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Values of the Dedekind Eta Function at Quadratic Irrationalities: Corrigendum

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Abstract. Habib Muzaffar of Carleton University has pointed out to the authors that in their paper [A] only the result

$$\pi_{K,d}(x) + \pi_{K^{-1},d}(x) = \frac{1}{h(d)} \frac{x}{\log x} + O_{K,d}\Big(\frac{x}{\log^2 x}\Big)$$

follows from the prime ideal theorem with remainder for ideal classes, and not the stronger result

$$\pi_{K,d}(x) = \frac{1}{2h(d)} \frac{x}{\log x} + O_{K,d}\Big(\frac{x}{\log^2 x}\Big)$$

stated in Lemma 5.2. This necessitates changes in Sections 5 and 6 of [A]. The main results of the paper are not affected by these changes. It should also be noted that, starting on page 177 of [A], each and every occurrence of $o(s-1)$ should be replaced by $o(1)$.

Sections 5 and 6 of [A] have been rewritten to incorporate the above mentioned correction and are given below. They should replace the original Sections 5 and 6 of [A].

5 Estimation of a Certain Infinite Product

Our aim in this section is to prove the following result. We make use of the ideas in [4, pp. 346–353].

Proposition 5.1 *Let $K \in H(d)$. Let ω be a complex number with $|\omega| = 1$. Then there exists a nonzero complex number $C(K, d, \omega)$ depending only on K, d and ω such that*

$$\prod_{\substack{p \\ (\frac{d}{p})=1 \\ K_p=K}} \Big(1 - \frac{\omega}{p^s}\Big) \prod_{\substack{p \\ (\frac{d}{p})=1 \\ K_p=K^{-1}}} \Big(1 - \frac{\omega}{p^s}\Big) = (s-1)^{\omega/h(d)} C(K, d, \omega)\big(1 + o(1)\big),$$

as $s \to 1^+$, where p runs through prime numbers.

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This proposition will be used in the proof of Proposition 6.1. In order to prove Proposition 5.1 we require a number of lemmas. For $x \in R$ and $K \in H(d)$ we set

$$\pi_{K,d}(x) := \sum_{\substack{p \le x \\ K_p = K}} 1,$$

$$\theta_{K,d}(x) := \sum_{\substack{p \le x \\ K_p = K}} \log p,$$

$$\kappa_{K,d}(x) := \sum_{\substack{p \le x \\ K_p = K}} \frac{\log p}{p},$$

$$\lambda_{K,d}(x) := \sum_{\substack{p \le x \\ K_p = K}} \frac{1}{p},$$

where p runs through prime numbers.

Lemma 5.2 *Let $K \in H(d)$. Then*

$$\pi_{K,d}(x) + \pi_{K^{-1},d}(x) = \frac{1}{h(d)} \frac{x}{\log x} + O_{K,d}\Big(\frac{x}{\log^2 x}\Big),$$

where the constant implied by the O-symbol depends on K and d, and not on x.

Proof We have

$$\pi_{K,d}(x) = \sum_{\substack{p \le x \\ K_p = K}} 1 = \sum_{\substack{p \le x \\ K_p = K \\ (\frac{d}{p}) = 1}} 1 + \sum_{\substack{p \le x \\ K_p = K \\ (\frac{d}{p}) = 0}} 1.$$

Clearly

$$0 \le \sum_{\substack{p \le x \\ K_p = K \\ (\frac{d}{p}) = 0}} 1 \le \sum_{p \mid d} 1 = \tau(d) = O_d(1).$$

Thus

$$\pi_{K,d}(x) = \sum_{\substack{p \le x \\ K_p = K \\ (\frac{d}{p}) = 1}} 1 + O_d(1).$$

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Hence

$$\pi_{K,d}(x) + \pi_{K^{-1},d}(x) = \sum_{\substack{p \le x \\ K_p = K \\ (\frac{d}{p})=1}} 1 + \sum_{\substack{p \le x \\ K_p = K^{-1} \\ (\frac{d}{p})=1}} 1 + O_d(1). \tag{5.1}$$

