AN INVERSION AND REPRESENTATION THEORY FOR THE LAPLACE INTEGRAL OF ABSTRACTLY-VALUED FUNCTIONS

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1. Introduction. The theory of the Laplace integral of abstractly-valued functions of a real or complex variable has been developed, in the last few years, to an extent that it is almost approaching the degree of completeness enjoyed by the classical theory of the Laplace integral of numerically-valued functions. In certain respects, however, there are still large gaps. One of the gaps occurs in representation theory.

In particular, there are no theorems giving conditions that an abstractedly-valued function be represented as the Laplace integral **(4)** of a function in $B_p([0, \infty); \mathfrak{X})$, the Banach space equivalent of $L_p(0, \infty)$. It is our purpose here to fill this gap.

Since, as in the numerically-valued case, this representation theory is developed in terms of a particular inversion operator for the Laplace integral, the opportunity was offered further to study some inversion operator for the transform. We have grasped this opportunity, and have developed the theory in terms of a certain "real" inversion operator given by the formula

$$\text{I} \qquad L_{\kappa,\tau}[f(\lambda)] = (\kappa e^{2\kappa}/\pi\tau) \int_0^\infty \eta^{-\frac{1}{2}} \cos(2\kappa\eta^{\frac{1}{2}}) f(\kappa(\eta + 1)/\tau)\, d\eta$$

or by the alternative formula, which we shall use occasionally,

$$\text{I}_a \qquad L_{\kappa,\tau}[f(\lambda)] = (2\kappa e^{2\kappa}/\pi\tau) \int_0^\infty \cos(2\kappa\eta) f(\kappa(\eta^2 + 1)/\tau)\, d\eta,$$

where the integrals are Bochner integrals.

The theory of this operator for numerically-valued functions has already appeared in print; see Rooney **(7)**.

The fact that the representation theory is stated in terms of a particular operator is no restriction, since the method is quite general, and will work quite well with any operator for which similar theorems hold in the numerically-valued case.

In §2 of this paper we define a slight generalization of the one-dimensional Bochner integral and prove one or two theorems concerning this generalization. In §3 we prove a lemma concerning the Lebesgue sets of abstractly-valued functions, while §4 is given over to a theorem enabling us to evaluate a singular integral that appears repeatedly in the theory.

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In §5 we prove an inversion theorem for the operator I.

§6 is devoted to certain lemmas preliminary to the representation theory, while in §7 we prove the "Fundamental Theorem" which gives sufficient conditions that the Laplace integral of $L_{\kappa,\tau}[f(\lambda)]$ should tend to $f(\lambda)$ as κ tends to infinity.

In §8, we give necessary and sufficient conditions that a function $f(\lambda)$ be the Laplace integral of a function in $B_p([0, \infty); \mathfrak{X})$, $1 < p < \infty$, where $\mathfrak{X}$ is a reflexive Banach space. In §9 we prove a similar theorem for $B_\infty([0, \infty); \mathfrak{X})$, $\mathfrak{X}$ now uniformly convex.

Lastly, in §10, we give sufficient conditions that a function in $H_1(\alpha; \mathfrak{X})$ be a Laplace integral, and we also have a theorem concerning $H_p(\alpha; \mathfrak{X})$, $p \neq 1$.

Throughout this paper we use the notation of Hille **(4)**. Also, whenever we speak of limits, we mean limits in the strong topology, unless otherwise specified.

2. Improper Bochner integrals. For the inversion theory of the operator $L_{\kappa,\tau}[f(\lambda)]$ we shall use a slightly generalized form of a one dimensional Bochner integral which we shall call the improper Bochner integral.

DEFINITION 2.1. Let $x(\alpha)$ be in $B([\lambda, \omega]; \mathfrak{X})$ for a fixed λ and all $\omega > \lambda$. If

$$\int_\lambda^\omega x(\alpha)\, d\alpha$$

converges to a limit y as $\omega \to \infty$, then we say that y is the *improper Bochner integral* of $x(\alpha)$ over $[\lambda, \infty)$, and we write

$$y = \int_\lambda^{\to\infty} x(\alpha)\, d\alpha.$$

We shall prove two theorems; the first giving sufficient conditions for the interchange of the order of integrations when one of the integrals involved is an improper Bochner integral and the other is a Bochner integral, and the second giving sufficient conditions for Bochner and improper Bochner integrals to be equivalent.

Let $E_{\omega,\zeta} = \{(\alpha, \beta) \mid 0 \leqslant \alpha \leqslant \omega; 0 \leqslant \beta \leqslant \zeta\}$.

THEOREM 2.1. *If*

(1) $x(\alpha, \beta)$ *is in* $B(E_{\omega,\zeta}; \mathfrak{X})$ *for a fixed* ζ *and all* $\omega > 0$,

(2) $\int_0^{\to\infty} x(\alpha, \beta)\, d\alpha$ *converges uniformly with respect to* β, $0 \leqslant \beta \leqslant \zeta$, *then*

(i) $\int_0^{\to\infty} \int_0^\zeta x(\alpha, \beta)\, d\beta\, d\alpha$ *converges*,

(ii) $\int_0^{\to\infty} x(\alpha, \beta)\, d\alpha$ *is in* $B([0, \zeta]; \mathfrak{X})$,

(iii) $$\int_0^{\to\infty} \int_0^\zeta x(\alpha, \beta)\, d\beta\, d\alpha = \int_0^\zeta \int_0^{\to\infty} x(\alpha, \beta)\, d\alpha\, d\beta.$$

Proof.

(i) It is sufficient to show that for every $\epsilon > 0$, $\omega(\epsilon)$ exists such that

$$\left\| \int_{\omega_1}^{\omega_2} \int_0^{\zeta} x(\alpha, \beta)\, d\beta\, d\alpha \right\| < \epsilon, \qquad \omega_2 > \omega_1 > \omega(\epsilon).$$

By (2), for every $\epsilon > 0$, $\omega_3(\epsilon)$ exists such that

$$\left\| \int_{\omega_1}^{\omega_2} x(\alpha, \beta)\, d\alpha \right\| < \epsilon, \qquad \omega_2 > \omega_1 > \omega_3(\epsilon).$$

Let $\omega(\epsilon) = \omega_3(\epsilon/\zeta)$. Then if $\omega_2 > \omega_1 > \omega(\epsilon)$,

$$\left\| \int_{\omega_1}^{\omega_2} \int_0^{\zeta} x(\alpha, \beta)\, d\beta\, d\alpha \right\| = \left\| \int_0^{\zeta} \int_{\omega_1}^{\omega_2} x(\alpha, \beta)\, d\alpha\, d\beta \right\|$$
$$\leqslant \int_0^{\zeta} \left\| \int_{\omega_1}^{\omega_2} x(\alpha, \beta)\, d\alpha \right\| d\beta < (\epsilon/\zeta) \cdot \zeta = \epsilon,$$

the interchange of integrations being justified by (1) and **(4**; Theorem 3.6.7**)**.

(ii) By **(4**; Theorem 3.2.3(3)**)**,

$$y(\beta) = \int_0^{\to\infty} x(\alpha, \beta)\, d\alpha$$

is a strongly measurable function of β. Thus by **(4**; theorem 3.5.2.**)**, it is sufficient to show that $\|y(\beta)\|$ is in $L(0, \zeta)$. Since $y(\beta)$ is strongly measurable, $\|y(\beta)\|$ is measurable (Lebesgue). Further,

$$||y(\beta)|| \leqslant \int_0^{\omega} ||x(\alpha, \beta)||\, d\alpha + \left\| \int_{\omega}^{\to\infty} x(\alpha, \beta)\, d\alpha \right\|$$
$$\leqslant \int_0^{\omega} ||x(\alpha, \beta)||\, d\alpha + \epsilon$$

for sufficiently large ω.

Thus

$$\int_0^{\zeta} ||y(\beta)||\, d\beta \leqslant \int_0^{\zeta} \int_0^{\omega} ||x(\alpha, \beta)||\, d\alpha\, d\beta + \epsilon\zeta$$

and the statement is proved.

(iii) By (i), for each $\epsilon > 0$, $\omega(\epsilon)$ exists such that for $\omega_1 > \omega(\epsilon)$

$$\left\| \int_{\omega_1}^{\to\infty} \int_0^{\zeta} x(\alpha, \beta)\, d\beta\, d\alpha \right\| < \tfrac{1}{2}\epsilon.$$

Thus

$$\Gamma = \left\| \int_0^{\zeta} \int_0^{\to\infty} x(\alpha, \beta)\, d\alpha\, d\beta - \int_0^{\to\infty} \int_0^{\zeta} x(\alpha, \beta)\, d\beta\, d\alpha \right\|$$
$$< \left\| \int_0^{\zeta} \int_0^{\to\infty} x(\alpha, \beta)\, d\alpha\, d\beta - \int_0^{\omega_1} \int_0^{\zeta} x(\alpha, \beta)\, d\beta\, d\alpha \right\| + \tfrac{1}{2}\epsilon$$
$$= \left\| \int_0^{\zeta} \int_{\omega_1}^{\to\infty} x(\alpha, \beta)\, d\alpha\, d\beta \right\| + \tfrac{1}{2}\epsilon$$

by (1) and (**4**, theorem 3.6.7). But by (2), for each $\epsilon > 0$, $\omega_2(\epsilon)$ exists such that for $\omega_3 > \omega_2(\epsilon)$,

$$\left\| \int_{\omega_3}^{\to\infty} x(\alpha, \beta)\, d\alpha \right\| < \epsilon.$$

Choose $\omega_1 > \max\{\omega(\epsilon), \omega_2(\epsilon/2\zeta)\}$. Then

$$\Gamma < \left\| \int_0^{\zeta} \int_{\omega_1}^{\to\infty} x(\alpha, \beta)\, d\alpha\, d\beta \right\| + \tfrac{1}{2}\epsilon$$
$$\leqslant \int_0^{\zeta} \left\| \int_{\omega_1}^{\to\infty} x(\alpha, \beta)\, d\alpha \right\| d\beta + \tfrac{1}{2}\epsilon = \epsilon.$$

Letting $\epsilon \to 0$, the conclusion is reached.

