	47040	33600	120960
				-24	-40	-120	-180	-360	-504	-840	-1120	-1680	-2160
			-96	-56	-440	-300	-1260	-936	-2856	-2240	-5600	-4560	-9936
			0	0	0	0	0	0	0	0	0	0	0
	12	8	28	22	50	42	78	68	112	100	152	138	198
	1	1	1	1	1	1	1	1	1	1	1	1	1
s	2	3	4	5	6	7	8	9	10	11	12	13	14

Proceeding in this way we are led to the approximations given in Table 6, which also includes similar ones for $\Phi(1, 2)$.

TABLE 6.

UPPER AND LOWER BOUNDS FOR $\Phi(1, 1)$ and $\Phi(1, 2)$.

	$\Phi(1, 1)$			$\Phi(1, 2)$	
s	(1)	(2)	s	(1)	(2)
4	0·52	0·74	4	0·196	0·205
8	0·612	0·653	8	0·1992	0·1998
12	0·6199	0·6284	12	0·19931	0·19954
16	0·6200	0·6227	16	0·19946	0·19953
20	0·6204	0·6217	20	0·19950	0·19952
24	0·6209	0·6216			

(1) and (2) refer to lower and upper bounds respectively.

It will be noticed that the rate of convergence for $\Phi(1, 1)$ is rather slow, and that after 24 terms we can only assert that $0{\cdot}6209<\Phi(1,1)<0{\cdot}6216$. For $\Phi(1, 2)$ the situation is better and twenty terms give accuracy in the fourth decimal place. Of course we could determine another set of upper bounds using (10), merely adding $s!\ s!/\{z^2k_{2s}\}$ to t_{2s}: however, there seems to be little improvement introduced in this way. According to Ser (1938) the values of the integrals are $\Phi(1, 1)=0{\cdot}62145$, $\Phi(1, 2) = 0{\cdot}199510$.

7. *Conclusion.*

We intend to develop on another occasion the expansion of *generalised* C.F.'s using the compound determinants of § 2, each element of these determinants being a *recurrent*. We shall give two

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types of recurrence formulae, and show that the evaluation of the convergents of a *generalised* C.F. of any order can be made a practical proposition.

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College of Technology,
Manchester.