As d is the discriminant of the imaginary quadratic field F, we have $F = \mathbb{Q}(\sqrt{d})$. In the ring O_F of integers of F the prime p factors into prime ideals as follows:

$$pO_F = \begin{cases} PP', P \neq P', N(P) = N(P') = p, & \text{if } (\frac{d}{p}) = 1, \\ P^2, P = P', N(P) = p, & \text{if } (\frac{d}{p}) = 0, \\ P, P = P', N(P) = p^2, & \text{if } (\frac{d}{p}) = -1, \end{cases} \tag{5.2}$$

where P' denotes the conjugate ideal of P and $N(P)$ denotes the norm of P from F to $\mathbb{Q}$. For primes p with $(\frac{d}{p}) = 1$ we distinguish between the prime ideals P and P' as follows. In this case the congruence $u^2 \equiv d \pmod{4p}$ has exactly two solutions satisfying $0 \le u < 2p$, and we denote the smaller of these two solutions by t. Then

$$P_1 = \left[p, \frac{-t + \sqrt{d}}{2} \right]$$

is a prime ideal of O_F uniquely determined by p and d. The conjugate ideal of P_1 is

$$P_1' = \left[p, \frac{-t - \sqrt{d}}{2} \right] = \left[p, \frac{t + \sqrt{d}}{2} \right].$$

Moreover,

$$P_1 P_1' = pO_F, \quad P_1 \neq P_1'. \tag{5.3}$$

Thus in the first line of (5.2) we have $P = P_1$ or P_1'. We let $C(F)$ denote the ideal class group of F. If A is an ideal of O_F we denote its class in $C(F)$ by $\bar{A}$. Recall that $H(d)$ denotes the form class group of discriminant d. Then it is well known that

$$\alpha \colon H(d) \to C(F)$$

defined by

$$\alpha([a, b, c]) = \overline{\left[a, \frac{-b + \sqrt{d}}{2} \right]}$$

is an isomorphism. For p a prime with $(\frac{d}{p}) = 1$ we have

$$\alpha(K_p) = \alpha\big([p, t, (t^2 - d)/4p]\big) = \overline{\left[p, \frac{-t + \sqrt{d}}{2} \right]} = \bar{P}_1.$$

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From (5.3) we see that

$$\overline{P_1'} = \overline{P_1}^{-1}$$

so that

$$\alpha(K_p)^{-1} = \overline{P_1'}.$$

Thus

$$\begin{aligned}
\sum_{\substack{p\le x\\ (\frac{d}{p})=1\\ K_p=K}} 1 + \sum_{\substack{p\le x\\ (\frac{d}{p})=1\\ K_p=K^{-1}}} 1 &= \sum_{\substack{p\le x\\ (\frac{d}{p})=1\\ \alpha(K_p)=\alpha(K)}} 1 + \sum_{\substack{p\le x\\ (\frac{d}{p})=1\\ \alpha(K_p)=\alpha(K)^{-1}}} 1\\
&= \sum_{\substack{p\le x\\ (\frac{d}{p})=1\\ \overline{P_1}=\alpha(K)}} 1 + \sum_{\substack{p\le x\\ (\frac{d}{p})=1\\ \overline{P_1'}=\alpha(K)}} 1\\
&= \sum_{\substack{p\le x\\ (\frac{d}{p})=1\\ P_1\in\alpha(K)}} 1 + \sum_{\substack{p\le x\\ (\frac{d}{p})=1\\ P_1'\in\alpha(K)}} 1\\
&= \sum_{\substack{P\\ P\ne P'\\ N(P)=p\le x\\ P\in\alpha(K)}} 1.
\end{aligned}$$

Hence

$$\sum_{\substack{p\le x\\ (\frac{d}{p})=1\\ K_p=K}} 1 + \sum_{\substack{p\le x\\ (\frac{d}{p})=1\\ K_p=K^{-1}}} 1 = \sum_{\substack{P\\ P\ne P'\\ N(P)=p\le x\\ P\in\alpha(K)}} 1. \tag{5.4}$$

Now by the prime ideal theorem with remainder for ideal classes of F (see for example [7, Corollary (i), p. 369]), we have with $h(F) = |C(F)|$

$$\sum_{\substack{P\\ N(P)\le x\\ P\in\alpha(K)}} 1 = \frac{\operatorname{li} x}{h(F)} + O_{F,\alpha(K)}(xe^{-B(F,\alpha(K))\sqrt{\log x}})$$

for some positive constant B depending only on the field F and the class $\alpha(K)$ of $C(F)$, that is,

$$\sum_{\substack{P\\ N(P)\le x\\ P\in\alpha(K)}} 1 = \frac{\operatorname{li} x}{h(d)} + O_{K,d}(xe^{-b(K,d)\sqrt{\log x}}) \tag{5.5}$$

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for some positive constant b depending only on the discriminant d and the class K of $H(d)$. Next

(5.6) $$\sum_{\substack{P \\ N(P)\le x \\ P\in\alpha(K)}} 1 = \sum_1 + \sum_2 + \sum_3,$$

where

(5.7)$_1$ $$\sum_1 := \sum_{\substack{P \\ P\neq P' \\ N(P)=p\le x \\ P\in\alpha(K)}} 1,$$