THEOREM 2.2. *If*
(1) $x(\alpha)$ *is in* $B([0, \omega]; \mathfrak{X})$ *for each* $\omega > 0$,
(2) $\|x(\alpha)\|$ *is in* $L(0, \infty)$,
then
(i) $x(\alpha)$ *is in* $B([0, \infty); \mathfrak{X})$,

(ii) $$\int_0^{\to\infty} x(\alpha)\, d\alpha \quad \textit{converges},$$

(iii) $$\int_0^{\infty} x(\alpha)\, d\alpha = \int_0^{\to\infty} x(\alpha)\, d\alpha.$$

Proof.
(i) By (**4**; theorem 3.5.2), it is sufficient to show that $x(\alpha)$ is strongly measurable over $[0, \infty)$.

By (1), $x(\alpha)$ is strongly measurable over $[0, \omega]$ for each $\omega > 0$. Let

$$1_\omega(\alpha) = \begin{cases} 1, & 0 \leqslant \alpha \leqslant \omega, \\ 0, & \alpha > \omega. \end{cases}$$

Obviously 1_ω is Lebesgue measurable for each positive ω. Then $x_\omega(\alpha) = 1_\omega(\alpha)x(\alpha)$ is strongly measurable over $[0, \omega]$ by (**4**; theorem 3.2.3(2)), and since $x_\omega(\alpha) = \theta$ for $\alpha > \omega$, $x_\omega(\alpha)$ is strongly measurable over $[0, \infty)$.

Obviously

$$x(\alpha) = \lim_{\omega\to\infty} x_\omega(\alpha),$$

and thus, by (**4**; theorem 3.2.3(3)), $x(\alpha)$ is strongly measurable over $[0, \infty)$.

(ii) Since $\|x(\alpha)\|$ is in $L(0, \infty)$, we have,

$$\left\| \int_{\omega_1}^{\omega_2} x(\alpha)\, d\alpha \right\| \leqslant \int_{\omega_1}^{\omega_2} \|x(\alpha)\|\, d\alpha \to 0, \qquad \omega_1, \omega_2 \to \infty.$$

(iii)

$$\left\| \int_0^{\to\infty} x(\alpha)\, d\alpha - \int_0^{\infty} x(\alpha)\, d\alpha \right\| = \left\| \int_{\omega_1}^{\to\infty} x(\alpha)\, d\alpha - \int_{\omega_1}^{\infty} x(\alpha)\, d\alpha \right\|$$
$$\leqslant \left\| \int_{\omega_1}^{\to\infty} x(\alpha)\, d\alpha \right\| + \int_{\omega_1}^{\infty} \|x(\alpha)\|\, d\alpha \to 0, \qquad \omega_1 \to \infty.$$

3. Lebesgue sets. Let $x(\alpha)$ be in $B(E_1; \mathfrak{X})$ where E_1 is the one-dimensional Euclidian space. Then by (**4**; page 48, formula (3.6.1)),

$$\lim_{\gamma\to 0} \frac{1}{\gamma}\int_{\xi}^{\xi+\gamma} ||x(\alpha) - x(\xi)||\, d\alpha = 0$$

almost everywhere.

Definition 3.1. We shall call the set where the above formula holds the *Lebesgue set* of $x(\alpha)$.

Theorem 3.1. *If $x(\alpha)$ is in $B(E_1; \mathfrak{X})$, and τ is in the Lebesgue set of $x(\alpha)$, then the Lebesgue set of $\|x(\alpha) - x(\tau)\|$ contains the Lebesgue set of $x(\alpha)$.*

Proof. Let ξ be in the Lebesgue set of $x(\alpha)$. Then

$$\frac{1}{\gamma}\int_{\xi}^{\xi+\gamma} \big|\, ||x(\alpha) - x(\tau)|| - ||x(\xi) - x(\tau)||\, \big|\, d\alpha$$
$$\leqslant \frac{1}{\gamma}\int_{\xi}^{\xi+\gamma} ||(x(\alpha) - x(\tau)) - (x(\xi) - x(\tau))||\, d\alpha$$
$$= \frac{1}{\gamma}\int_{\xi}^{\xi+\gamma} ||x(\alpha) - x(\xi)||\, d\alpha \to 0, \qquad \gamma \to 0,$$

and ξ is in the Lebesgue set of $\|x(\alpha) - x(\tau)\|$.

4. A theorem on a certain singular integral. The following theorem will be needed both in the inversion and representation theories.

Theorem 4.1. *If*
(1) $x(\alpha)$ *is in* $B([0, \omega]; \mathfrak{X})$ *for each* $\omega > 0$,

(2) $$\int_0^{\to\infty} e^{-\lambda\alpha} x(\alpha)\, d\alpha$$

converges for $\lambda = \lambda_0 > 0$, *then*
(i) *for each* $\tau > 0$ *and for all* $\kappa > \lambda_0 \tau$,

$$I_\kappa = e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_0^{\to\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})}\, \alpha^{-\frac{1}{2}}\, x(\alpha)\, d\alpha$$

converges,

(ii) $$\lim_{\kappa\to\infty} I_\kappa = x(\tau)$$

for each $\tau > 0$ *in the Lebesgue set of* $x(\alpha)$.

Proof.
(i) Let $\kappa > \lambda_0\tau$, let $\omega_2 > \omega_1$, and let

$$M = \sup_{0\leqslant\omega<\infty} \left\| \int_0^{\omega} e^{-\lambda_0\alpha}\, x(\alpha)\, d\alpha \right\|.$$

By (2), $M < \infty$. Thus

$$\left\| \int_{\omega_1}^{\omega_2} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha)\, d\alpha \right\|$$

$$= \left\| e^{-\kappa(\omega_2\tau^{-1}+\tau\omega_2^{-1})+\lambda_0\omega_2} \omega_2^{-\frac{1}{2}} \int_{\omega_1}^{\omega_2} e^{-\lambda_0\beta} x(\beta)\, d\beta \right.$$

$$\left. - \int_{\omega_1}^{\omega_2} \left\{ \frac{d}{d\alpha} (e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})+\lambda_0\alpha} \alpha^{-\frac{1}{2}} \right\} \int_{\omega_1}^{\alpha} e^{-\lambda_0\beta} x(\beta)\, d\beta\, d\alpha \right\|$$

$$\leqslant 2M \left\{ e^{-\kappa(\omega_2\tau^{-1}+\tau\omega_2^{-1})+\lambda_0\omega_2} \omega_2^{-\frac{1}{2}} + \int_{\omega_2}^{\omega_1} d(e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})+\lambda_0\alpha} \alpha^{-\frac{1}{2}}) \right\}$$

$$= 2M\, e^{-\kappa(\omega_1\tau^{-1}+\tau\omega_1^{-1})+\lambda_0\omega_1} \omega_1^{-\frac{1}{2}} \to 0, \qquad \omega_1, \omega_2 \to \infty.$$

(ii) It should be noted that

$$e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_0^{\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}}\, d\alpha = 1.$$

Let $\tau > 0$ be in the Lebesgue set of $x(\alpha)$, and let $\omega > \tau$. Then,

$$||I_\kappa - x(\tau)|| = \left\| e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_0^{\to\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}}(x(\alpha) - x(\tau))\, d\alpha \right\|$$

$$\leqslant e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \left\{ \int_0^{\omega} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}}\, ||x(\alpha) - x(\tau)||\, d\alpha \right.$$

$$\left. + \left\| \int_\omega^{\to\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha)\, d\alpha \right\| + ||x(\tau)|| \int_\omega^{\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}}\, d\alpha \right\}$$

$$= \Gamma_1 + \Gamma_2 + \Gamma_3.$$

Let $\kappa_0 > \lambda_0$ and $\kappa > \kappa_0$. Then

$$\Gamma_3 = ||x(\tau)||\, e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_\omega^{\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}}\, d\alpha$$

$$= ||x(\tau)||\, e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_\omega^{\infty} e^{-\kappa_0(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} e^{-(\kappa-\kappa_0)(\alpha\tau^{-1}+\tau\alpha^{-1})}\, d\alpha$$

$$\leqslant ||x(\tau)||\, e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} e^{-(\kappa-\kappa_0)(\omega\tau^{-1}+\tau\omega^{-1})} \int_\omega^{\infty} e^{-\kappa_0(\alpha\tau^{-1}+\tau\alpha^{-1})}\, d\alpha$$

$$= ||x(\tau)||(\kappa/\pi\tau)^{\frac{1}{2}} e^{-\kappa_0(\omega\tau^{-1}+\tau\omega^{-1})} e^{-\kappa(\omega\tau^{-1}+\tau\omega^{-1}-2)} \int_\omega^{\infty} e^{-\kappa_0(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}}\, d\alpha$$

$$\to 0, \qquad \kappa \to \infty.$$

since $\alpha\tau^{-1} + \tau\alpha^{-1} - 2$ attains its minimum value of 0 at $\alpha = \tau$, and $\omega > \tau$.