(5.7)$_2$ $$\sum_2 := \sum_{\substack{P \\ P=P' \\ N(P)=p\le x \\ P\in\alpha(K)}} 1,$$

(5.7)$_3$ $$\sum_3 := \sum_{\substack{P \\ P=P' \\ N(P)=p^2\le x \\ P\in\alpha(K)}} 1.$$

Clearly

(5.8) $$0 \le \sum_2 \le \sum_{p\mid d} 1 = \tau(d) = O_d(1)$$

and

(5.9) $$0 \le \sum_3 \le \sum_{p\le x^{1/2}} 1 = O(x^{1/2}).$$

Hence, by (5.1), (5.4), (5.7), (5.6), (5.5), (5.8), (5.9), we have

$$\begin{aligned}
\pi_{K,d}(x) &+ \pi_{K^{-1},d}(x) \\
&= \sum_1 + O_d(1) \\
&= \sum_{\substack{P \\ N(P)\le x \\ P\in\alpha(K)}} 1 - \sum_2 - \sum_3 + O_d(1) \\
&= \frac{\mathrm{li}(x)}{h(d)} + O_{K,d}(xe^{-b(K,d)\sqrt{\log x}}) + O_d(1) + O(x^{1/2}) \\
&= \frac{1}{h(d)}\left\{\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)\right\} + O_{K,d}\left(\frac{x}{\log^2 x}\right) + O_d(1) + O(x^{1/2}) \\
&= \frac{1}{h(d)}\frac{x}{\log x} + O_{K,d}\left(\frac{x}{\log^2 x}\right);
\end{aligned}$$

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where we note that

$$\operatorname{li} x = x/\log x + O(x/\log^2 x) \quad \text{and} \quad \exp\bigl(-b(K,d)\sqrt{\log x}\bigr) = O_{K,d}(1/\log^2 x).$$

This completes the proof of Lemma 5.2. ■

Lemma 5.3 *Let $K \in H(d)$. Then*

$$\theta_{K,d}(x) + \theta_{K^{-1},d}(x) = \frac{1}{h(d)} x + O_{K,d}\Bigl(\frac{x}{\log x}\Bigr).$$

Proof By partial summation we have

$$\theta_{K,d}(x) = \pi_{K,d}(x)\log x - \int_2^x \frac{\pi_{K,d}(t)}{t}\,dt, \quad x \geq 2,$$

see for example [4, Theorem 421, p. 346]. Thus

$$\theta_{K,d}(x) + \theta_{K^{-1},d}(x) = \bigl(\pi_{K,d}(x) + \pi_{K^{-1},d}(x)\bigr)\log x - \int_2^x \frac{\pi_{K,d}(t) + \pi_{K^{-1},d}(t)}{t}\,dt$$

and the result follows using Lemma 5.2. ■

Lemma 5.4 *Let $K \in H(d)$. Then*

$$\kappa_{K,d}(x) + \kappa_{K^{-1},d}(x) = \frac{1}{h(d)}\log x + O_{K,d}(\log\log x).$$

Proof By partial summation we have

$$\kappa_{K,d}(x) = \frac{\theta_{K,d}(x)}{x} + \int_2^x \frac{\theta_{K,d}(t)}{t^2}\,dt,$$

and similarly for K^{-1}. Thus

$$\kappa_{K,d}(x) + \kappa_{K^{-1},d}(x) = \frac{\theta_{K,d}(x) + \theta_{K^{-1},d}(x)}{x} + \int_2^x \frac{\theta_{K,d}(t) + \theta_{K^{-1},d}(t)}{t^2}\,dt$$

and the result follows on using Lemma 5.3. ■

Lemma 5.5 *Let $K \in H(d)$. Then there exists a constant $c(K,d)$ depending only on K and d such that*

$$\lambda_{K,d}(x) + \lambda_{K^{-1},d}(x) = \frac{1}{h(d)}\log\log x + c(K,d) + O_{K,d}\Bigl(\frac{1}{\log\log x}\Bigr).$$

Proof Set

$$\kappa_{K\cdot d}(x) + \kappa_{K^{-1},d}(x) = \frac{1}{h(d)}\log x + \tau_{K,d}(x).$$

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By Lemma 5.4 we have $\tau_{K,d}(x) = O_{K,d}(\log\log x)$. Next, by partial summation, we have

$$\lambda_{K,d}(x) = \frac{\kappa_{K,d}(x)}{\log x} + \int_2^x \frac{\kappa_{K,d}(t)}{t\log^2 t}\,dt$$

and similarly for K^{-1}. Thus

$$\lambda_{K,d}(x) + \lambda_{K^{-1},d}(x) = \frac{\kappa_{K,d}(x) + \kappa_{K^{-1},d}(x)}{\log x} + \int_2^x \frac{\kappa_{K,d}(t) + \kappa_{K^{-1},d}(t)}{t\log^2 t}\,dt.$$

Appealing to Lemma 5.4, we obtain

$$\begin{aligned}\lambda_{K,d}(x) + \lambda_{K^{-1},d}(x) &= \frac{1}{h(d)} + O_{K,d}\Big(\frac{\log\log x}{\log x}\Big)\\ &\quad + \frac{1}{h(d)}(\log\log x - \log\log 2) + \int_2^x \frac{\tau_{K,d}(t)}{t\log^2 t}\,dt.\end{aligned}$$

As $\tau_{K,d}(t) = O_{K,d}(\log\log t)$ the integrals $\int_2^\infty \frac{\tau_{K,d}(t)}{t\log^2 t}\,dt$ and $\int_x^\infty \frac{\tau_{K,d}(t)}{t\log^2 t}\,dt$ are convergent. Moreover

$$\int_x^\infty \frac{\tau_{K,d}(t)\,dt}{t\log^2 t} = O_{K,d}\Big(\frac{1}{\log\log x}\Big),$$

so

$$\lambda_{K,d}(x) + \lambda_{K^{-1},d}(x) = \frac{1}{h(d)}\log\log x + c(K,d) + O_{K,d}\Big(\frac{1}{\log\log x}\Big),$$

with

$$c(K,d) = \frac{1}{h(d)}(1 - \log\log 2) + \int_2^\infty \frac{\kappa_{K,d}(t) + \kappa_{K^{-1},d}(t) - \frac{1}{h(d)}\log t}{t\log^2 t}\,dt. \qquad \blacksquare$$

Lemma 5.6 *Let $K \in H(d)$. Then*

$$\sum_{\substack{p\\ K_p=K}} \frac{1}{p^s} + \sum_{\substack{p\\ K_p=K^{-1}}} \frac{1}{p^s} = -\frac{1}{h(d)}\log(s-1) + \Big(c(K,d) - \frac{\gamma}{h(d)}\Big) + o(1),$$

as $s \to 1^+$.

Proof Let δ be a real number satisfying $0 < \delta < 1/4$. By partial summation we have

$$\sum_{\substack{p\le x\\ K_p=K}} \frac{1}{p^{1+\delta}} = \frac{\lambda_{K,d}(x)}{x^\delta} + \delta\int_2^x \frac{\lambda_{K,d}(t)}{t^{1+\delta}}\,dt, \quad x \ge 2.$$

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Let $x \to \infty$. By the definition of $\lambda_{K,d}(x)$ we obtain

$$\sum_{\substack{p \\ K_p = K}} \frac{1}{p^{1+\delta}} = \delta \int_2^\infty \frac{\lambda_{K,d}(t)}{t^{1+\delta}}\, dt.$$

We set

$$\lambda_{K,d}(x) + \lambda_{K^{-1},d}(x) = \frac{1}{h(d)} \log\log x + c(K,d) + E_{K,d}(x).$$

By Lemma 5.5 we have $E_{K,d}(x) = O_{K,d}(1/\log\log x)$, say

$$|E_{K,d}(x)| \le \frac{e(K,d)}{\log\log x}, \quad x > e\,(= 2.7182818284\cdots),$$

for some positive number $e(K,d)$. Then

$$\begin{aligned}\sum_{\substack{p \\ K_p = K}} \frac{1}{p^{1+\delta}} + \sum_{\substack{p \\ K_p = K^{-1}}} \frac{1}{p^{1+\delta}} &= \delta \int_2^\infty \frac{\lambda_{K,d}(t) + \lambda_{K^{-1},d}(t)}{t^{1+\delta}}\, dt \\ &= \delta \int_2^\infty \frac{\frac{1}{h(d)} \log\log t + c(K,d) + E_{k,d}(t)}{t^{1+\delta}}\, dt \\ &= \frac{\delta}{h(d)} \int_2^\infty \frac{\log\log t}{t^{1+\delta}}\, dt + \frac{c(K,d)}{2^\delta} + \delta \int_2^\infty \frac{E_{K,d}(t)}{t^{1+\delta}}\, dt,\end{aligned}$$

as

$$\delta \int_2^\infty \frac{dt}{t^{1+\delta}} = \frac{1}{2^\delta}.$$

Now

$$\Big| \int_1^2 \frac{\log\log t}{t^{1+\delta}}\, dt \Big| \le \int_1^2 \frac{|\log\log t|}{t}\, dt = \text{constant}$$