$$\Gamma_2 = \left\| e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_\omega^{\to\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha)\, d\alpha \right\|$$

$$= \left\| e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_\omega^{\infty} \left\{ \frac{d}{d\alpha}(e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})+\lambda_0\alpha} \alpha^{-\frac{1}{2}} \right\} \int_\omega^{\alpha} e^{-\lambda_0\beta} x(\beta)\, d\beta\, d\alpha \right\|$$

$$\leqslant 2M(\kappa/\pi\tau)^{\frac{1}{2}} e^{-\kappa(\omega\tau^{-1}+\tau\omega^{-1}-2)+\lambda_0\omega} \omega^{-\frac{1}{2}} \to 0, \qquad \kappa \to \infty,$$

where we have used Theorem 2.2 after the integration by parts. By Widder (**8**, p. 278, theorem 2b, corollary 2b.1), $\Gamma_1 \to 0$ as $\kappa \to \infty$ if τ is in the Lebesgue set of $\|x(\alpha) - x(\tau)\|$. However, since τ is in the Lebesgue set of $x(\alpha)$, we have, by Theorem 3.1 that τ is in the Lebesgue set of $\|x(\alpha) - x(\tau)\|$, and thus the theorem is proved.

COROLLARY. *If $e^{-\lambda_0\alpha} x(\alpha)$ is in $B([0, \infty); \mathfrak{X})$, then*
(i) *for each $\tau > 0$ and all $\kappa > \lambda_0\tau$,*

$$e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})}\alpha^{-\frac{1}{2}}x(\alpha)$$

is in $B([0, \infty); \mathfrak{X})$,
(ii) $\lim_{\kappa\to\infty} I_\kappa = x(\tau)$ *for each $\tau > 0$ in the Lebesgue set of $x(\alpha)$ where*

$$I_\kappa = e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_0^\infty e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})}\alpha^{-\frac{1}{2}}x(\alpha)\,d\alpha.$$

Proof.
(i) This follows from the fact that for $\kappa > \lambda_0\tau$,

$$e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})}\alpha^{-\frac{1}{2}} \leqslant N e^{-\lambda_0\alpha}.$$

(ii) This now follows from Theorem 2.2 and the preceding theorem.

5. Inversion of the Laplace Transformation. We define

$$\overrightarrow{L}_{\kappa,\tau}[f(\lambda)] = (\kappa e^{2\kappa}/\pi\tau)\int_0^{\to\infty} \eta^{-\frac{1}{2}}\cos(2\kappa\eta^{\frac{1}{2}})f(\kappa(\eta+1)/\tau)\,d\eta.$$

The following theorem provides sufficient conditions for the inversion of a Laplace transformation which involves improper Bochner integrals by

$$\overrightarrow{L}_{\kappa,\tau}[f(\lambda)].$$

THEOREM 5.1. *If $x(\alpha)$ is in $B([0, \omega]; \mathfrak{X})$ for each $\omega > 0$, and if*

$$f(\lambda) = \int_0^{\to\infty} e^{-\lambda\alpha}x(\alpha)\,d\alpha$$

converges uniformly in λ for $\lambda > \gamma > 0$, then for each $\tau > 0$ and all $\kappa > \gamma\tau$, $\overrightarrow{L}_{\kappa,\tau}[f(\lambda)]$ exists, and

$$\lim_{\kappa\to\infty}\overrightarrow{L}_{\kappa,\tau}[f(\lambda)] = x(\tau)$$

at each point $\tau > 0$ of the Lebesgue set of $x(\alpha)$.

Proof. We shall show that

$$\overrightarrow{L}_{\kappa,\tau}[f(\lambda)] = e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}}\int_0^{\to\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})}\alpha^{-\frac{1}{2}}x(\alpha)\,d\alpha,$$

and the conclusion will follow from Lemma 4.1.

Let κ and τ be fixed and $\kappa > \gamma\tau$. Choose λ_0, $\gamma < \lambda_0 < \kappa/\tau$. Let

$$M = \sup_{0\leqslant\omega<\infty}\left\|\int_0^\omega e^{-\lambda_0\alpha}x(\alpha)\,d\alpha\right\|.$$

By hypothesis, $M < \infty$. Consider

$$I_\omega = (\kappa\, e^{2\kappa}/\pi\tau) \int_0^\omega \eta^{-\frac{1}{2}} \cos\,(2\kappa\eta^{\frac{1}{2}})\, f(\kappa(\eta+1)/\tau)\, d\eta$$

$$= (\kappa\, e^{2\kappa}/\pi\tau) \int_0^\omega \eta^{-\frac{1}{2}} \cos\,(2\kappa\eta^{\frac{1}{2}})\, d\eta \int_0^{\to\infty} e^{-\kappa(\eta+1)\alpha/\tau}\, x(\alpha)\, d\alpha.$$

Since $f(\lambda)$ converges uniformly for $\lambda > \gamma$ and $\kappa > \gamma\tau$, we may, by Theorem 2.1, interchange the order of integrations. Thus

$$I_\omega = (\kappa\, e^{2\kappa}/\pi\tau) \int_0^{\to\infty} e^{-\kappa\alpha/\tau}\, x(\alpha)\, d\alpha \int_0^\omega e^{-\kappa\eta\alpha/\tau}\, \eta^{-\frac{1}{2}} \cos\,(2\kappa\eta^{\frac{1}{2}})\, d\eta$$

$$= (\kappa/\pi\tau)^{\frac{1}{2}}\, e^{2\kappa} \int_0^{\to\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})}\, \alpha^{-\frac{1}{2}}\, x(\alpha)\, d\alpha \left\{ \pi^{-\frac{1}{2}} \int_{-\sigma}^{\bar{\sigma}} e^{-\theta^2}\, d\theta \right\},$$

by an obvious change of variable, where $\sigma = (\alpha\kappa\omega/\tau)^{\frac{1}{2}} + i(\kappa\tau/\alpha)^{\frac{1}{2}}$. Then

$$\Gamma_\omega = \left\| (\kappa/\pi\tau)^{\frac{1}{2}}\, e^{2\kappa} \int_0^{\to\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})}\, \alpha^{-\frac{1}{2}}\, x(\alpha)\, d\alpha - I_\omega \right\|$$

$$\leqslant (\kappa/\pi\tau)^{\frac{1}{2}}\, e^{2\kappa} \left\{ \int_0^\delta e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})}\, \alpha^{-\frac{1}{2}}\, \|x(\alpha)\| \cdot \left| 1 - \pi^{-\frac{1}{2}} \int_{-\sigma}^{\bar{\sigma}} e^{-\theta^2}\, d\theta \right| d\alpha \right.$$

$$\left. + \left\| \int_\delta^{\to\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})}\, \alpha^{-\frac{1}{2}}\, x(\alpha) \left[1 - \pi^{-\frac{1}{2}} \int_{-\sigma}^{\bar{\sigma}} e^{-\theta^2}\, d\theta \right] d\alpha \right\| \right\}$$

$$= \Gamma_\omega^{(1)} + \Gamma_\omega^{(2)} \qquad (\delta > 0).$$

Now

$$\lim_{\omega\to\infty} \left\{ 1 - \pi^{-\frac{1}{2}} \int_{-\sigma}^{\bar{\sigma}} e^{-\theta^2}\, d\theta \right\} = 0$$

almost everywhere for α in $[0, \delta]$. Thus by (**4**; theorem 3.2.1) the limit equals zero almost uniformly in $[0, \delta]$. Further, since

$$\int_{-\sigma}^{\bar{\sigma}} e^{-\theta^2}\, d\theta = \{(\kappa\tau/\alpha) + (\alpha\kappa\omega/\tau)\}^{-1}\, e^{(\kappa\tau/\alpha)-(\alpha\kappa\omega/\tau)} \{ -(\kappa\tau/\alpha)^{\frac{1}{2}} \sin\,(2\kappa\omega^{\frac{1}{2}})$$

$$- (\alpha\kappa\omega/\tau)^{\frac{1}{2}} \cos\,(2\kappa\omega^{\frac{1}{2}})\} - \tfrac{1}{2} \int_{-\sigma}^{\bar{\sigma}} \theta^{-2}\, e^{-\theta^2}\, d\theta,$$

and

$$\left| \int_{-\sigma}^{\bar{\sigma}} \theta^{-2}\, e^{-\theta^2}\, d\theta \right| \leqslant M\, \alpha^{\frac{1}{2}}\, e^{\kappa\tau/\alpha},$$

where M is independent of α and ω for α in $[0, \delta]$, we have then

$$e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})}\, \alpha^{-\frac{1}{2}}\, \|x(\alpha)\| \cdot \left| 1 - \pi^{-\frac{1}{2}} \int_{-\sigma}^{\bar{\sigma}} e^{-\theta^2}\, d\theta \right| \leqslant N\, e^{-\kappa\alpha/\tau},$$