so that

$$\delta \int_1^2 \frac{\log\log t}{t^{1+\delta}}\, dt = O(\delta).$$

Further, putting $t = e^{u/\delta}$, we obtain

$$\begin{aligned}\delta \int_1^\infty \frac{\log\log t}{t^{1+\delta}}\, dt &= \int_0^\infty e^{-u} \log\Big(\frac{u}{\delta}\Big)\, du \\ &= \int_0^\infty e^{-u} \log u\, du - \log\delta \int_0^\infty e^{-u}\, du \\ &= -\gamma - \log\delta,\end{aligned}$$

as

$$\int_0^\infty e^{-u} \log u\, du = -\gamma,$$

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see for example [3, p. 602]. Hence

$$\delta \int_2^\infty \frac{\log\log t}{t^{1+\delta}}\,dt = -\gamma - \log\delta + O(\delta).$$

Now set $T = e^{1/\sqrt{\delta}}$ so that

$$\log T = 1/\sqrt{\delta},\ \log\log T = \frac{1}{2}|\log\delta|, \quad T > e^2.$$

We also set

$$g(K,d) = \int_2^{e^2} \frac{|E_{K,d}(t)|}{t}\,dt.$$

Then

$$\begin{aligned}
\left|\delta \int_2^\infty \frac{E_{K,d}(t)}{t^{1+\delta}}\,dt\right| &\le \delta \int_2^{e^2} \frac{|E_{K,d}(t)|}{t^{1+\delta}}\,dt + \delta \int_{e^2}^{T} \frac{|E_{K,d}(t)|}{t^{1+\delta}}\,dt + \delta \int_T^\infty \frac{|E_{K,d}(t)|}{t^{1+\delta}}\,dt \\
&\le \delta \int_2^{e^2} \frac{|E_{K,d}(t)|}{t}\,dt + \delta \frac{e(K,d)}{\log\log(e^2)} \int_{e^2}^{T} \frac{dt}{t^{1+\delta}} \\
&\quad + \delta \frac{e(K,d)}{\log\log T} \int_T^\infty \frac{dt}{t^{1+\delta}} \\
&\le \delta g(K,d) + \delta \frac{e(K,d)}{\log 2} \int_{e^2}^{T} \frac{dt}{t} + \delta \frac{e(K,d)}{\log\log T} \frac{1}{\delta T^\delta} \\
&\le \delta g(K,d) + \delta \frac{e(K,d)}{\log 2} \log T + \frac{e(K,d)}{\log\log T} \\
&\le \delta g(K,d) + 2\delta e(K,d)\log T + \frac{2e(K,d)}{|\log\delta|} \\
&= g(K,d)\delta + 2e(K,d)\sqrt{\delta} + \frac{2e(K,d)}{|\log\delta|},
\end{aligned}$$

so that

$$\delta \int_2^\infty \frac{E_{K,d}(t)}{t^{1+\delta}}\,dt = o(1), \quad \text{as } \delta \to 0^+.$$

Hence

$$\begin{aligned}
\sum_{\substack{p \\ K_p = K}} \frac{1}{p^{1+\delta}} + \sum_{\substack{p \\ K_p = K^{-1}}} \frac{1}{p^{1+\delta}} &= \frac{1}{h(d)}\big(-\gamma - \log\delta + O(\delta)\big) + c\big(K,d)(1+o(1)\big) + o(1) \\
&= -\frac{1}{h(d)}\log\delta + \left(c(K,d) - \frac{\gamma}{h(d)}\right) + o(1),
\end{aligned}$$

as $\delta \to 0^+$. Finally we set $s = 1 + \delta$ to obtain the asserted result. ∎

Lemma 5.7 *Let $K \in H(d)$. Let ω be a complex number such that $|\omega| = 1$.*

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(i) The series

$$\sum_{\substack{p \\ K_p=K}} \Big(\sum_{n=2}^{\infty} \frac{\omega^n}{np^n} \Big)$$

converges.

(ii) Denoting the sum of the series in (i) by $A(K, d, \omega)$, we have

$$\sum_{\substack{p \\ K_p=K}} \Big(\sum_{n=2}^{\infty} \frac{\omega^n}{np^{ns}} \Big) = A(K, d, \omega) + o(1), \quad \text{as } s \to 1^+.$$

Proof For $s \geq 1$ we have

$$\Big| \sum_{n=2}^{\infty} \frac{\omega^n}{np^{ns}} \Big| \leq \sum_{n=2}^{\infty} \frac{1}{np^{ns}} \leq \sum_{n=2}^{\infty} \frac{1}{np^n} \leq \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{p^n} = \frac{1}{2} \frac{1/p^2}{1 - 1/p} \leq \frac{1}{p^2},$$

so the series $\sum_{p:\, K_p=K} (\sum_{n=2}^{\infty} \omega^n / np^{ns})$ is uniformly convergent for $s \geq 1$. Thus, in particular, $\sum_{p:\, K_p=K} (\sum_{n=2}^{\infty} \omega^n / np^n)$ converges, proving (i). Moreover, the uniform convergence ensures that

$$\lim_{s \to 1^+} \sum_{\substack{p \\ K_p=K}} \Big(\sum_{n=2}^{\infty} \frac{\omega^n}{np^{ns}} \Big) = \sum_{\substack{p \\ K_p=K}} \Big(\sum_{n=2}^{\infty} \frac{\omega^n}{np^n} \Big) = A(K, d, \omega),$$

proving (ii). We note that $\overline{A(K, d, \omega)} = A(K, d, \bar{\omega})$. ■

Lemma 5.8 *Let $K \in H(d)$. Let ω be a complex number with $|\omega| = 1$. Then there exists a nonzero complex number $B(K, d, \omega)$ depending only on K, d and ω such that*

$$\prod_{\substack{p \\ K_p=K}} (1 - \omega p^{-s}) \prod_{\substack{p \\ K_p=K^{-1}}} (1 - \omega p^{-s}) = (s-1)^{\omega/h(d)} B(K, d, \omega) \big(1 + o(1)\big),$$

as $s \to 1^+$.

Proof Let s be a real number with $s > 1$. We have

$$\sum_{\substack{p \\ K_p=K}} |-\omega p^{-s}| = \sum_{\substack{p \\ K_p=K}} p^{-s} \leq \sum_{p} p^{-s} \leq \zeta(s)$$

so that the infinite series $\sum_{p:\, K_p=K} -\omega p^{-s}$ converges absolutely. Hence the infinite product $\prod_{p:\, K_p=K} (1 - \omega p^{-s})$ converges absolutely and thus converges. Similarly the

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corresponding product, over the primes p such that $K_p = K^{-1}$, converges. We have, noting that $|\omega p^{-s}| < 1$, and appealing to Lemmas 5.6 and 5.7(ii),

$$\begin{aligned}
&\prod_{\substack{p\\ K_p=K}} (1-\omega p^{-s}) \prod_{\substack{p\\ K_p=K^{-1}}} (1-\omega p^{-s})\\
&\quad= \prod_{\substack{p\\ K_p=K}} e^{\log(1-\omega p^{-s})} \prod_{\substack{p\\ K_p=K^{-1}}} e^{\log(1-\omega p^{-s})}\\
&\quad= \exp\Big\{ \sum_{\substack{p\\ K_p=K}} \log(1-\omega p^{-s}) + \sum_{\substack{p\\ K_p=K^{-1}}} \log(1-\omega p^{-s}) \Big\}\\
&\quad= \exp\Big\{ -\sum_{\substack{p\\ K_p=K}} \sum_{n=1}^{\infty} \frac{\omega^n}{n} p^{-ns} - \sum_{\substack{p\\ K_p=K^{-1}}} \sum_{n=1}^{\infty} \frac{\omega^n}{n} p^{-ns} \Big\}\\
&\quad= \exp\Big\{ -\omega \sum_{\substack{p\\ K_p=K}} p^{-s} - \omega \sum_{\substack{p\\ K_p=K^{-1}}} p^{-s} - \sum_{\substack{p\\ K_p=K}} \sum_{n=2}^{\infty} \frac{\omega^n}{n} p^{-ns}\\
&\qquad\qquad - \sum_{\substack{p\\ K_p=K^{-1}}} \sum_{n=2}^{\infty} \frac{\omega^n}{n} p^{-ns} \Big\}\\
&\quad= \exp\Big\{ -\omega\Big(-\log(s-1)/h(d) + \big(c(K,d) - \gamma/h(d)\big) + o(1)\Big)\\
&\qquad\qquad - \big(A(K,d,\omega) + A(K^{-1},d,\omega) + o(1)\big) \Big\}\\
&\quad= (s-1)^{\omega/h(d)} B(K,d,\omega)\big(1+o(1)\big), \quad \text{as } s \to 1^+,
\end{aligned}$$

where

$$B(K,d,\omega) := \exp\Big(\omega\big(\gamma/h(d) - c(K,d)\big) - \big(A(K,d,\omega) + A(K^{-1},d,\omega)\big)\Big) \neq 0.$$