and this bound is an integrable function of α. Thus by (**4**; theorem 3.3.6),

$$\lim_{\omega\to\infty} \Gamma_\omega^{(1)} = 0.$$

Also,

$$J_\omega^{(3)} = (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \int_\delta^{\to\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha)\left\{1 - \pi^{-\frac{1}{2}} \int_{-\sigma}^{\bar\sigma} e^{-\theta^2} d\theta\right\} d\alpha$$

converges uniformly in ω for $\omega > 0$. For, for every $\omega_2 > \omega_1 > \delta$,

$$\left\| \int_{\omega_1}^{\omega_2} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha)\left\{1 - \pi^{-\frac{1}{2}} \int_{-\sigma}^{\bar\sigma} e^{-\theta^2} d\theta\right\} d\alpha \right\|$$

$$= \left\| 2(\kappa/\pi\tau)^{\frac{1}{2}} \int_{\omega_1}^{\omega_2} e^{-\kappa\alpha/\tau} x(\alpha)\, d\alpha \int_{\sqrt{\omega}}^{\infty} e^{-\kappa\alpha\phi^2/\tau} \cos(2\kappa\phi)\, d\phi \right\|$$

$$= 2(\kappa/\pi\tau)^{\frac{1}{2}} \left\| e^{(\lambda_0-\kappa/\tau)\omega_2} \int_{\sqrt{\omega}}^{\infty} e^{-\kappa\omega_2\phi^2/\tau} \cos(2\kappa\phi)\, d\phi \int_{\omega_1}^{\omega_2} e^{-\lambda_0\alpha} x(\alpha)\, d\alpha \right.$$

$$+ \int_{\omega_1}^{\omega_2} e^{(\lambda_0-\kappa/\tau)\alpha}\left\{-(\lambda_0 - \kappa/\tau) \int_{\sqrt{\omega}}^{\infty} e^{-\alpha\kappa\phi^2/\tau} \cos(2\kappa\phi)\, d\phi\right.$$

$$\left.\left. + (\kappa/\tau) \int_{\sqrt{\omega}}^{\infty} e^{-\alpha\kappa\phi^2/\tau} \phi^2 \cos(2\kappa\phi)\, d\phi\right\} \int_{\omega_1}^{\alpha} e^{-\lambda_0\beta} x(\beta)\, d\beta\, d\alpha \right\|$$

$$\leqslant 4(\kappa/\pi\tau)^{\frac{1}{2}} M\left\{ e^{(\lambda_0-\kappa/\tau)\omega_2} \int_0^\infty e^{-\kappa\omega_2\phi^2/\tau} d\phi \right.$$

$$\left. + \int_{\omega_1}^{\omega_2} e^{(\lambda_0-\kappa/\tau)\alpha}\left[-(\lambda_0 - \kappa/\tau)\int_0^\infty e^{-\alpha\kappa\phi^2/\tau} d\phi + (\kappa/\tau)\int_0^\infty e^{-\alpha\kappa\phi^2/\tau}\phi^2 d\phi\right] d\alpha \right\}$$

$$= 4(\kappa/\pi\tau)^{\frac{1}{2}} M\left\{ e^{(\lambda_0-\kappa/\tau)\omega_2} \int_0^\infty e^{-\kappa\omega_2\phi^2/\tau} d\phi + \int_{\omega_2}^{\omega_1} d\left(e^{(\lambda_0-\kappa/\tau)\alpha} \int_0^\infty e^{-\alpha\kappa\phi^2/\tau} d\phi\right)\right\}$$

$$= 4(\kappa/\pi\tau)^{\frac{1}{2}} M e^{(\lambda_0-\kappa/\tau)\omega_1} \int_0^\infty e^{-\omega_1\kappa\phi^2/\tau} d\phi \to 0, \qquad \omega_1, \omega_2 \to \infty.$$

Thus we may choose δ so large that

$$\Gamma_\omega^{(2)} = \| J_\omega^{(3)} \| < \epsilon$$

for any $\epsilon > 0$ and thus

$$\lim_{\omega\to\infty} \Gamma_\omega = 0$$

and the theorem is proved.

6. Some Lemmas. We now prove two lemmas which are preliminary to the representation theory.

LEMMA 6.1 *If*
(1) $\lambda^{-1}f(\lambda)$ *is in* $B([\delta, \infty); \mathfrak{X})$ *for each* $\delta > 0$,

(2) $$\psi(\xi) = \int_\xi^\infty \lambda^{-1} \|f(\lambda)\|\, d\lambda = O(\xi^{-m}),$$

with $m > \frac{1}{2}$, *as* $\xi \to \infty$, *and* $\psi(\xi) = O(e^{\gamma/\xi})$ *with* $\gamma > 0$ *as* $\xi \to 0+$, *then for each* $\kappa > 0$, *and almost all* $\tau > 0$,

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$$L_{\kappa,\tau}[f(\lambda)] = (\kappa e^{2\kappa}/\pi\tau)\int_0^\infty \eta^{-\frac{1}{2}} \cos(2\kappa\eta^{\frac{1}{2}})f(\kappa(\eta+1)/\tau)\,d\eta$$

exists. In particular, $L_{\kappa,\tau}[f(\lambda)]$ *exists when* κ *and* τ *are positive, and* κ/τ *is in the Lebesgue set of* $f(\lambda)$.

Proof. The proof is the same as in (**7**; §3, lemma 2, page 440) if $|f(s)|$ is replaced by $\|f(\lambda)\|$ wherever it appears, and certain minor notational changes are made.

LEMMA 6.2. *If*

(1) $$\lambda^{-1}f(\lambda) \text{ is in } B([\delta,\infty);\mathfrak{X}) \qquad \text{for each } \delta > 0,$$

(2) $$\psi(\xi) = \int_\xi^\infty \lambda^{-1}\,||f(\lambda)||\,d\lambda = O(\xi^{-m})$$

with $m > 0$, *as* $\xi \to \infty$, *and* $\psi(\xi) = O(e^{\gamma/\xi})$ *with* $\gamma > 0$ *as* $\xi \to 0+$, *then for each* $\epsilon > 0$,

$$\begin{aligned}\int_\xi^\infty \lambda^{-1}e^{-\epsilon\lambda}\,||f(\lambda)||\,d\lambda &= O(\xi^{-n}) \quad \text{for every } n>0, \text{ as } \xi\to\infty\\ &= O(e^{\gamma/\xi}) \qquad\qquad \text{as } \xi\to 0+.\end{aligned}$$

Proof.

$$\begin{aligned}\int_\xi^\infty \lambda^{-1}e^{-\epsilon\lambda}\,||f(\lambda)||\,d\lambda &\leqslant e^{-\epsilon\xi}\int_\xi^\infty \lambda^{-1}\,||f(\lambda)||\,d\lambda\\ &= O(\xi^{-n}), \qquad \xi\to\infty\\ &= O(e^{\gamma/\xi}), \qquad \xi\to 0+.\end{aligned}$$

7. A fundamental theorem. The following theorem is fundamental in the representation theory.

THEOREM 7.1. *If*

(1) $$\lambda^{-1}f(\lambda) \text{ is in } B([\delta,\infty);\mathfrak{X}) \qquad \text{for all } \delta > 0,$$

(2) $$\psi(\xi) = \int_\xi^\infty \eta^{-1}\,||f(\eta)||\,d\eta = O(\xi^{-m})$$

with $m > \frac{1}{2}$, *as* $\xi \to \infty$, *and* $\psi(\xi) = O(e^{\gamma/\xi})$, *with* $\gamma > 0$, *as* $\xi \to 0+$,

(3) $e^{-\zeta\tau}L_{\kappa,\tau}[f(\lambda)]$ *is in* $B([0,\infty);\mathfrak{X})$ *for* $\zeta > \gamma_1$, *and all* $\kappa > \kappa_0$,

then

$$\lim_{\kappa\to\infty}\int_0^\infty e^{-\zeta\tau}L_{\kappa,\tau}[f(\lambda)]\,d\tau = f(\zeta)$$

at every point $\zeta > \gamma_1$, *of the Lebesgue set of* $f(\lambda)$.

Proof. $L_{\kappa,\tau}[f(\lambda)]$ exists by Lemma 6.1 and has a Laplace transform when $\zeta > \gamma_1$, by (3). To prove the assertion we shall use (**4**; theorem 3.6.7) and the corollary to Theorem 4.1.