We note that $\overline{B(K,d,\omega)} = B(K,d,\bar{\omega})$. ∎

Proof of Proposition 5.1 Let $K \in H(d)$. If p is a prime with $K_p = K$ then $(\frac{d}{p}) = 0$ or 1. Hence

$$\begin{aligned}
&\prod_{\substack{p\\ (\frac{d}{p})=1\\ K_p=K}} (1-\omega p^{-s}) \prod_{\substack{p\\ (\frac{d}{p})=1\\ K_p=K^{-1}}} (1-\omega p^{-s})\\
&\qquad= \frac{\prod_{\substack{p\\ K_p=K}} (1-\omega p^{-s}) \prod_{\substack{p\\ K_p=K^{-1}}} (1-\omega p^{-s})}{\prod_{\substack{p\\ (\frac{d}{p})=0\\ K_p=K}} (1-\omega p^{-s}) \prod_{\substack{p\\ (\frac{d}{p})=0\\ K_p=K^{-1}}} (1-\omega p^{-s})}
\end{aligned}$$

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$$= \frac{(s-1)^{\omega/h(d)} B(K,d,\omega)\big(1+o(1)\big)}{\prod_{\substack{p \\ (\frac{d}{p})=0 \\ K_p=K}} (1-\omega p^{-1})\big(1+o(1)\big) \prod_{\substack{p \\ (\frac{d}{p})=0 \\ K_p=K^{-1}}} (1-\omega p^{-1})\big(1+o(1)\big)}$$

$$= (s-1)^{\omega/h(d)} C(K,d,\omega)\big(1+o(1)\big), \quad \text{as } s \to 1^+,$$

where

$$C(K,d,\omega) := \frac{B(K,d,\omega)}{\prod_{\substack{p \\ (\frac{d}{p})=0 \\ K_p=K}} (1-\omega p^{-1}) \prod_{\substack{p \\ (\frac{d}{p})=0 \\ K_p=K^{-1}}} (1-\omega p^{-1})} \neq 0.$$

We note that $\overline{C(K,d,\omega)} = C(K,d,\bar{\omega})$. ■

6 The Quantity $j(K,d)$

In this section we make use of Proposition 5.1 to determine the limiting behaviour of the infinite product

$$\prod_{\substack{p \\ (\frac{d}{p})=1}} \left(1 - \frac{f(K,K_p)}{p^s}\right)\left(1 - \frac{f(K,K_p)^{-1}}{p^s}\right)$$

as $s \to 1^+$ for $K(\neq I) \in H(d)$. We prove

Proposition 6.1 *If $K(\neq I) \in H(d)$ then*

$$\lim_{s\to 1^+} \prod_{\substack{p \\ (\frac{d}{p})=1}} \left(1 - \frac{f(K,K_p)}{p^s}\right)\left(1 - \frac{f(K,K_p)^{-1}}{p^s}\right)$$

exists and is a nonzero real number which we denote by $j(K,d)$.

Proof Let s be a real number with $s > 1$. Then, by (2.21), (2.20), (2.19), and Proposition 5.1, we obtain

$$\prod_{\substack{p \\ (\frac{d}{p})=1}} \left(1 - \frac{f(K,K_p)}{p^s}\right)\left(1 - \frac{f(K,K_p)^{-1}}{p^s}\right)$$

$$= \prod_{\substack{p \\ (\frac{d}{p})=1}} \left(1 - \frac{\exp(2\pi i[K,K_p])}{p^s}\right)\left(1 - \frac{\exp(-2\pi i[K,K_p])}{p^s}\right)$$

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$$= \prod_{\substack{p \\ (\frac{d}{p})=1}} \left[1 - \frac{\exp\bigl(2\pi i \sum_{j=1}^{\ell} \operatorname{ind}_{A_j}(K) \operatorname{ind}_{A_j}(K_p)/h_j\bigr)}{p^s}\right]$$

$$\times \left[1 - \frac{\exp\bigl(-2\pi i \sum_{j=1}^{\ell} \operatorname{ind}_{A_j}(K) \operatorname{ind}_{A_j}(K_p)/h_j\bigr)}{p^s}\right]$$

$$= \prod_{b_1,\dots,b_\ell=0}^{h_1-1,\dots,h_\ell-1} \prod_{\substack{p \\ (\frac{d}{p})=1 \\ K_p = A_1^{b_1}\cdots A_\ell^{b_\ell}}} \left[1 - \frac{\exp(2\pi i \sum_{j=1}^{\ell} k_j b_j/h_j)}{p^s}\right]$$

$$\times \left[1 - \frac{\exp(-2\pi i \sum_{j=1}^{\ell} k_j b_j/h_j)}{p^s}\right]$$