Operating formally we have

$$\begin{aligned}
\int_0^\infty e^{-\zeta\tau} L_{\kappa,\tau}[f(\lambda)]\, d\tau &= (\kappa e^{2\kappa}/\pi) \int_0^\infty e^{-\zeta\tau}\tau^{-1} d\tau \int_0^\infty \eta^{-\frac{1}{2}} \cos(2\kappa\eta^{\frac{1}{2}}) f(\kappa(\eta+1)/\tau)\, d\eta \\
&= (2\kappa e^{2\kappa}/\pi) \int_0^\infty e^{-\zeta\tau}\tau^{-1} d\tau \int_0^\infty \cos(2\kappa\xi) f(\kappa(\xi^2+1)/\tau)\, d\xi \\
&= (2\kappa e^{2\kappa}/\pi) \int_0^\infty \cos(2\kappa\xi)\, d\xi \int_0^\infty e^{-\zeta\tau}\, \tau^{-1} f(\kappa(\xi^2+1)/\tau)\, d\xi \\
&= (2\kappa e^{2\kappa}/\pi) \int_0^\infty \cos(2\kappa\xi)\, d\xi \int_0^\infty e^{-\kappa\zeta\beta(\xi^2+1)}\, \beta^{-1} f(\beta^{-1})\, d\beta \\
&= (2\kappa e^{2\kappa}/\pi) \int_0^\infty e^{-\kappa\zeta\beta}\, \beta^{-1} f(\beta^{-1})\, d\beta \int_0^\infty e^{-\kappa\zeta\beta\xi^2} \cos(2\kappa\xi)\, d\xi \\
&= (\kappa/\pi\zeta)^{\frac{1}{2}}\, e^{2\kappa} \int_0^\infty e^{-\kappa(\zeta\beta+(\zeta\beta)^{-1})}\, \beta^{-3/2} f(\beta^{-1})\, d\beta \\
&\to f(\zeta), \qquad \kappa \to \infty.
\end{aligned}$$

These formal calculations will be justified if the two interchanges of integrations are justified, and the conditions of Lemma 4.1 are met.

For the first interchange of integrations it is sufficient to show that

$$\int_0^\infty |\cos(2\kappa\xi)|\, d\xi \int_0^\infty e^{-\kappa\zeta\beta(\xi^2+1)}\, \beta^{-1}\, ||f(\beta^{-1})||\, d\beta < \infty.$$

But by (2) and (**7**; §3, lemma 1, page 438), if $\kappa\zeta > \gamma$, the inner integral is $O(\xi^{-2m})$ as $\xi \to \infty$, and $m > \frac{1}{2}$.

For the second interchange it is sufficient to show that

$$\int_0^\infty e^{-\kappa\zeta\beta}\, \beta^{-1}\, ||f(\beta^{-1})||\, d\beta \int_0^\infty |e^{-\kappa\zeta\beta\xi^2} \cos(2\kappa\xi)|\, d\xi < \infty.$$

But this is so since the inner integral is less than $\frac{1}{2}(\pi/(\kappa\zeta\beta))^{\frac{1}{2}}$ and

$$\int_0^\infty e^{-\kappa\zeta\beta}\, \beta^{-3/2}\, ||f(\beta^{-1})||\, d\beta < \infty$$

by (1), (2), and (**7**; §3, lemma 1, page 438).

8. Representation of abstractly-valued functions by Laplace transformations of functions in $B_p([0, \infty); \mathfrak{X})$, $1 < p < \infty$. In this section we find necessary and sufficient conditions that a function $f(\lambda)$ on $(0, \infty)$ to a Banach space $\mathfrak{X}$ be represented as the Laplace integral of a function in $B_p([0, \infty); \mathfrak{X})$ where $1 < p < \infty$.

In order to obtain such conditions for these general classes of functions, we find it necessary to postulate some sort of compactness condition on $B_p([0, \infty); \mathfrak{X})$. We have chosen the weakest condition possible, namely weak compactness of the unit sphere in $B_p([0, \infty); \mathfrak{X})$. By Bochner and Taylor (**1**), Pettis (**6**), and Eberlein (**3**), a necessary and sufficient condition for this compactness is that $\mathfrak{X}$ be reflexive.

There is one other complication that must be dealt with. That is that a function $f(\lambda)$ may be the Laplace transform of a function $x(\alpha)$ in $B_p([0, \infty); \mathfrak{X})$, $1 < p \leqslant 2$, and yet $L_{\kappa,\tau}[f(\lambda)]$ may not exist. For example, let $f(\lambda) = (\lambda + 1)^{-1/3}$. To overcome this difficulty we resort to what is essentially Cauchy's method of summation.

We define

$$L^{\epsilon}_{\kappa,\tau}[f(\lambda)] = L_{\kappa,\tau}[e^{-\epsilon\lambda} f(\lambda)].$$

The following theorem gives the necessary and sufficient conditions in question.

THEOREM 8. *If $\mathfrak{X}$ is a reflexive Banach space, then the following conditions are necessary and sufficient for $f(\lambda)$ to be equal almost everywhere for $\lambda > 0$ to the Laplace integral of a function in $B_p([0, \infty); \mathfrak{X})$, p fixed, $1 < p < \infty$.*

(1) $\lambda^{-1} f(\lambda)$ *is in* $B([\delta, \infty); \mathfrak{X})$ *for all* $\delta > 0$,

(2) $$\int_{\xi}^{\infty} \lambda^{-1} \|f(\lambda)\| \, d\lambda = O(\xi^{-m}) \quad \text{with } m > 0 \text{ as } \xi \to \infty$$
$$= O(e^{\gamma/\xi}) \quad \text{with } \gamma > 0 \text{ as } \xi \to 0+,$$

(3) $\|L^{\epsilon}_{\kappa,\cdot}[f(\lambda)]\|_p < M_p$, *where M_p is independent of ϵ and κ, p fixed,* $\kappa > \kappa_0$, $\epsilon > 0$.

Proof of necessity. Suppose

$$f(\lambda) = \int_0^{\infty} e^{-\lambda\alpha} x(\alpha) \, d\alpha$$

a.e. for $\lambda > 0$, and $x(\alpha)$ is in $B_p([0, \infty); \mathfrak{X})$. Then using Hölder's inequality we have

$$\lambda^{-1}\|f(\lambda)\| \leqslant \lambda^{-1} \int_0^{\infty} e^{-\lambda\alpha} \|x(\alpha)\| \, d\alpha \leqslant \lambda^{-1} \left\{ \int_0^{\infty} e^{-q\lambda\alpha} d\alpha \right\}^{1/q} \left\{ \int_0^{\infty} \|x(\alpha)\|^p \, d\alpha \right\}^{1/p}$$
$$= A\lambda^{-1-1/q},$$

so that (1) is necessary. Thus

$$\int_{\xi}^{\infty} \lambda^{-1} \|f(\lambda)\| \, d\lambda \leqslant B\xi^{-1/q},$$

so that (2) is necessary. Now

$$\begin{aligned} L^{\epsilon}_{\kappa,\tau}[f(\lambda)] &= L_{\kappa,\tau}[e^{-\epsilon\lambda} f(\lambda)] \\ &= (2\kappa e^{2\kappa}/\pi\tau) \int_0^{\infty} \cos(2\kappa\eta) \, e^{-\epsilon\kappa(\eta^2+1)/\tau} d\eta \int_0^{\infty} e^{-\kappa(\eta^2+1)\alpha/\tau} x(\alpha) \, d\alpha \\ &= (2\kappa e^{2\kappa}/\pi\tau) \int_0^{\infty} e^{-\kappa(\alpha+\epsilon)/\tau} x(\alpha) \, d\alpha \int_0^{\infty} e^{-\kappa(\alpha+\epsilon)\eta^2/\tau} \cos(2\kappa\eta) \, d\eta \\ &= (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \int_0^{\infty} e^{-\kappa((\alpha+\epsilon)\tau^{-1}+\tau(\alpha+\epsilon)^{-1})} (\alpha + \epsilon)^{-\frac{1}{2}} x(\alpha) \, d\alpha, \end{aligned}$$

the interchange of integration being justified by the fact that

$$\int_0^\infty e^{-\kappa(\alpha+\epsilon)/\tau} \| x(\alpha) \| d\alpha \int_0^\infty e^{-\kappa(\alpha+\epsilon)\eta^2/\tau} | \cos (2\kappa\eta) | d\eta$$
$$\leqslant B \int_0^\infty e^{-\kappa\alpha/\tau} \| x(\alpha) \| d\alpha.$$

Thus

$$\| L^{\epsilon}_{\kappa,\tau} [f(\lambda)] \| \leqslant (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \int_0^\infty e^{-\kappa((\alpha+\epsilon)\tau^{-1}+\tau(\alpha+\epsilon)^{-1})} (\alpha + \epsilon)^{-\frac{1}{2}} \| x(\alpha) \| d\alpha$$
$$\leqslant (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \left\{ \int_0^\infty e^{-\kappa((\alpha+\epsilon)\tau^{-1}+\tau(\alpha+\epsilon)^{-1})} (\alpha + \epsilon)^{-\frac{1}{2}} \| x(\alpha) \|^p d\alpha \right\}^{1/p}$$
$$\cdot \left\{ \int_0^\infty e^{-\kappa((\alpha+\epsilon)\tau^{-1}+\tau(\alpha+\epsilon)^{-1})} (\alpha + \epsilon)^{-\frac{1}{2}} d\alpha \right\}^{1/q}$$
$$\leqslant (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \left\{ \int_0^\infty e^{-\kappa((\alpha+\epsilon)\tau^{-1}+\tau(\alpha+\epsilon)^{-1})} (\alpha + \epsilon)^{-\frac{1}{2}} \| x(\alpha) \|^p d\alpha \right\}^{1/p}$$
$$\cdot \left\{ \int_0^\infty e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} d\alpha \right\}^{1/q}$$
$$= (\kappa/\pi\tau)^{\frac{1}{2}p} e^{2\kappa/p} \left\{ \int_0^\infty e^{-\kappa(\alpha+\epsilon)\tau^{-1}+\tau(\alpha+\epsilon)^{-1})} (\alpha+\epsilon)^{-\frac{1}{2}} \| x(\alpha) \|^p d\alpha \right\}^{1/p}.$$

Hence,

$$\| L^{\epsilon}_{\kappa,\cdot} [f(\lambda)] \|_p$$
$$\leqslant \left\{ (\kappa/\pi)^{\frac{1}{2}} e^{2\kappa} \int_0^\infty \int_0^\infty e^{-\kappa((\alpha+\epsilon)\tau^{-1}+\tau(\alpha+\epsilon)^{-1})} (\tau(\alpha + \epsilon))^{-\frac{1}{2}} \| x(\alpha) \|^p d\tau\, d\alpha \right\}^{1/p}$$
$$= \left\{ \int_0^\infty \| x(\alpha) \|^p d\alpha \right\}^{1/p} = \| x(\cdot) \|_p$$

and (3) is necessary.