$$= \prod_{b_1,\dots,b_\ell=0}^{h_1-1,\dots,h_\ell-1} (s-1)^{\frac{1}{h(d)} \exp(2\pi i \sum_{j=1}^{\ell} k_j b_j/h_j)}$$

$$\times C\Bigl(A_1^{b_1}\cdots A_\ell^{b_\ell}, d, \exp\Bigl(2\pi i \sum_{j=1}^{\ell} k_j b_j/h_j\Bigr)\Bigr)\bigl(1+o(1)\bigr)$$

$$= (s-1)^{\frac{1}{h(d)} \prod_{j=1}^{\ell} (\sum_{b_j=0}^{h_j-1} \exp(2\pi i k_j b_j/h_j))}$$

$$\times \prod_{b_1,\dots,b_\ell=0}^{h_1-1,\dots,h_\ell-1} C\Bigl(A_1^{b_1}\cdots A_\ell^{b_\ell}, d, \exp\Bigl(2\pi i \sum_{j=1}^{\ell} k_j b_j/h_j\Bigr)\Bigr)\bigl(1+o(1)\bigr).$$

As $K \neq I$, at least one of $k_1, \dots, k_\ell$ is nonzero, say k_j, in which case $0 < k_j < h_j$ and

$$\sum_{b_j=0}^{h_j-1} \exp(2\pi i k_j b_j/h_j) = 0.$$

Thus

$$\prod_{\substack{p \\ (\frac{d}{p})=1}} \left(1 - \frac{f(K, K_p)}{p^s}\right)\left(1 - \frac{f(K, K_p)^{-1}}{p^s}\right)$$

$$= \prod_{b_1,\dots,b_\ell=0}^{h_1-1,\dots,h_\ell-1} C\Bigl(A_1^{b_1}\cdots A_\ell^{b_\ell}, d, \exp\Bigl(2\pi i \sum_{j=1}^{\ell} k_j b_j/h_j\Bigr)\Bigr)\bigl(1+o(1)\bigr)$$

$$= \prod_{L \in H(d)} C\bigl(L, d, f(K, L)\bigr)\bigl(1+o(1)\bigr),$$

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as $s \to 1^+$, by (2.18)–(2.21). Hence

$$\lim_{s\to 1^+} \prod_{\substack{p \\ (\frac{d}{p})=1}} \left(1 - \frac{f(K,K_p)}{p^s}\right)\left(1 - \frac{f(K,K_p)^{-1}}{p^s}\right)$$

exists and is equal to

$$j(K,d) := \prod_{L\in H(d)} C\big(L,d,f(K,L)\big). \tag{6.1}$$

Since each $C\big(L,d,f(K,L)\big)$ with $L \in H(d)$ is a nonzero complex number, $j(K,d)$ is a nonzero complex number. However, by the limit form above, $j(K,d)$ is real as $f(K,K_p)^{-1} = \overline{f(K,K_p)}$. Hence $j(K,d)$ is a nonzero real number. ■

Again from the limit form of $j(K,d)$ above, we see that

$$j(K,d) = j(K^{-1},d). \tag{6.2}$$

It is convenient to set

$$m(K,d) := \frac{t_1(d)}{j(K,d)}, \quad K \in H(d), \tag{6.3}$$

where $t_1(d)$ is defined in (2.32). Thus, appealing to (2.32), Proposition 6.1, and (6.3), we obtain

$$m(K,d) = \frac{\prod_{p:\,(\frac{d}{p})=1}(1-\frac{1}{p^2})}{\lim_{s\to 1^+}\prod_{p:\,(\frac{d}{p})=1}(1-\frac{f(K,K_p)}{p^s})(1-\frac{f(K,K_p)^{-1}}{p^s})}$$

so that

$$m(K,d) = \frac{\lim_{s\to 1^+}\prod_{p:\,(\frac{d}{p})=1}(1-\frac{1}{p^{2s}})}{\lim_{s\to 1^+}\prod_{p:\,(\frac{d}{p})=1}(1-\frac{f(K,K_p)}{p^s})(1-\frac{f(K,K_p)^{-1}}{p^s})}$$

that is

$$m(K,d) = \lim_{s\to 1^+} \prod_{\substack{p \\ (\frac{d}{p})=1}} \frac{(1-\frac{1}{p^s})(1+\frac{1}{p^s})}{(1-\frac{f(K,K_p)}{p^s})(1-\frac{f(K,K_p)^{-1}}{p^s})}. \tag{6.4}$$

From (6.2) and (6.3) we deduce that

$$m(K,d) = m(K^{-1},d). \tag{6.5}$$

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