Proof of sufficiency. By (1), (2), (3), lemma 6.2, and theorem 7.1,

$$e^{-\epsilon\zeta} f(\zeta) = \lim_{\kappa\to\infty} \int_0^\infty e^{-\zeta\tau} L^{\epsilon}_{\kappa,\tau} [f(\lambda)]\, d\tau$$

a.e. for $\zeta > 0$.

By **(1)** and **(6)**, if $\mathfrak{X}$ is a reflexive Banach space, $B_p([0, \infty); \mathfrak{X})$, $1 < p < \infty$ is reflexive, and by **(3)** a reflexive Banach space has a weakly compact unit sphere. Thus $B_p([0, \infty); \mathfrak{X})$ has a weakly compact unit sphere, and thus, since

$$\| L^{\epsilon}_{\kappa,\cdot} [f(\lambda)] \|_p < M_p,$$

for each $\epsilon > 0$ there exists an element $x_\epsilon(\cdot)$ of $B_p([0, \infty); \mathfrak{X})$ and an increasing unbounded sequence $\{{}_\epsilon\kappa_i\}$ such that for every y^* in $B_p{}^*([0, \infty); \mathfrak{X})$,

$$\lim_{i\to\infty} y^*(L^{\epsilon}_{\epsilon\kappa_i,\cdot} [f(\lambda)]) = y^*(x_\epsilon(\cdot)).$$

Further, since

$$| y^*(L^{\epsilon}_{\epsilon\kappa_i,\cdot} [f(\lambda)]) | \leqslant \| y^* \| . \| L^{\epsilon}_{\epsilon\kappa_i,\cdot} [f(\lambda)] \|_p$$

we have, by **(4**; theorem 2.12.3**)**, $\|x_\epsilon(\cdot)\|_p \leqslant M_p$.

Let x^* be an arbitrary element of $\mathfrak{X}^*$. Then if $g(\cdot)$ is an arbitrary element of $B_p([0, \infty); \mathfrak{X})$,

$$x^*\left(\int_0^\infty e^{-\zeta\alpha} g(\alpha)\, d\alpha\right) = \int_0^\infty e^{-\zeta\alpha} x^*(g(\alpha))\, d\alpha = y_\zeta^*\,(g(\cdot))$$

defines an element

$$y_\zeta^* \text{ of } B_p^*([0, \infty);\ \mathfrak{X})$$

for each $\zeta > 0$. For, y_ζ^* is obviously linear, and using Hölder's inequality we have,

$$|y_\zeta^*\,(g(\cdot))| = \left|\int_0^\infty e^{-\zeta\alpha} x^*(g(\alpha))\, d\alpha\right| \leqslant \left\{\int_0^\infty e^{-q\zeta\alpha} d\alpha\right\}^{1/q} \left\{\int_0^\infty |x^*(g(\alpha))|^p d\alpha\right\}^{1/p}$$
$$\leqslant (q\zeta)^{-1/q}\, ||\, x^*\, ||\left\{\int_0^\infty ||\, g(\alpha)\, ||^p\, d\alpha\right\}^{1/p} = (q\zeta)^{-1/q}\, ||\, x^*\, ||\; ||\, g(\cdot)\, ||_p$$

so that $y^*{}_\zeta$ is bounded for each $\zeta > 0$.

Thus we have, for almost every $\zeta > 0$, each $\epsilon > 0$, and every x^* in $\mathfrak{X}^*$,

$$x^*(e^{-\epsilon\zeta} f(\zeta)) = x^*\left(\lim_{i\to\infty} \int_0^\infty e^{-\zeta\tau} L_{\epsilon\kappa_{i,\tau}}^{\epsilon}\,[f(\lambda)]\, d\tau\right),$$

$$\lim_{i\to\infty} y_\zeta^*\,(L_{\epsilon\kappa_{i},\,\cdot}^{\epsilon}\,[f(\lambda)]) = y_\zeta^*\,(x_\epsilon(\cdot)) = x^*\left(\int_0^\infty e^{-\zeta\tau} x_\epsilon(\tau)\, d\tau\right).$$

Thus for each $\epsilon > 0$, and all $\zeta > 0$ not in a set Σ_ϵ of measure zero,

$$e^{-\epsilon\zeta} f(\zeta) = \int_0^\infty e^{-\zeta\tau} x_\epsilon(\tau)\, d\tau.$$

Now since $\|x_\epsilon(\cdot)\|_p \leqslant M_p$ for each $\epsilon > 0$, and since $B_p([0, \infty); \mathfrak{X})$ has a weakly compact unit sphere, there exists an element $x(\cdot)$ of $B_p([0, \infty); \mathfrak{X})$ and a sequence $\{\epsilon_i\}$ with

$$\lim_{i\to\infty} \epsilon_i = 0$$

such that for every y^* in $B_p^*([0, \infty); \mathfrak{X})$,

$$\lim_{i\to\infty} y^*(x_{\epsilon_i}(\cdot)) = y^*(x(\cdot)).$$

Let

$$\Sigma = \bigcup_i \Sigma_{\epsilon_i}.$$

Then Σ has measure zero. Further, let ζ be in $(0, \infty) - \Sigma$. Then for each x^* in $\mathfrak{X}^*$

$$x^*(f(\zeta)) = \lim_{i\to\infty} x^*(e^{-\epsilon_i\zeta} f(\zeta)) = \lim_{i\to\infty} x^*\left(\int_0^\infty e^{-\zeta\tau} x_{\epsilon_i}(\tau)\, d\tau\right)$$
$$= \lim_{i\to\infty} y_\zeta^*\,(x_{\epsilon_i}(\cdot)) = y_\zeta^*\,(x(\cdot)) = x^*\left(\int_0^\infty e^{-\zeta\tau} x(\tau)\, d\tau\right).$$

Thus

$$f(\zeta) = \int_0^\infty e^{-\zeta\tau} x(\tau)\, d\tau$$

a.e. for $\zeta > 0$, and $x(\cdot)$ is in $B_p([0, \infty); \mathfrak{X})$.

9. Representation of Abstractly-valued functions by Laplace Transformations of functions in $B_\infty([0, \infty); \mathfrak{X})$. The following two theorems give necessary and sufficient conditions that a function on $(0, \infty)$ to a uniformly convex Banach space $\mathfrak{X}$ be equal almost everywhere to a Laplace integral of a function in $B_\infty([0, \infty); \mathfrak{X})$.

It will be noted that the representation theorem is stated in terms of $L_{\kappa,\tau}[f(\lambda)]$. The theorem could equally well be stated in terms of $L^{\epsilon}_{\kappa,\tau}[f(\lambda)]$, but we have chosen this form for simplicity.

THEOREM 9.1. *If $\{T_\sigma\}$, $0 < \sigma < \infty$, is a set of linear transformations on a separable Banach space $\mathfrak{X}$ to a reflexive Banach space $\mathfrak{Y}$, and if $\|T_\sigma\| \leqslant M$ independent of σ, for all $\sigma > 0$, then there exists an increasing unbounded sequence $\{\sigma_i\}$ and a linear transformation T on $\mathfrak{X}$ to $\mathfrak{Y}$ with $\|T\| \leqslant M$, such that*

$$\lim_{i\to\infty} y^*(T_{\sigma_i}(x)) = y^*(T(x))$$

for every x in $\mathfrak{X}$ and every y^* in $\mathfrak{Y}^*$.

Proof. Let $D = \{x_n\}$ be a countable set dense in $\mathfrak{X}$. Since $\mathfrak{Y}$ is reflexive, it has, by **(3)**, a weakly compact unit sphere, so that there exists an increasing unbounded sequence $\sigma_{i,1}$ and an element y_1 of $\mathfrak{Y}$ such that for every y^* in $\mathfrak{Y}^*$,

$$\lim_{i\to\infty} y^*(T_{\sigma_{i,1}}(x_1)) = y^*(y_1).$$

Further, there exists an increasing unbounded sequence $\{\sigma_{i,2}\} \subseteq \{\sigma_{i,1}\}$ and an element y_2 of $\mathfrak{Y}$ such that for every y^* in $\mathfrak{Y}^*$,

$$\lim_{i\to\infty} y^*(T_{\sigma_{i,2}}(x_2)) = y^*(y_2).$$

Inductively, there exists an increasing unbounded sequence $\{\sigma_{i,n}\} \subseteq \{\sigma_{i,n-1}\}$ and an element y_n of $\mathfrak{Y}$ such that for every y^* in $\mathfrak{Y}^*$

$$\lim_{i\to\infty} y^*(T_{\sigma_{i,n}}(x_n)) = y^*(y_n).$$

Thus, using the diagonal sequence, we have for every y^* in $\mathfrak{Y}^*$,

$$\lim_{i\to\infty} y^*(T_{\sigma_{i,i}}(x_j)) = y^*(y_j).$$

Further

$$| y^*(y_j) | = \lim_{i\to\infty} | y^*(T_{\sigma_{i,i}}(x_j)) | \leqslant || y^* || \cdot M \cdot || x_j ||,$$

so that, by **(4**; theorem 2.12.3**)**, $\|y_n\| \leqslant M\|x_n\|$.

We define $\sigma_i = \sigma_{i,i}$, $T(x_n) = y_n$, $T(\alpha x_m + \beta x_n) = \alpha y_m + \beta y_n$. The last part of the definition of T is consistent with the first part, since, if $x_l = \alpha x_m + \beta x_n$, then for every y^* in $\mathfrak{Y}^*$,

$$\begin{aligned} y^*(y_l) &= \lim_{i\to\infty} y^*(T_{\sigma_i}(x_l)) = \lim_{i\to\infty} y^*(T_{\sigma_i}(\alpha x_m + \beta x_n)) \\ &= \lim_{i\to\infty} \{\alpha y^*(T_{\sigma_i}(x_m)) + \beta y^*(T_{\sigma_i}(x_n))\} = \alpha y^*(y_m) + \beta y^*(y_n) \\ &= y^*(\alpha y_m + \beta y_n), \end{aligned}$$

so that $y_l = \alpha y_m + \beta y_n$, or $T(x_l) = \alpha T(x_m) + \beta T(x_n)$. Obviously we have on this linear manifold, $\|T\| \leqslant M$.

Now let x be an arbitrary element of $\mathfrak{X}$. Then there is a sequence $\{x_{n_j}\} \subseteq D$ such that

$$\lim_{j\to\infty} x_{n_j} = x.$$

Further, if $y_{n_j} = T(x_{n_j})$,

$$\lim_{j\to\infty} y_{n_j}$$

exists; for,

$$\| y_{n_j} - y_{n_k} \| = \| T(x_{n_j} - x_{n_k}) \| \leqslant M \| x_{n_j} - x_{n_k} \| \to 0, \qquad j, k, \to \infty.$$

Further if $\bar{x}_{n_l}$ is any other sequence whose limit is x, and if $\bar{y}_{n_l} = T(\bar{x}_{n_l})$, then

$$\| y_{n_j} - \bar{y}_{n_l} \| = \| T(x_{n_j} - \bar{x}_{n_l})\| \leqslant M \| x_{n_j} - \bar{x}_{n_l} \| \to 0, \qquad j, l \to \infty$$

since the sequences have the same limit. We define

$$T(x) = \lim y_{n_j}.$$

It is evident that T is bounded and linear on $\mathfrak{X}$, and, in fact $\|T\| \leqslant M$.

Also we have

$$\lim_{i\to\infty} y^*(T_{\sigma_i}(x)) = y^*(T(x))$$

for every y^* in $\mathfrak{Y}^*$. For,

$$| y^*(T_{\sigma_i}(x)) - y^*(T(x))|$$
$$= | y^*(T_{\sigma_i}(x - x_{n_j})) + (y^*(T_{\sigma_i}(x_{n_j})) - y^*(T(x_{n_j}))) + y^*(T(x - x_{n_j})) |$$
$$\leqslant 2 \| y^* \| \cdot M \cdot \| x - x_{n_j} \| + | y^*(T_{\sigma_i}(x_{n_j})) - y^*(T(x_{n_j})) | \to 0, \quad i, j \to \infty.$$

Thus, the theorem is proved.

THEOREM 9.2. *If $\mathfrak{X}$ is a uniformly convex Banach space, then the following conditions are necessary and sufficient that $f(\lambda)$ be equal almost everywhere, for $\lambda > 0$, to a Laplace integral of a function in $B_\infty([0, \infty); \mathfrak{X})$.*

(1) $\lambda^{-1} f(\lambda)$ *is in* $B_\infty([\delta, \infty); \mathfrak{X})$ *for all* $\delta > 0$,

(2)
$$\int_\xi^\infty \lambda^{-1} \| f(\lambda) \| \, d\lambda = O(\xi^{-m}) \quad \text{with } m > \tfrac{1}{2}, \quad \xi \to \infty$$
$$= O(e^{\gamma/\xi}) \quad \text{with } \gamma > 0, \quad \xi \to 0+,$$

(3)
$$\| L_{\kappa,\cdot} [f(\lambda)] \|_\infty \leqslant M_\infty, \qquad \kappa > \kappa_0.$$

Proof of necessity. Suppose

$$f(\lambda) = \int_0^\infty e^{-\lambda\tau} x(\tau) \, d\tau,$$

where $x(\tau)$ is in $B_\infty([0, \infty); \mathfrak{X})$. Then for $\lambda > 0$,

$$\lambda^{-1}\|f(\lambda)\| \leqslant \lambda^{-1} \int_0^\infty e^{-\lambda\tau}\|x(\tau)\| \, d\tau \leqslant \lambda^{-1}\| x(\cdot)\|_\infty \int_0^\infty e^{-\lambda\tau} \, d\tau = \lambda^{-2}\|x(\cdot)\|_\infty,$$

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so that (1) is necessary. Further then,

$$\int_{\xi}^{\infty} \lambda^{-1} \| f(\lambda) \| \, d\lambda \leqslant \xi^{-1} \| x(\cdot) \|_{\infty},$$

so that (2) is necessary. Finally, we have

$$\begin{aligned} L_{\kappa,\tau}[f(\lambda)] &= (2\kappa e^{2\kappa}/(\pi\tau)) \int_0^{\infty} \cos(2\kappa\eta)\, d\eta \int_0^{\infty} e^{-\kappa(\eta^2+1)\alpha/\tau} x(\alpha)\, d\alpha \\ &= (2\kappa e^{2\kappa}/\pi\tau) \int_0^{\infty} e^{-\kappa\alpha/\tau} x(\alpha)\, d\alpha \int_0^{\infty} e^{-\kappa\eta^2\alpha/\tau} \cos(2\kappa\eta)\, d\eta \\ &= (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \int_0^{\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha)\, d\alpha, \end{aligned}$$

the interchange of integrations being justified by the fact that

$$\begin{aligned} &\int_0^{\infty} |\cos(2\kappa\eta)|\, d\eta \int_0^{\infty} e^{-\kappa(\eta^2+1)\alpha/\tau} \| x(\alpha) \|\, d\alpha \\ &\qquad \leqslant (\tau \| x(\cdot) \|_{\infty}/\kappa) \int_0^{\infty} (\eta^2+1)^{-1}\, d\eta = \pi\tau \| x(\cdot) \|_{\infty}/2\kappa < \infty. \end{aligned}$$

Thus

$$\begin{aligned} \| L_{\kappa,\tau}[f(\lambda)] \| &\leqslant (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \int_0^{\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} \| x(\alpha) \|\, d\alpha \\ &\leqslant \| x(\cdot) \|_{\infty} (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \int_0^{\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}}\, d\alpha \\ &= \| x(\cdot) \|_{\infty}. \end{aligned}$$

Hence

$$\| L_{\kappa,\cdot}[f(\lambda)] \|_{\infty} = \sup_{0 \leqslant \tau < \infty} \| L_{\kappa,\tau}[f(\lambda)] \| \leqslant \| x(\cdot) \|_{\infty}.$$

Proof of sufficiency. By (1), (2), (3) and Theorem 7.1, we have, for almost all $\zeta > 0$,

$$f(\zeta) = \lim_{\kappa \to \infty} \int_0^{\infty} e^{-\zeta\tau} L_{\kappa,\tau}[f(\lambda)]\, d\tau.$$

By **(5)**, a uniformly convex Banach space is reflexive, so that $\mathfrak{X}$ is reflexive. Let ϕ be in $L_1(0, \infty)$. Define

$$T_\kappa(\phi) = \int_0^{\infty} \phi(\tau) L_{\kappa,\tau}[f(\lambda)]\, d\tau.$$

Then

$$\begin{aligned} \| T_\kappa(\phi) \| &\leqslant \int_0^{\infty} |\phi(\tau)| \cdot \| L_{\kappa,\tau}[f(\lambda)] \|\, d\tau \\ &\leqslant \| L_{\kappa,\cdot}[f(\lambda)] \|_{\infty} \int_0^{\infty} |\phi(\tau)|\, d\tau \leqslant M_{\infty} \| \phi \|_1. \end{aligned}$$

Thus $T_\kappa(\phi)$ is a set of linear transformations on a separable space, $L_1(0, \infty)$, to a reflexive space $\mathfrak{X}$, and so, by Theorem 9.1, there is an increasing unbounded sequence $\{\kappa_i\}$, and a linear transformation T on L_1 to $\mathfrak{X}$, such that for every x^* in $\mathfrak{X}^*$, and every ϕ in L_1,

$$\lim_{i\to\infty} x^*(T_{\kappa_i}(\phi)) = x^*(T(\phi)).$$

But by (2), every bounded linear transformation T on L_1 to a uniformly convex Banach space $\mathfrak{X}$ is of the form

$$T(\phi) = \int_0^\infty \phi(\tau)\, x(\tau)\, d\tau,$$

where $x(\tau)$ is in $B_\infty([0, \infty); \mathfrak{X})$. Thus $x(\tau)$ in $B_\infty([0, \infty); \mathfrak{X})$ exists so that T has the above form, and then we must have for every x^* in $\mathfrak{X}^*$,

$$\lim_{i\to\infty} x^*\left(\int_0^\infty \phi(\tau)\, L_{\kappa_i,\tau}[f(\lambda)]\, d\tau\right) = \lim_{i\to\infty} x^*(T_{\kappa_i}(\phi))$$
$$= x^*(T(\phi)) = x^*\left(\int_0^\infty \phi(\tau)\, x(\tau)\, d\tau\right).$$

Let $\phi(\tau) = e^{-\zeta\tau}$. Then, for almost all $\zeta > 0$,

$$x^*(f(\zeta)) = \lim_{i\to\infty} x^*\left(\int_0^\infty e^{-\zeta\tau} L_{\kappa_i,\tau}[f(\lambda)]\, d\tau\right)$$
$$= \lim_{i\to\infty} x^*(T_{\kappa_i}(e^{-\zeta\tau})) = x^*(T(e^{-\zeta\tau}))$$
$$= x^*\left(\int_0^\infty e^{-\zeta\tau} x(\tau)\, d\tau\right).$$

so that for almost all $\zeta > 0$,

$$f(\zeta) = \int_0^\infty e^{-\zeta\tau} x(\tau)\, d\tau.$$

10. Representation Theorems for $f(\lambda)$ in $H_p(\alpha; \mathfrak{X})$. The class $H_p(\alpha; \mathfrak{X})$ is defined in (4; definition 10.4). The following two theorems give the conditions under which a function in $H_p(\alpha; \mathfrak{X})$ can be represented as a Laplace integral.

THEOREM 10.1. *If $f(\lambda)$ is in $H_1(\alpha; \mathfrak{X})$ where $\alpha > 0$, then*

$$\lim_{\kappa\to\infty} L_{\kappa,\tau}[f(\lambda)]$$

exists and equals

$$g(\tau) = (2\pi i)^{-1} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\tau\mu} f(\mu)\, d\mu$$

and

$$f(\lambda) = \int_0^\infty e^{-\lambda\tau} g(\tau)\, d\tau.$$

Proof. By (4; theorem 10.4.1) we have, for $\Re\lambda > \alpha$,

$$f(\lambda) = (2\pi i)^{-1} \int_{\alpha - i\infty}^{\alpha + i\infty} f(\mu)(\lambda - \mu)^{-1}\, d\mu.$$

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Thus

$$L_{\kappa,\tau}[f(\lambda)] = (2\kappa e^{2\kappa}/\pi\tau)\int_0^\infty \cos(2\kappa\eta)\, f(\kappa(\eta^2+1)/\tau)\, d\eta$$
$$= (\kappa e^{2\kappa}/\pi^2\tau)\int_0^\infty \cos(2\kappa\xi)d\xi \int_{-\infty}^\infty f(\alpha+i\eta)\{(\kappa(\xi^2+1)/\tau)-(\alpha+i\eta)\}^{-1}d\eta$$
$$= (e^{2\kappa}/\pi^2)\int_{-\infty}^\infty f(\alpha+i\eta)\, d\eta \int_0^\infty \cos(2\kappa\xi)\{\xi^2+(1-\tau(\alpha+i\eta)/\kappa)\}^{-1}d\xi$$

the interchange of integrations being valid for $\kappa > \alpha\tau$ since

$$\int_{-\infty}^\infty ||f(\alpha+i\eta)||\, d\eta \int_0^\infty |\cos(2\kappa\xi)\{\xi^2+(1-\tau(\alpha+i\eta)/\kappa)\}^{-1}|\, d\xi$$
$$\leqslant \int_{-\infty}^\infty ||f(\alpha+i\eta)||\, d\eta \int_0^\infty (\xi^2+(1-\tau\alpha/\kappa))^{-1}\, d\xi$$
$$\leqslant \tfrac{1}{2}\pi(1-\tau\alpha/\kappa)^{-\frac{1}{2}}\, ||f||_1 < \infty.$$

Thus if $\kappa > \alpha\tau$,

$$L_{\kappa,\tau}[f(\lambda)] = (e^{2\kappa}/\pi)\int_{-\infty}^\infty e^{-2\kappa(1-\tau(\alpha+i\eta)/\kappa)^{1/2}}\,(1-\tau(\alpha+i\eta)/\kappa)^{-\frac{1}{2}} f(\alpha+i\eta)\, d\eta.$$

It is evident that for each η and each $\tau > 0$,

$$\lim_{\kappa\to\infty} e^{2\kappa(1-(1-\tau(\alpha+i\eta)/\kappa)^{1/2})}\,(1-\tau(\alpha+i\eta)/\kappa)^{-\frac{1}{2}} = e^{\tau(\alpha+i\eta)}.$$

Further, a lengthy but straightforward calculation shows that the maximum value of

$$|e^{2\kappa(1-(1-\tau(\alpha+i\eta)/\kappa)^{1/2})}\,(1-\tau(\alpha+i\eta)/\kappa)^{-\frac{1}{2}} - e^{\tau(\alpha+i\eta)}|$$

occurs at $\eta = 0$. Thus

$$||L_{\kappa,\tau}[f(\lambda)] - g(\tau)||$$
$$= ||(2\pi)^{-1}\int_{-\infty}^\infty (e^{2\kappa(1-(1-\tau(\alpha+i\eta)/\kappa)^{1/2})}(1-\tau(\alpha+i\eta)/\kappa)^{-\frac{1}{2}} - e^{\tau(\alpha+i\eta)})\, f(\alpha+i\eta)\, d\eta\, ||$$
$$\leqslant (2\pi)^{-1}|e^{2\kappa(1-(1-\tau\alpha/\kappa)^{1/2})}(1-\tau\alpha/\kappa)^{-\frac{1}{2}} - e^{\tau\alpha}|\int_{-\infty}^\infty ||f(\alpha+i\eta)||\, d\eta \to 0, \quad \kappa\to\infty.$$

Thus

$$\lim_{\kappa\to\infty} L_{\kappa,\tau}[f(\lambda)] = (2\pi i)^{-1}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\tau\mu} f(\mu)\, d\mu = g(\tau).$$

Hence we have, for $\Re\lambda > \alpha$

$$\int_0^\infty e^{-\lambda\tau} g(\tau)\, d\tau = (2\pi)^{-1}\int_0^\infty e^{-\lambda\tau}\, d\tau \int_{-\infty}^\infty e^{\tau(\alpha+i\eta)} f(\alpha+i\eta)\, d\eta$$
$$= (2\pi)^{-1}\int_{-\infty}^\infty f(\alpha+i\eta)\, d\eta \int_0^\infty e^{-(\lambda-(\alpha+i\eta))\tau}\, d\tau$$
$$= (2\pi)^{-1}\int_{-\infty}^\infty f(\alpha+i\eta)(\lambda-(\alpha+i\eta))^{-1}\, d\eta$$
$$= (2\pi i)^{-1}\int_{\alpha-i\infty}^{\alpha+i\infty} f(\mu)(\lambda-\mu)^{-1}\, d\mu = f(\lambda).$$

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The interchange of integrations is valid for $\Re\lambda > \alpha$ since

$$\int_{-\infty}^{\infty} || e^{\tau(\alpha+i\eta)} f(\alpha + i\eta) || \, d\eta = e^{\alpha\tau} \int_{-\infty}^{\infty} || f(\alpha + i\eta) || \, d\eta.$$

THEOREM 10.2. *If*

(1) $\qquad f(\lambda)$ *is in* $H_p(\alpha; \mathfrak{X})$, $p > 1$, $\alpha > 0$,

(2) $\qquad p^{-1} + q^{-1} = 1,$

(3) $\qquad \beta q > 1,$

then

$$f(\lambda) = \lambda^{\beta} \int_0^{\infty} e^{-\lambda\tau} g_\beta(\tau) \, d\tau,$$

where

$$g_\beta(\tau) = \lim_{\kappa\to\infty} L_{\kappa,\tau} [\lambda^{-\beta} f(\lambda)] = (2\pi i)^{-1} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\tau\mu} \mu^{-\beta} f(\mu) \, d\mu.$$

Proof. $\lambda^{-\beta} f(\lambda)$ is in $H_1(\alpha; \mathfrak{X})$; for using Hölder's inequality we have

$$\int_{-\infty}^{\infty} || (\alpha + i\eta)^{-\beta} f(\alpha + i\eta) || \, d\eta$$
$$\leqslant \left\{ \int_{-\infty}^{\infty} || f(\alpha + i\eta) ||^p \, d\eta \right\}^{1/p} \left\{ \int_{\infty}^{-\infty} | \alpha + i\eta) |^{-\beta q} \, d\eta \right\}^{1/q} < \infty.$$

Thus applying the previous theorem to $\lambda^{-\beta} f(\lambda)$, we have

$$\lim_{\kappa\to\infty} L_{\kappa,\tau} [\lambda^{-\beta} f(\lambda)] = (2\pi i)^{-1} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\tau\mu} \mu^{-\beta} f(\mu) \, d\mu = g_\beta(\tau),$$
$$f(\lambda) = \lambda^{\beta} \int_0^{\infty} e^{-\lambda\tau} g_\beta(\tau) \, d\tau.$$

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